Square Roots and Cube Roots

4.1

***Mathematics 10*, pages 152–161**

Suggested Timing

80–100 min

Materials

- square dot paper
- isometric dot paper
- ruler

Blackline Masters

Master 3 Square Dot Paper
Master 4 Isometric Dot Paper
BLM 4–3 Chapter 4 Warm-Up
BLM 4–5 Chapter 4 Unit 2 Project
BLM 4–6 Section 4.1 Extra Practice

Mathematical Processes

✓ Communication (C)
✓ Connections (CN)
✓ Mental Math and Estimation (ME)
✓ Problem Solving (PS)
✓ Reasoning (R)
✓ Technology (T)
✓ Visualization (V)

Specific Outcome

AN1 Demonstrate an understanding of factors of whole numbers by determining the:

- prime factors
- greatest common factor
- least common multiple
- square root
- cube root.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 4, 5, 7, 9, 10, 13, 15, 20
Typical	#1, 4, 6–10, 11 *or* 12, four of 13–17, 20, 21
Extension/Enrichment	#1, 4, 6, 7, 13, 15, 18–21

Unit Project Note that #13 and 15 are Unit 2 project questions.

Planning Notes

Have students complete the warm-up questions on **BLM 4–3 Chapter 4 Warm-Up** to reinforce prerequisite skills needed for this section.

As a class, read and discuss the opening text about how workers such as painters and designers apply math when working with area and volume. Ask students why it is important for painters and designers to take accurate measurements. Tell students that tradespeople need to be effective problem solvers. Ask students to discuss what strategies a painter and a designer might use to solve the problems posed.

Investigate Square Roots and Cube Roots

In this Investigate, students use patterning to connect the side length of a square with the square root of a number (area) and the edge length of a cube with the cube root of a number (volume).

Have students work in pairs. Make **Master 3 Square Dot Paper** and **Master 4 Isometric Dot Paper** available to students. They may find it easier to draw cubes using the isometric dot paper.

Circulate as students work and note the way they are recording area and volume. Ask what units are used for measuring area and volume. Use the following prompts:

- How are the procedures for finding the area of a square and the volume of a cube different? similar?
- How could you estimate the side length of a square with an area of 20 square units using your results?
- How could you use your calculator to determine the area of a square? the volume of a cube?

Have students discuss #3 in pairs, then in large groups. As a class, have students discuss their response to #3c).

In a follow-up class discussion, have students discuss how different types of numbers have certain features that can be grouped together. One way to look at numbers is to create geometric representations. Ask students for examples of numbers that form squares and cubes.

You might reinforce the idea that mathematicians have been relating to numbers using geometric representations for a long time by mentioning that Pythagoras (about 580–500 B.C.E) believed that all of nature is based on numbers and a mathematical structure. For instance, you might show an array of 9 beads and explain that the number 9 can be represented concretely as a perfect square. You might help students make the connection between numbers and their geometric representation by asking the following questions:

- For 5^7, we say "5 to the 7th." For 6^4, we say "6 to the 4th." For 7^2, why do we say "7 squared" instead of "7 to the 2nd?"
- For 4^3, why do we say "4 cubed?"

Explain that during the chapter, they will learn about numbers that form rectangles (golden ratio) and diagonals of rectangles (irrational numbers).

Meeting Student Needs

- Some students may not relate to the opening scenario. For example, in most Northern communities, house paint is not readily available. It must be ordered and shipped by air freight or on the annual resupply. You might adapt the context by asking students how a painter knows how much paint to order for the local school.
- Review concepts of length, area, and volume of rectangular prisms briefly before beginning the Investigate. The Investigate relies on students having a clear understanding of the differences between these concepts. They also need to be clear about using the correct corresponding units.
- Consider having students create posters illustrating perfect squares composed of smaller blocks, or tape sections of the classroom floor, if it is tiled. Display the posters in the classroom.
- For #2, encourage students to use manipulatives such as building blocks (e.g., connect-a-cubes, LEGO®) to build models of perfect cubes.

Enrichment

- Challenge students to devise a method for predicting perfect cubes in large numbers.

Gifted

- Have students look for a pattern in the number of perfect cubes that appear as the base values increase.
- Elephants' ears have a high surface area and act to cool the elephant's blood. Challenge students to give reasons why the volume-to-surface-area ratio of an animal might affect mammals in hot climates. Why might animals in the north be expected to have small ears?

Common Errors

- Some students may confuse the concepts of volume and surface area of a cube.

R_x Help students recall the meaning of volume and the formula for finding the volume of a cube. Recall that volume is always measured in cubic units and surface area is measured in square units. Use a model of a cube to help students to visualize the difference. Have students verbalize their understanding.

