Volume

2.3

Mathematics 10, **pages 80–91**

Suggested Timing

100–120 min

Materials

- conical cup
- paper
- scissors
- tape or glue
- sand, rice, or popcorn

Blackline Masters

BLM 2–3 Chapter 2 Warm-Up
BLM 2–5 Chapter 2 Unit 1 Project
BLM 2–9 Section 2.3 Extra Practice
TM 2–3 How to Do Page 91 #19 Using TI-Nspire™
TM 2–4 How to Do Page 91 #19 Using Microsoft® Excel

Mathematical Processes

✓ Communication (C)
✓ Connections (CN)
✓ Mental Math and Estimation (ME)
✓ Problem Solving (PS)
✓ Reasoning (R)
✓ Technology (T)
✓ Visualization (V)

Specific Outcomes

M3 Solve problems, using SI and imperial units, that involve the surface area and volume of 3-D objects, including:

- right cones
- right cylinders
- right prisms
- right pyramids
- spheres.

AN3 Demonstrate an understanding of powers with integral and rational exponents.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2, 4–9, 15, 16, 21
Typical	#1–5, 7–11, 13–16, 20, 21
Extension/Enrichment	#8, 11–21

Unit Project The Unit 1 project questions are, 15, 16, and 21.

Planning Notes

Have students complete the warm-up questions on **BLM 2–3 Chapter 2 Warm-Up** to reinforce prerequisite skills needed for this section.

As a class, read and discuss the introductory text and photograph. Reinforce that citizens need to understand mathematics in order to make decisions about Canada's natural resources. Use the opportunity to discuss examples of people working in the natural resources sector and how they apply knowledge of volume. Have students discuss different examples of units of volume that they are familiar with.

Investigate Volume

In this Investigate, students explore the relationship between the volume of a right cylinder and the volume of a right cone with the same radius and height. They construct a right cylinder, and then test their prediction of the relationship between the volumes of these two objects.

Have students work individually to make a prediction about the relationship between the volumes of the cone and cylinder before they construct the model of the cylinder and test their prediction. Have them tape the base of the cylinder securely to its lateral surface in order to avoid spills.

As students work, circulate and consider using some of the following prompts to help them extend their thinking:

- How did you arrive at your prediction?
- Does your result depend on the type of material used to fill the cup? Explain.
- Would this relationship hold for other combinations of right cones and right cylinders (with the same height and the same base area as the cone)? Explain.
- How could your result help lead to a formula for the volume of a right cone?
- What do you predict would happen if you repeated this experiment for a right rectangular prism and a right pyramid?

Have students work with a partner to answer the Reflect and Respond question. Have the pairs discuss their response with another pair of students, and then report to the whole class. Students who did not determine the expected relationship will benefit from the class discussion. You might have students discuss the possible reasons for any discrepancies.

Meeting Student Needs

- Make 3-D models of right cylinders and right cones available to students to help them visualize volume.
- Help students orally recall what they learned about volume from earlier math courses. You might have them use a reflection/math journal and respond to one or more of the following prompts:
  - In your own words, define volume.
  - Compare volume with surface area.
  - Why is volume always expressed in cubic units?

ELL

- Explain that *potash* is the common name for potassium chloride, which is used in fertilizers and manufactured products, such as soap and glass.

Enrichment

- Challenge students to research careers in the natural resource sector and how people working in these careers apply knowledge of volume. For example, they might research careers in mining, forestry, and the oil and gas industry.

Common Errors

- Students may confuse area measurement units with volume units.

R_x Emphasize that volume is always measured in cubic units, such as cubic millimetres, cubic centimetres, cubic metres, and cubic kilometres to represent the amount of space an object occupies. Have students develop a 2D model showing area and a 3D model showing volume. Have them talk aloud about the differences between the models and what they understand about the difference between area and volume.

WWW Web Link

For information about careers in the natural resource sector, go to www.mhrmath10.ca and follow the links.

Answers

Investigate Volume

4. b) Example: The cylinder holds three times as much as the cone.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about the volume of a right cone.	• As there will likely be variations in the dimensions of the cones and cylinders that students create, have students discuss the results of their experiment. • Have students use the class discussion to help them get a clear understanding that the volume of a right cone is one third of the volume of the related right cylinder.

Link the Ideas

This section follows up on what students did in the Investigate by introducing how the formula for the volume of a right cone is derived from the formula for the volume of a right cylinder. Walk through the explanation with students. Begin by asking students to recall the formula for the volume of a cylinder.

