3.2 The Sine and Cosine Ratios

***Mathematics 10*, pages 114–124**

Suggested Timing

100–120 min

Materials

- protractor
- ruler
- calculator
- 1 m of foam pipe insulation, cut lengthwise
- marble or small steel ball
- eight to ten thick books or bricks or a chair
- masking tape
- measuring tape
- table

Blackline Masters

BLM 3–3 Chapter 3 Warm-Up
BLM 3–4 Chapter 3 Unit 1 Project
BLM 3–7 Section 3.2 Extra Practice

Mathematical Processes

✓ Communication (C)
✓ Connections (CN)
Mental Math and Estimation (ME)
✓ Problem Solving (PS)
✓ Reasoning (R)
✓ Technology (T)
✓ Visualization (V)

Specific Outcomes

M4 Develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1a), c), e), 2a), c), 3a), b), e), f), 4a), 5, 6a)–c), 7, 8, 13
Typical	#1a), c), 2a), c)–e), 3a), b), e), f), 4a), 5, 6a)–c), 7, 8, 11, 13, 17
Extension/Enrichment	#10, 12–17

Planning Notes

Have students complete the warm-up questions on **BLM 3–3 Chapter 3 Warm-Up** to reinforce prerequisite skills needed for this section. If you have posted the chapter outcomes, have students consider those that relate to this section.

You might begin the class by asking students if they have ever been on a suspension bridge. What was it like? About how long was it? How would an engineer calculate the distance a bridge would have to span? Ask, “Would guessing be a good strategy to solve this problem?” Note that there are few bridges built in some parts of Canada because of the geography. For example, in Canada’s Arctic, ice and permafrost can make bridge construction difficult. If you are teaching in one of these areas, you may want to ask students if there is a river or a ravine near their community over which it would be useful to have a bridge.

Before moving on to the investigation, you may want to discuss the meaning of the word *solve*. Students need to understand that this means to find all missing sides and angles.

Investigate Trigonometric Ratios

Accuracy in the drawing of the triangles is important, so suggest that students draw the triangles with a ruler (alternatively, the triangles could be drawn using grid paper). You could ask the class how, if they know only one angle in a right triangle, they can be certain the triangles are similar.

When they are measuring the side lengths, partners should discuss the best unit of measurement to use. Which would be most accurate? Which would be least accurate?

When students fill in the charts and put the ratios in their lowest terms, they should share the results of the charts with the class. Have them discuss and compare patterns and generalizations.

Meeting Student Needs

- This Investigate works best if it is done in pairs or small groups. One student can be the recorder and could create the required tables on a computer. The other students create the triangles needed for the investigation. All students should be involved in the measurements and conclusions.
- This section assumes that students recall angle properties learned in previous math courses. You may want to review parallel line rules, such as alternate interior angles, corresponding angles, and interior angles. Use geometric diagrams that review these properties as well as vertically opposite angles and the angle sum of triangles.
- You could discuss traditional navigation methods. For example, Inuit have many different methods of navigation, such as looking at the stars, sun, and moon, as well as considering the wind direction, snow drifts, landmarks, and land shapes. What affect do you think these technologies are having on the traditional skills that are being used and passed down?
- **BLM 3–4 Chapter 3 Unit 1 Project** includes all of the unit project questions for this chapter. These provide a beginning for the Unit 1 project report.

Common Errors

- Some students may not take accurate measurements.

R_x Encourage students to draw the triangle with a sharp pencil and straight edge. The side lengths should be measured twice and checked by another student.

- Some students may not draw the angles accurately.

R_x It might be a good idea to review the use of a protractor to draw angles. Students are often unsure whether to use the inner or outer scale. Also, students must be sure that the other angle measures exactly 90º. Encourage students to start with the 90º angle and then draw their chosen angle adjacent to it.

Answers

Investigate Trigonometric Ratios

7. Example: The opposite to hypotenuse ratio in each triangle is equal, as is the adjacent to hypotenuse ratio.

8. Example: The ratio of the opposite to the hypotenuse in one table is the same as the ratio of the adjacent to the hypotenuse in the other table.

9. Example: Since they are all similar triangles, the ratios for this angle will always be the same, regardless of the size of the triangle.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the investigation. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Remind students that their answers are relative estimates. • All students might benefit from writing their results on a common chart on the board. Doing so may provide a better opportunity for students to see the pattern. • It will be important for students to understand that the rules for their calculator mode (degree), angle calculations, and side lengths will be the same for cosine and sine as it was for tangent.

Link the Ideas

The tangent ratio was introduced in the last section. This section introduces two other trigonometric ratios: sine and cosine. Both of these ratios are based on the theory of similar triangles, which was developed in the tangent ratio. You might want to review labelling the sides of the triangle in relation to the reference angle and establish the ratios for sine and cosine.

Example 1

In this example, students write out the trigonometric ratios for sine and cosine, using both acute angles.

Have students work with a partner to do the Your Turn. Partner can explain to each other how they completed one of the questions.