Answers

Investigate Square Roots and Cube Roots

1. a), b)

Side Length	Area in Exponential Form	Area
1	1^2	1
2	2^2	4
3	3^2	9
4	4^2	16
5	5^2	25
6	6^2	36

c) The area is the square of the side length so $A = s^2$.

2. a), b)

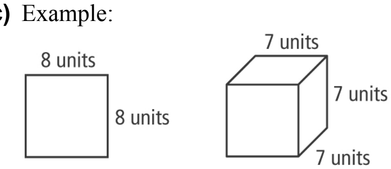
Edge Length	Volume in Exponential Form	Volume
1	1^3	1
2	2^3	8
3	3^3	27
4	4^3	64
5	5^3	125
6	6^3	216

c) The volume is the cube of the edge length so $V = s^3$.

3. a) Take the square root of the area to get the side length or $s = \sqrt{A}$.

b) Take the cube root of the volume to get the edge length or $s = \sqrt[3]{V}$.

c) Example:



$A = 64$ units^2 $V = 343$ units^3

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to show a pictorial representation of the square and the cube of a number. For example, using 5, show a square with 25 small squares inside it, and a cube with 125 small squares inside it. • Ensure that students understand that the square root is derived from a 2-D shape and the cube root is derived from a 3-D object (initially). • For students who struggle with #3c), encourage them to rewrite the numbers in prime factorization form. Help them recall that this involves writing a number as the product of its prime factors.

Link the Ideas

It is important that students have a clear understanding of the terms in this section in order to apply the terminology correctly later with radicals. As a class, have students discuss the terms *perfect square*, *square root*, *perfect cube*, and *cube root*. You might have students recall what they know about perfect squares and square roots. Reinforce using the example that 25 is a perfect square because it forms a square.

After students have discussed the examples of perfect squares, ask the following questions:

- What is another example of a perfect square?
- How do you know it is a perfect square?

Help students clarify any misunderstandings by asking questions such as the following:

- What is the difference between squaring a number and taking the square root of a number?
- Using the number 9, what is the difference between squaring 9 and taking the square root of 9?

Using an example such as $\sqrt{9}$, explain that this means the square root of 9. The 2 (in the index) is understood. This is important so students get a clear understanding of and can differentiate between square roots and higher-order radicals. Also, use an example with a leading negative sign, such as (-4^2) and (-4^3), and point out the negative sign. Explain how the negative sign affects the final answer.

Direct students to the example showing how perfect cubes and cube roots are related. You might reinforce that the cube root of 27 equals the cube root of (3)(3)(3), which equals the cube root of 3 to the power of 3, which, therefore, equals 3.

After students have discussed the examples of perfect cubes, ask the following questions:

- What is another example of a perfect cube?
- How do you know it is a perfect cube?
- What is the difference between cubing a number and taking the cube root of a number?
- Using the number 8, what is the difference between cubing 8 and taking the cube root of 8?

After the discussion, use the following prompts:

- What did you learn by discussing these terms?
- What, if any, misconceptions did you correct?

Example 1

In this Example, students determine whether given numbers are perfect squares, perfect cubes, both, or neither.

Before students consider the strategy shown for each problem, encourage them to try other strategies such as guess and check, prime factorization, and using a calculator. Have student pairs compare their answers with another pair of students who used different strategies. Have them correct any discrepancies.

As a class, walk through the provided solutions. For part b), discuss the term *prime factorization* and consider working through a second example of prime factorization using a number such as 1000 or 216 that is not both a perfect square and a perfect cube. This will illustrate how to use prime factors to identify a perfect cube.

For part c), you might model using prime factorization to further illustrate that since the factors do not form groups of two, 356 is not a perfect square. Since the factors do not form groups of three, 356 is not a perfect cube.

Since students have tried different strategies, you might ask which strategy they prefer and why.

Make the following points:

- Prime factorization is an effective method to establish that a number is a perfect square if the prime factors can be equally distributed into two groups. A number is a perfect cube if the prime factors can be equally distributed into three groups.
- A calculator is useful to check if an answer is correct.

Direct students to the Did You Know? on page 155 that explains how square roots and cube roots were studied and applied by ancient civilizations. You might ask students what they know about how ancient civilizations applied this knowledge.

Have students work in small groups to complete the Your Turn questions. Challenge each member of the group to use a different method to answer the questions and then discuss their solutions. Alternatively, students could work individually and use prime factorization for all three questions. Have them check their answers using a calculator.

Example 2

In this Example, students calculate the cube root of a large number that is a perfect cube.