Students may benefit from a visual representation of the relationship between the volumes of a cone and cylinder by watching the video mentioned in the Web Link on page 81 in the student resource.

Have students consider the two formulas for the volume of a cone: $V = \frac{1}{3}Bh$ and $V = \frac{1}{3}\pi r^2 h$. Discuss how these two formulas are related. (The base of a cone is a circle; the area of a circle is πr^2.) Once students realize this, they will have to remember only the first formula. You may wish to have them put a diagram and sample formula in their Foldable.

Have students look at the visuals on page 81 and discuss how the volume of a right rectangular pyramid might compare to the volume of the related right rectangular prism. Lead this discussion toward the formula for the volume of a right pyramid. You might ask students to recall how to determine the volume of a right prism. Walk through the explanation with students. Begin by clarifying the term *apex*.

Have students use a model of a right rectangular prism (with the same base and height) and a right rectangular pyramid to help them understand the relationship between the volumes of these two objects. Have students make a prediction and then consider questions such as the following:

- How did you arrive at your prediction?
- If you fill the pyramid with material, predict how many cups of material it would take to fill the prism.
- Use the model and show how the volume of the prism is related to the volume of the pyramid.
- How could your result help lead to a formula for the volume of a pyramid?

Have students consider how the formula for the surface area of a right cone and a right pyramid are related. Again, have students consider the two different formulas for the volume of a right pyramid and discuss the similarities and differences. Once students understand the formula, they will have to remember only the fact that, to solve for the volume of a right prism, they determine the volume of the related right rectangular prism and divide by 3. Understanding is more important than memorization.

Once students have considered the relationship between the volume of a right cone and a right pyramid, have them consider the visual of the right cylinder and the sphere. Ask the following questions:

- How do you determine the volume of a right cylinder?
- What suggests that the sphere is more than one-third the volume of the related right cylinder?
- What fraction of the related cylinder does the explanation suggest it holds?
- How can you use this information to develop a formula for the volume of a sphere?

Again, make sure that students understand the steps of the calculation. Rather than memorizing a formula, they may prefer to think through the calculation each time. For example:

1. Determine the volume of a related cylinder with the radius of the circle and the height of the diameter of the circle.
2. Multiply the answer by $\frac{2}{3}$.

It is far more important for students to understand what they are doing and to use formulas they understand than to memorize formulas without understanding.

Have students use their Foldable and record how to determine the volume of a right cone, a right pyramid, and a sphere.

Example 1

This Example follows up the Investigate by calculating the volume of a right cylinder and a right cone by using a formula and volume relationships. Walk through the example. Extend students' thinking by asking the following questions:

- What is the relationship between the diameter and the radius of a right cylinder? Is this relationship the same for a right cone?
- Explain how you know the formula for the volume of a cylinder is $V = \pi r^2 h$.
- Is the value for pi squared in the formula? Explain.
- If you know the volume of a cylinder, then based on your investigation, what do you expect the volume of the related cone to be?
- Which method do you prefer for calculating the volume of a cone? Why?
- Give a situation when using a formula would be more convenient than using volume relationships. Explain.
- Give a situation when using volume relationships would be more convenient than using a formula. Explain.

Direct students to the note beside the solution for part a). Discuss the difference between writing the solution as 75π cm^3 versus 235.6 cm^3.

For the Your Turn, students determine volumes given radius and height. Ask students what is different about what they have been given compared to the problem in the Example. How can they handle this difference?

Consider having students estimate the volume of the cylinder and the cone using the respective formulas.

Have students use the method of their choice to solve each problem. Have them compare their answer with that of a classmate who used another method. Ask which method they prefer. Alternatively, encourage students to solve part a), then to solve part b) using two different methods. In this case, have students compare their solutions for part b).

Example 2

This Example determines the volume of a right pyramidal greenhouse with a square base using a formula. As a class, walk through the Example. Extend students' thinking by asking questions such as the following:

- Does the square base of the right pyramid affect the way you use the formula?
- How could you write a formula that would apply only to the volume of a square-based pyramid? What would the formula look like?
- Would such a formula be useful? Why or why not?
- Does it matter which side is labelled *l* and which is labelled *w*? For that matter, does it matter which side is labelled *h*? Explain.
- What is the volume of the pyramid in cubic feet? How did you make the conversion?
- How else could you solve this question? (Encourage students to develop strategies that they are comfortable with and understand.)

For the Your Turn, students determine the volume of the smaller greenhouses at the Muttart Conservatory. For part a), students determine the volume of a small pyramid-shaped greenhouse with a given side length and height. As there are several ways to correctly solve this problem, you may wish to have students discuss their answers in pairs or small groups, and then report to the whole class.