Example 2

This example allows students the chance to familiarize themselves with how to use a calculator to calculate sine and cosine ratios, and how to find an angle given the sine or cosine ratio. If the angle measure is given and they need to find the ratio, they use the sin or cos button on their calculator. If the ratio is given and they need to find the angle, they use the inverse of sine ($\sin^{-1}$) or the inverse of cosine ($\cos^{-1}$).

In the Your Turn, have students work in pairs. Ask them to discuss with each other what is different about recording trigonometric ratios compared to recording degrees. If students have different calculators, encourage them to show each other how their particular calculator works.

Example 3

In this example, students are given a contextual framework and asked to solve for a missing angle. Sketching a diagram is an effective way to organize information. From the diagram, students should identify the reference angle and from this, label the sides of the triangle as hypotenuse, adjacent, and opposite. They must decide which ratio to use to solve the problem based on what was given.

Have students draw a diagram for the Your Turn, identify what they need to do, and then determine the answer.

Example 4

In this example, students are given a contextual framework and asked to solve for a missing side. Again, sketching a diagram is an effective way to organize information.

In the Your Turn, have students sketch a diagram to show the scenario, describe the diagram to a partner, and then discuss how to solve for the missing value.

Key Ideas

Use the following prompts to help students clarify their understanding of the Key Ideas:

- What diagram(s) would help you remember the difference between the sine and cosine?
- What do you need to do to determine each ratio?
- What calculator key sequences can you use to determine the value of the sine ratio? cosine ratio?

Meeting Student Needs

- You could have students create a mnemonic of their own to remember the three trigonometric ratios. Each student could add their acronym to their Foldable (bookmark, recipe card) for easy reference.
- Before studying Example 1, some students may benefit from a quick review of changing fractions to decimals.

Enrichment

- To give students the opportunity to measure sides and angles with and without technology, consider having them work in pairs to construct their own right triangles. They could use a protractor to precisely measure the right angle and one other internal angle. They can then use a ruler to measure either the adjacent or opposite side. Using what they have learned, they can use either the cosine or sine ratio (depending on which side they measured) to find the unknown side length. They can measure this side with a ruler to check their calculation. Pairs could then make two other right triangles, measuring all three sides for each. They could use $\sin^{-1}$ to find the internal angles on one of the triangles, and $\cos^{-1}$ to find the internal angles on the other. They can check their calculations by measuring with a protractor.

Gifted

- You could have students research the graphical representation of the sine and cosine ratios, for reference angles of 0º to 360º (see Web Link below).

  For the sine ratio, direct students' attention to the fact that it oscillates from –1 to +1. This means the absolute value of the sine ratio can never be greater than 1. You may also want to guide their attention to the fact that the graph shows that the sine ratio is 0 when the reference angle is 0º, 180º, and 360º. The sine ratio is exactly 1 when the reference angle is 90º.

  For the cosine curve, the ratio oscillates from –1 to +1. This means the absolute value of the cosine ratio can never be greater than the absolute value of 1. Students should also notice that the graph shows that the cosine ratio is 0 when the reference angle is 90º and 270º. The cosine ratio is exactly 1 when the reference angle is 0º and 360º.

Web Link

For links to graphical representations of the sine and cosine ratios, have students visit www.mhrmath10.ca, and follow the links.

Common Errors

- Some students may label the sides of the triangle incorrectly.

Rx You can review with students how to label a triangle using the reference angle. The hypotenuse and opposite side are often the easier sides for students to identify. The adjacent side would be the remaining side of the right triangle.

- Some students may write the sine or cosine ratios incorrectly.

Rx You can review the primary trigonometric ratios using memory aides. Encourage students to write a memory aid that works for them at the top of the page and for each question write out the trigonometric ratio before substituting known values.

Web Link

Students can visit www.mhrmath10.ca and follow the links in order to find out more about the northern and southern lights, and how the parallax is used in astronomy.

Answers

Example 1: Your Turn

a) $\frac{5}{13}$ **b)** $\frac{5}{13}$ **c)** $\frac{12}{13}$ **d** $\frac{12}{13}$

Example 2: Your Turn

a) 0.8660, 0.5000, 0.7071 **b)** 26°, 78°

Example 3: Your Turn

45.1°

Example 4: Your Turn

423.8 m

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Suggest that students write out the sine and cosine ratios for each angle before starting to solve. • Remind students to have their calculators in degree mode.
Example 2 Have students do the Your Turn related to Example 2.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Have students recall how to use the calculator to calculate sine and cosine ratios, and how to find an angle given the sine or cosine ratio. • Suggest that students write a quick rule as to when to use the *sin* and *cos* buttons, and when to use the inverse buttons on their calculator. These are concepts that many students confuse. Posting the ratios and a quick reminder as to when each is used would be helpful. • Provide students who are having difficulty with remediation and additional questions.
Example 3 Have students do the Your Turn related to Example 3.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Encourage students to verbalize which is the reference angle and what the relationship of the sides is to this angle. • Help students recall how identifying the sides determines the trigonometric ratio to use. Ask the following questions: – Which sides are used to calculate cosine? – Which sides are used to calculate sine? • Encourage students to draw a diagram and label the sides in relation to the reference angle.
Example 4 Have students do the Your Turn related to Example 4.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Because students will be working with both sine and cosine questions, encourage them to draw a diagram and label the sides with the reference angle from the problem. This should be standard practice for students who struggle without visuals.