Use the information in the Did You Know? about uranium to help set the context. Explain that uranium is a silvery-grey element that occurs in nature in the form of minerals. It is often used as a fuel in nuclear power plants.

Before having students consider the solution, ask them to explain how they know that the question requires working with cube roots and not square roots. You may wish to have students work in pairs and use their own strategies to solve the problem. Use the following prompts to assist students:

- How did you determine the dimensions of the cube?
- How did you determine the cube root?
- How else could you determine the cube root?

You might encourage students to use prime factorization to solve and then check with a calculator.

Walk through the given solutions. Have students compare their solutions and discuss any discrepancies with their partner.

Have students complete the Your Turn questions using the methods of their choice and then explain their methods to a classmate.

Key Ideas

The Key Ideas summarize determining square roots of whole numbers that are perfect squares, and cube roots of whole numbers that are perfect cubes. Check understanding by asking students for an additional example of a perfect square and a perfect cube. Check that students understand the process of prime factorization. Ask the following questions:

- How do you rewrite the number 36 as a product of prime factors?
- How can you use prime factorization for 36 to determine if 36 is a perfect square? a perfect cube?

Have students use their Foldable to record their own definition and example for each Key Term. Additionally, have them create their own summary of the Key Ideas and include it in their Foldable.

Meeting Student Needs

- Review the difference between *factors* and *prime factors*. Factors are all the numbers that a number is divisible by, whereas prime factors are the primes that divide evenly into a number. Have students recall that 1 is not a prime number. The reason is because 1 is divisible by 1 and 1, which is not two distinct (different) factors.
- Some students may benefit from reviewing divisibility rules for the prime numbers 2, 3, and 5. Remind students that the number 7 has no rule.
- Some students may benefit from practising prime factorization using the interactive tool described in the related Web Link at the end of this section.
- Ensure that students know how to use their calculator for operations involving square roots and cube roots. They need to identify and be able to use the keys to square, cube, take the square root, and take the cube root.
- Some students may benefit from working in pairs to develop a summary of the methods for determining that a number is a perfect square or a perfect cube. Encourage them to practise using all the methods.
- At the end of each lesson, consider using one of the following tools to informally assess student understanding of new concepts and determine where students need clarification:
  – Develop an exit slip with a few key questions to be answered by students during the last 5 to 10 min of class. Have them turn in the exit slips as they leave. Use the assessment to help determine where to begin in the next class or which students require assistance.

– Use Fist-to-Five at the end of the lesson. Ask students how well they understand a key concept. Be specific. Ask each student to show the number of fingers that correspond to their level of understanding. The responses will range from a fist, which indicates no understanding, to five fingers, which indicates complete understanding. Generally, fewer than three fingers indicates a need for revisiting a concept.

- For the Did You Know? on page 155, clarify that wet clay was formed into flat shapes to create tablets. While the tablets were still wet, the Babylonians used a stylus to make wedge-shaped letters on the surface. The clay was then sun dried or kiln fired.

Enrichment

- For the Example 1: Your Turn, assign the following additional questions:
  – Which of the following are both perfect squares and perfect cubes?
  216 400 1024 46 656
  – Think of a larger perfect square that is also a perfect cube.
- Challenge students to think of a number that contains only the number 5 in its prime factorization and that is both a perfect square and a perfect cube. Ask them to explain how they solved the problem and whether there is more than one solution.

Gifted

- For Example 2, have students use the information in the Did You Know? on page 156 in the student resource as a springboard to research the annual uranium production in Canada or the world to develop and solve a similar problem.
- Challenge students to research the smallest number expressible as the sum of two positive cubes in two different ways ($1729 = 1^3 + 12^3$ or $9^3 + 10^3$). They may find the Web Link at the end of this section interesting.

Common Errors

- Some students may struggle with determining the prime factorization of a number.

R_x Review the procedure and the divisibility rules for 2, 3, 5, and 10.

Web Link

For a factor tree tool that illustrates both prime factorization and common factors, go to www.mhrmath10.ca and follow the links.

To read a story about the discovery of the smallest number expressible as the sum of two positive cubes in two different ways, go to www.mhrmath10.ca and follow the links.