For part b), have students identify the shape of the greenhouse. Ask the following questions:

- Will this greenhouse be taller or shorter than a pyramid with the same volume?
- How can a visual help you understand this problem?

Students may predict the height as the reciprocal of the actual height. In other words, they may use the correct ratio, but in the wrong direction, to answer the question. To help students who have difficulty understanding what this question is asking, draw a right rectangular pyramid and a right rectangular prism (with identical-sized bases) on the board, and ask students when the two volumes would be equal.

Example 3

This Example determines an unknown dimension when given the volume of a sphere. In the solution, a cube root is determined, which is a new procedure and a new concept for students. As you discuss the example, consider asking how the rules for the order of operations help solve equations.

Many students will need help to extract the cube root using their calculator. Be prepared to help them find the correct procedure, and then practise the skill using an example.

For the Your Turn part a), students determine a cube root. Have students estimate the cube root using a Guess and Check strategy such as (7)(7)(7) = 343. They may write a note in their Foldable about how to find cube roots on their calculator.

For part b), students determine the diameter of a sphere given its volume. Ask the following questions:

- How can you use mental math to estimate the answer? (Discuss different strategies.)
- What strategies can you use to determine the answer to this question? Encourage students to develop as many different strategies as they can.
- What unit should your answer be in?

Have students who use different strategies compare answers and review each others' calculations if their solutions do not agree.

Example 4

Example 4 determines the volume of a composite object made up of a sphere and a right cone. The volume of a composite object is generally more straightforward than the surface area, as frequently the total volume is the sum of the individual volumes.

As students discuss the Example, be prepared to ask questions such as the following:

- How is the radius related to the diameter?
- At what point in the solution do you substitute for pi? Explain why.
- Do you think the total volume would change if the objects were joined in a different location? Explain your thinking.

Students are required to keystroke a third power in this example. Review how to do this on their calculators.

For the Your Turn, students determine the volume of a building composed of a cylinder and half a sphere. Ask students how the volume of the roof of the building is related to the volume of the corresponding sphere. Have them explain why.

Consider asking students to compare the method for calculating the volume of this building with the method for calculating its surface area.

Key Ideas

The Key Ideas summarize calculating the volume of a right cone, a right pyramid, and a sphere. You might have students use index cards to prepare their own summary of the Key Ideas, including an example for each 3-D object. Also, have them include notes on how to keystroke cube roots. They can refer to Example 3 for a specific example to record. Beside each step, have them describe the reason for the step and provide the keystroke sequence (where applicable).

Meeting Student Needs

- Make 3-D models of right prisms, right cylinders, right cones, right pyramids, and spheres available to students to help them visualize volume problems.
- Provide hands-on learning opportunities for each 3-D object to students who would benefit. Pairing students of similar ability will encourage them to explore together. Encourage student pairs to compare their findings with another pair of students.
- Have students create a poster that shows the relationship between the volume of a right cylinder and a right cone, the volume of a right prism and a right pyramid, and the volume of a sphere and the related right cylinder.
- Have students explain the difference between the slant height and the apex in a right pyramid.
- For the Example 3: Your Turn, coach students who try to evaluate and round 288π when solving, and lose degrees of accuracy. Explain that when given the exact value of a volume, it is necessary to maintain the exactness when solving a problem. Show the correct solution.

$$\begin{aligned} 288\pi &= \frac{4}{3}\pi r^3 \\ (3)(288\pi) &= 4\pi r^3 \\ 864\pi &= 4\pi r^3 \\ \frac{864\pi}{4\pi} &= \frac{4\pi r^3}{4\pi} \\ 216 &= r^3 \end{aligned}$$

- Have students verbalize the process for calculating the volume of right prisms, right cylinders, right pyramids, right cones, and spheres.

Common Errors

- Students may use incorrect keystrokes when entering $\frac{12\ 566.4}{4\pi}$.

R$_x$ Remind students of the proper sequencing and entries for keying the expression.
Correct keystrokes: 12566.4 ÷ (4 × π)

- Students may not understand cubes and cube roots.

R$_x$ Remind students that $4^3 = (4)(4)(4)$
$= 64$
When asked for the cube of 64, they are being asked to find a number whose cube is 64.
(___)(___)(___) = 64
Relate this calculation to a 3D cube. The volume of the cube is 64 units cubed. You are asking them to find the length of one side.
At this level, many students already know the perfect squares of the numerals from 1 to 10. Encourage them to use this information to determine and record the cubes of the same numbers. This will help them with many calculations at this level.