Check Your Understanding

Practise

Questions #1 and 3 will give students much needed practice using their calculators. Give students time to explain what is happening when they use their calculators and why they are pressing the buttons they are pressing. Encourage students to understand the difference between the ratio and the angle when using their calculator.

In #4 to 6 students should sketch and label the diagrams to help them decide which ratio to use and how to write the ratio.

Apply

Observe how students solve #7. Some will immediately go to the Pythagorean relationship to find the missing side. Others might use the tangent ratio to find the missing angle and then either the sine or cosine ratio to find the missing side. Discuss with students which method they use and encourage students to share their thinking.

For #8 to 10, students need to read through each problem to be able to gather all the information that is needed to solve it. Encourage them to sketch out the problem, list the components that they are given, and state what it is they are asked to find.

For #11, students need to draw a diagram for each scenario. You might also want to discuss why it is important to avoid using a calculated measurement to find another measurement whenever possible.

For #12, students may struggle with how to place the information on one right triangle. Since the question tells them the sag in the center is 4600 m long, students will need to bisect some of the measurements.

Watch what method students use to solve #13. You may want to provide guidance if any of your students are having difficulty.

Question #14 involves multiple triangles. You could encourage students to draw two separate diagrams for this question: one for the beginner slope and one for the expert slope.

Extend

For #15, students will need to put together all that they have learned about using the sine and cosine ratios. You might want to suggest to them that they list the components that they know, and what they need to find out. They will need to plan out how they will logically progress through this problem.

For #16, you may want to ask students how they could use the diameter to help them solve this problem. Also suggest that diagramming this problem would be very useful in solving the question.

Create Connections

Question #17 involves the Olympic event of ski jumping. For the ski ramps at Whistler, the length of the in-run is 116 m with an angle of 35º. This activity will allow students to see how the length of the run and the slope of the take-off ramp will affect the distance that a ball will travel. Students should work in small groups, where someone will measure the distance the ball travels, someone will let the ball roll down the ramp, and someone will control the slope of the take-off ramp. At the end of the activity, discuss as a class how the distance may be improved and what factors are involved. Once students have completed this question, you might want to have a discussion about ski jumping. What factors have they discovered in the question that may affect the length of a ski jump? You might also ask them if they can think of other factors that are not reflected in their experiment with the ramp and marble.

Meeting Student Needs

- Provide **BLM 3–7 Section 3.2 Extra Practice** to students who would benefit from more practice.
- For #11, some students may need help with the diagram, particularly those that have not played golf. You may want to discuss some of the terms, such as teeing area, fairway, and hole/flag.
- For #12, some students may not be familiar with gondolas and zip lines. You may want to have a discussion, asking questions such as the following:
  - Who has been on a gondola?
  - Why does the line sag?
  - Who has ever gone on a zip line?
  - Does the zip line sag like the gondola line?
  - How could engineers take into account the sag when constructing such lines?
- For #13, if students are struggling, you may want to suggest that they complete part b) before part a). If they find the angle first, the side length can be found using either the tangent or sine ratio.
- Students who are unfamiliar with skiing may have difficulty with #14. You could have a discussion, asking questions such as the following:
  - How many of you ski?

– Do any of you ski the difficult, or black diamond, runs?
– What makes a run difficult?
– What is the steepest run you have ever been on?

- For #16, you may want to remind students what it means for a triangle to be equilateral. Some students may also need to be reminded what *diameter* means. This may be a good question to have students work on in pairs so that they can strategize together about how to solve this problem.
- For #17, you may want to review the sport of ski jumping; some students may be unfamiliar with this event. Ski jumping is an event where skiers go down an in-run to a take-off ramp, attempting to jump as far as possible.

Web Link

You may suggest students visit www.mhrmath10.ca to learn how trigonometry is reflected in sports. They can also find information on the gondola at Whistler Blackcomb (in support of #12), and about the sculptures of Floyd Wanner (in support of #13).

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply #1a), c), e), 2a), c), 3a), b), e), f), 4a), b), 5a), b), 6a)–c), 7, 8. Students who have no problem with these questions can go on to the remaining questions.	• Students who have difficulty with either #1 or 2 should review Example 1, and be coached through the solution to the assigned questions. To check for understanding, use one of the additional parts in each question. Review the identification of the sides. • For questions #3 and 4, direct students to Example 2. Providing them an opportunity to try additional questions, and then the unassigned ones in #3 and 4, will help check for understanding. • Questions #5 and #6 are linked to Examples 3 and 4 respectively. Reviewing them and the mode and entry of values into the calculator would be beneficial. • Encourage all students to make it standard practice to draw a diagram reflecting the question, labelling the sides and angles before looking at what the question is asking them to find. • It would be beneficial for all students to revisit the term *angle of elevation*.
Assessment *as* Learning	
Create Connections Have all students do questions #17.	• It might be beneficial to have students of mixed ability work in groups of 3 or 4 to complete this mini lab. • Students will need sufficient space to complete the trials. Borrowing some material from the science lab may be required.