Answers

Example 1: Your Turn

a) Perfect cube; $125 = 5^3$

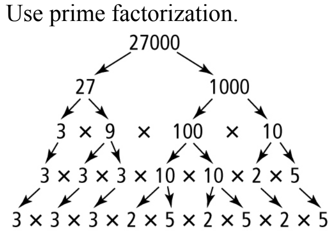
b) Perfect square; $196 = 14^2$

c) Both a perfect square and a perfect cube; $4096 = 64^2$ and 16^3. Look for a variety of methods for a)–c), such as using diagrams, prime factorization, or a calculator.
Example for c): $4096 = (2)(2)(2)(2)(2)(2)(2)(2)(2)(2)(2)(2)$
$= 2^{12}$
Square rooting gives two groups of $(2)(2)(2)(2)(2)(2) = 2^6$ or 64.
Cube rooting gives three groups of $(2)(2)(2)(2)$ or 2^4 or 16.

Example 2: Your Turn

a) 14 m

b) 30 in. by 30 in. by 30 in. Look for two methods. Example:
- Use prime factorization.



For a cube root, look for triplets, so one of 3, one of 2, and one of 5. Therefore, the cube root is $(3)(2)(5) = 30$.
- Use a calculator. Example:
C 27 000 2nd $\sqrt[x]{y}$ 3 = 30.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Encourage students to use more than one method. Encourage them to record and store multiple methods in their Foldable. • Some students may benefit from using grid paper and isometric dot paper to help visualize side lengths and their squares, and edge lengths and their cubes, respectively. • Encourage students to use prime factorization for large numbers. • Some students may benefit from reviewing factor trees. Point out that for large numbers, not all students will start with the same factors, but that the factors at the end will be the same. Explain that when there are two equal groups of factors, you have found the square root. When there are three equal groups of factors, you have found the cube root. Have students compare their factor trees with those of a partner.
Example 2 Have students do the Your Turn related to Example 2.	• Provide a similar problem to students who would benefit from more practice. Allow them to work with a partner and talk through their thinking. • For each of parts a) and b), have students verbalize what they are finding and how they know. • If students are using a calculator, you might ask them to explain what the index on the radical indicates.

Check Your Understanding

Practise

For #5 and 6, encourage students to use prime factorization in their solutions for the larger numbers.

For #9 and 10, tell students to check that the units in their solutions are correct. For #9, you may need to coach students to realize that the problem involves calculating the perimeter of the rug, not the length of one side.

For #10, some students may struggle with using prime factorization if they do not check division by 7. Check that students who use a calculator use it correctly to find the cube root.

Apply

It may not be necessary to assign all questions to all students. Allow students some choice in the questions they need to do. Remind students to read the problems carefully and to note the given units of measurement and the units of measurement needed in the solution.

For #11a), students use mental estimation to estimate a lower and upper boundary for the square root of a large number. You might prompt students who have difficulty with this to make a reasonable guess of the square root of 1000.

For #14, although it is appropriate for students to use a calculator to find the cube root of 46 656, encourage them to use prime factorization.

For #16, some students may benefit from recalling the formula for the surface area of a cube ($SA = 6s^2$). You may wish to take the opportunity to show students how to construct the formula using a net of a cube.

Extend

For #17b), encourage students to isolate the variable, t^2, prior to taking the square root on both sides of the equation.

For #18, students may need the hint to use the Pythagorean relationship.

For #19, the volume of the basketball is expressed in terms of pi. Discuss with students why it would be impossible to measure the volume of a basketball with exact precision. Students may need the hint to use the formula of a sphere to help solve the problem.

Create Connections

For #20, consider having students work in pairs before developing an individual response.

For #21, have students discuss their solutions in a class discussion.

Unit Project

The Unit 2 project questions, #13 and 15, provide an opportunity for students to solve problems involving square roots and cube roots.

For #13, check that students note the different units used. Encourage students to sketch a diagram to help visualize what the problem is asking. You may need to clarify that the classroom wall is not a square but a rectangle. Point out that the shape of the mural does not matter, as it is the shapes made by the square tiles that are important. Reinforce that it is the design of the mural (not the overall dimensions of the mural) that needs to be a geometric representation of square roots.

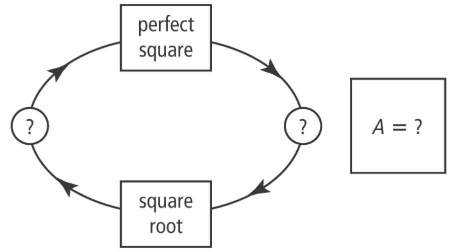
Students will approach #13 in a variety of ways. You might have them work in pairs before developing an individual response. Afterward, have students compare and discuss their designs in a class discussion. For instance, a design might involve squares with diagonals (e.g., a square that is 1 unit by 1 unit would have a diagonal that is $\sqrt{2}$ units in length). This activity and follow-up discussion will help set up students' learning about the golden ratio in section 4.4.