WWW Web Link

For a video showing the relationship between the volume of a right cone and the volume of a right cylinder, go to www.mhrmath10.ca and follow the links.

Answers

Example 1: Your Turn

a) 6912 cm^3 **b)** 2304 cm^3

Example 2: Your Turn

a) 2281.5 m^3 **b)** 6 m

Example 3: Your Turn

a) 7 **b)** 120 mm

Example 4: Your Turn

5235.59 m^3

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work with a partner. • Some students may require individual coaching. Using the same diagram but a different set of dimensions, provide a similar problem to solve. Have students explain how to determine the volumes.
Example 2 Have students do the Your Turn related to Example 2.	• Encourage students to verbalize their thinking. • Some students may require individual coaching. Using the same diagram but a different set of dimensions, provide a similar problem to solve. Have students explain how to determine the volume for part a).
Example 3 Have students do the Your Turn related to Example 3.	• For part a), it may be beneficial to review key strokes on a calculator for calculating the square and cube root of a number. • For part b), students may require individual coaching. Have students talk through the process of solving for an unknown dimension. Have them verbalize how to divide by pi using the order of operations, and then how to determine the cube root. • Give students a similar problem to solve. Allow them to work with a partner and talk through their thinking.
Example 4 Have students do the Your Turn related to Example 4.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Use a model of a right cylinder and a semi-sphere to help students identify, and sketch and label the individual components. • Using the same diagram but a different set of dimensions, provide a similar problem for students who would benefit from more practice.

Check Your Understanding

Practise

Question 1 provides an opportunity for students to demonstrate their understanding of calculating the volume of right prisms, right cylinders, right pyramids, right cones, and spheres. Check that students are able to determine the volume of each object.

For #2, note that this is the first time that students are given more information than is required to determine the volumes. They will need to identify the extraneous information. You might ask students to identify in what context the extraneous measurements would be needed.

Both #3 and 4 involve composite objects. For #4, students identify the correct solution to a problem. You may wish to have students work individually to develop their own solution to the problem before working with a partner to identify the error. Note that in part b), students may choose their own method. Ensure they can justify their method.

For #5, students determine the missing dimension, given the volume for four objects.

Apply

The Apply questions provide a variety of contexts for students to solve problems involving volume of 3-D objects. Encourage students to draw and label diagrams for word problems when diagrams are not provided.

Reinforce that drawing a sketch is an important step in solving problems involving volume, even when it is not specified. Encourage students to draw a sketch for #6, 7, 10, and 12.

For #6, you might point out that 1 cm^3 = 1 mL and 1000 cm^3 = 1 L. How many litres of oil can this section of pipe hold?

For #7, students find the cube root of a calculated volume. It may help students crystallize their thinking if you ask whether it is possible to determine the dimensions of a right rectangular prism, if only the volume is given. Some students may solve by finding the volume of the rectangular prism, and then cube root the answer. However, encourage them to solve by trial and error. Ask what number times itself, and times itself again, yields 216 $in.^3$.

Ask students to read the Did You Know? related to #8.

For #11, you might use prompts since it involves two half spheres, one inside the other, whose volumes must be subtracted to find the volume of snow. Ask students how to deal with the size of the vent.

Have students read #14, which involves a composite object. Have them explain what they are being asked to do. Then, ask how they will deal with the overlap of the sphere and the right rectangular prism.

Extend

These are multi-step problems.

For #17, students describe the relationship between a right cone and a sphere. They may benefit from exploring the interactive Web Link described at the end of this section, which allows them to enter various radii and heights, and then determine the volume of the related cone, cylinder, and sphere, in order to understand the relationship.

For #18, students solve a problem that requires using formulas for volume and the relationship $V = Bh$, where B is the area of the base. In this case, they need to use the lateral surface area of a cone as the base of an object, and its thickness to determine the volume of chocolate lining the cone.

For #19, students use technology to explore how changing the radius of a sphere changes its volume. You may wish to have students use **TM 2–3 How to Do Page 91 #19 Using TI-Nspire™** or **TM 2–4 How to Do Page 91 #19 Using Microsoft® Excel** to do this question. Because this relationship is not intuitive, students should be given an opportunity to discuss their work in pairs or small groups.

Create Connections

For #21, students have an opportunity to apply their understanding about volume by using SI and imperial measurements to estimate and determine volume of an object related to their Unit 1 project.

You will need to decide whether students will work individually or in small groups, and if they can choose an object from within the school or from home.