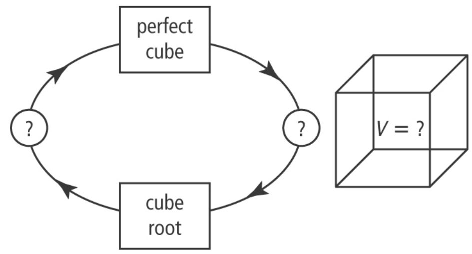
For #15, you might explain that sculptor Tony Bloom works mostly in steel, copper, bronze, and aluminum. He designed the WaterWork, a water-driven fountain for the head office of the BC Hydro Authority in Vancouver, BC. His work has been exhibited in North America, Europe, and Japan, and he has been the recipient of national, provincial, and civic awards.

For #13 and 15, encourage students to use a combination of sketches, words, and numbers to explain how the mural and the sculpture are geometric representations.

Meeting Student Needs

- Allow students to work in pairs.
- For #13, you may need to remind students to convert to the same units.
- For #19, review the formulas for the volume of a cube and of a sphere.
- Some students may benefit from using an example and a diagram to explain the relationship between perfect squares and square roots and between perfect cubes and cube roots. They may find the diagrams shown helpful. Have students use their example and the diagram to explain the relationships to a classmate.





- Provide **BLM 4–6 Section 4.1 Extra Practice** to students who would benefit from more practice.

ELL

- For #12, use the photograph and the related Did You Know? to describe star quilts.
- For #13, explain that a *mural mosaic* is a design constructed from smaller pieces and is displayed on a wall.
- Teach the following terms in context: *recycling depot, bales, copper leaf, meteorologists, tornadoes,* and *hurricanes.*

Enrichment

- Give students one of the following challenges:
  - Research how people working in some of the following occupations apply knowledge of square roots and cube roots: carpenters, electricians, machinists, architects, civil engineers, computer scientists, surveyors, chemists, physicists, and biologists. Allow students to present their findings in a format of their choice.
  - Research the Dominion Land Survey system and how it works. Mention that land surveyors used the Dominion Land Survey (DLS) as the method to divide much of Western Canada into 1 mi^2 sections, mainly for agricultural purposes. This survey system and its related terms, including meridians, baselines, townships, sections, and quarter sections, are an important part of rural culture in the Prairies today. Have students prepare and present a report of their findings.

– Some students may be interested in researching the 1869 Red River Rebellion or 1885 Battle of Batoche. The Red River is in present-day Manitoba. The Battle of Batoche took place in Saskatchewan. Both conflicts were precipitated by surveyors encroaching on Métis lands to break up the seigneurial land system and replace it with the imperial township grid system. One of the issues involved grievances about federal land surveyors using the DLS system to divide the river-front lots owned by the Métis, and their refusal to acknowledge traditional Métis land holdings.

Web Link

For information about the Dominion Land Survey system, go to www.mhrmath10.ca and follow the links.

For information about the Battle of Batoche, go to www.mhrmath10.ca and follow the links.

For information about the Red River Rebellion, go to www.mhrmath10.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 4, 5, 7, 9, 10, 13, and 15. Students who have no problems with these questions can go on to the remaining questions.	• For #1, students may benefit from reviewing the fractional forms, when to square the denominator, when to square the numerator, and when to square both the numerator and the denominator. Provide additional coaching and use #2 to check for understanding. • For #4, ensure students understand the difference between an index of 2 and 3. Model examples to help students conclude that the square root is one of two identical values and the cube root is one of three identical values. Provide additional coaching and use #3 to check for understanding. • Provide additional coaching with Example 1 to students who need support with #5 and 7. Work with them to correct their errors and then have them do #6 and 8 on their own. • Provide additional coaching with Example 2 to students who need support with #9 and 10. Work with them to correct their errors, and then have them try solving for 64 m^2 for #9 and 216 m^3 for #10. • Encourage students to draw diagrams for #9, 10, and 13. • Some students may find it helpful to list the squares of the numbers 1 to 20 and the cubes of the numbers 1 to 10 in their Foldable. However, ensure they are able to determine square roots and cube roots.
Unit 2 Project If students complete #13 and 15, which are related to the Unit 2 project, take the opportunity to assess how their understanding of the chapter outcomes is progressing.	• You may wish to provide students with **BLM 4–5 Chapter 4 Unit 2 Project**, and have them finalize their answers. • Remind them to store all project-related materials in their project portfolio.
Assessment *as* Learning	
Create Connections Have all students complete #20.	• Encourage students to verbalize their thinking. • Allow students to work with a partner to discuss the question, and then have them provide an individual response orally or in written form.