Students work with both SI and imperial measurements, and there is plenty of opportunity for mental mathematics. Have students present their findings in a whole class discussion, which is likely to be interesting, as a variety of objects, units, and methods will be chosen.

Unit Project

The Unit 1 project questions, #15, 16, and 21 give students opportunities to solve problems involving the volume of 3-D objects.

For #15, students estimate the volume of a cell phone.

For #16, students work with an MP3 player and a vinyl record to compare the storage capacity per cubic centimetre for an MP3 player and a vinyl record. This gives students an opportunity to apply much of what they learned during the chapter, including linear conversions, volume calculations, and volume conversions.

For #21, remind students that even though they work in small groups, they will be responsible for providing an individual response with their findings.

Meeting Student Needs

- For #4, students may benefit from practice identifying errors in solutions involving less complex formulas and fewer steps. Have them develop their own solution to the problem before identifying the error in the given solution.
- If you wish to make the spreadsheet work in #19 accessible to all students, use the Web Link below.
- Provide **BLM 2–9 Section 2.3 Extra Practice** to students who would benefit from more practice.

Enrichment

- As an extension for #8, invite students who are interested to research jewellery making and report on how knowledge of surface area and volume are important skills.
- Give students one of the following challenges:
  – The size of the star cluster Westerlund 1 is given as 6 light years in diameter. Research the distance of a light year in kilometres, and then determine the volume of Westerland 1.
  – The human brain is an amazing information storage device. Estimate the volume of the human brain. Compare the estimate of the memory capacity of the brain and the current capacity of memory devices. What are the implications for the future?

Gifted

- Building on the work done in 2.1, where students investigated the area of a circle as being the maximum area of an object with a given perimeter, ask students to speculate if a sphere contains the maximum volume within its surface area. Challenge them to support their ideas without using outside sources, but by building on their understandings of the relationships between shape and measurements.

Common Errors

- Students may forget the formulas for volume.

R_x Have students refer to the formulas in their Foldable. Encourage them to accompany each formula with a diagram and a worked example showing all the steps.

- Students may forget to use cubic units for volume.

R_x Emphasize that volume is always measured in cubic units. Demonstrate by showing the three dimensions on a solid and indicate that multiplying a unit for each dimension results in a cubic value.

- Students may give the incorrect volume when finding volumes of parts of objects.

R_x Remind students to read the question carefully and highlight (in their notebook) the part(s) of the object asked for.

@WWW Web Link

For information about a career in jewellery making or gemology, go to www.mhrmath10.ca and follow the links.

For a video that describes the cell phone development by Motorola, go to www.mhrmath10.ca and follow the links.

For an interactive activity that allows students to explore the relationship between a right cone and a sphere, go to www.mhrmath10.ca and follow the links. Students enter various radii and heights, and then determine the volume of the related right cone, right cylinder, and sphere.

To make #19 available to more students, have students go online to www.mhrmath10.ca and download one of the prepared spreadsheets. By entering the stretch ratio and radius, students can get the calculations, determine a pattern, and make predictions.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2, and 4 to 9. Students who have no problems with these questions can go on to the remaining Apply questions.	• Provide additional coaching to students who have difficulty with calculating volume for one or more of the objects in #1. Coach them through corrections and clarify any misunderstandings before allowing them to proceed. • Refer students who have difficulty calculating volume for a particular object to the relevant worked example. Then, provide a similar problem to solve by changing the dimensions on an existing diagram. • For #2, have students draw a sketch and verbalize what dimensions are needed to determine the volume of each object. They might find referring to their Foldable or to the worked examples helpful. • Have students explain how to determine the volume of a right rectangular prism, and then consider how to handle what is missing in #5a). You may wish to have them talk through that calculation, and then try b) on their own using a similar method. Next, have them talk through a solution to c), and then try d) on their own. • Encourage students to draw sketches for #6 and 7.
Unit 1 Project If students complete #15, 16, and 21, which are related to the Unit 1 project, take the opportunity to assess how their understanding of the chapter outcomes is progressing.	• Consider asking students to draw labelled sketches for each question. • For #16, encourage students to sketch and label the dimensions of the MP3 player and the record. They will need to convert the volume of the record into cubic centimetres. • For #21, have students brainstorm some possibilities of objects related to music. • You may wish to provide students with **BLM 2–5 Chapter 2 Unit 1 Project**, and have them finalize their answers. Have them store the work in their project portfolio.
Assessment *as* Learning	
Create Connections Have all students complete #21.	• Give students an opportunity to explain how they arrived at their estimates and determined the volume of their object. Ask them to discuss how they used mental math in answering the questions.