4.3 Rational Exponents

***Mathematics 10,* pages 174–183**

Suggested Timing

80–100 min

Blackline Masters

BLM 4–3 Chapter 4 Warm-Up
BLM 4–8 Section 4.3 Extra Practice

Mathematical Processes

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- Visualization (V)

Specific Outcome

AN3 Demonstrate an understanding of powers with integral and rational exponents.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 3a)–c), 4a), c), e), 5a)–c), 6, 7, 9–11, 18, 19
Typical	#1, 3a)–c), 4a), c), e), 5a)–c), 6 or 7, two of 8, 10, 12, *or* 13, three of 9, 11, 14, *or* 15, 18, 19
Extension/Enrichment	#3–5, 7, two of 11–15, 16–19

Planning Notes

Have students complete the warm-up questions on **BLM 4–3 Chapter 4 Warm-Up** to reinforce prerequisite skills needed for this section.

Discuss the opening text and visual about the piano keyboard. Explain that a piano keyboard has 88 musical pitches organized in ascending order of pitch. The lowest note is at the far left of the instrument, the highest note on the far right. Each black key and white key on a keyboard produces a pitch or note. The distance between one key and the next adjacent key is the smallest difference in pitch between two notes that humans can hear.

Highlight that the piano keyboard is an example of the connection between mathematics and music and that this relationship supports ideas in the Unit 2 project.

Investigate Rational Exponents

In this Investigate, students revisit the use of the exponent laws to develop the concept and the meaning of a rational exponent. The key idea is for students to connect the definitions of square root (and cube root) from section 4.1 with the notation using powers with rational exponents. Students explore two different ways of evaluating powers with integral exponents. They predict values of powers with rational exponents and check their predictions using technology.

Have students work individually or with a partner to work through #1 to 4. For #1, you might have students try their own example that they can evaluate without a calculator, using both notations (e.g., $25^{\frac{1}{2}}$). Encourage students to write one of the powers in #2 using the two forms shown in #1. As you circulate, have students verbalize the connection between square roots and powers with the fractional exponent $\frac{1}{2}$. Consider using the following prompts:

- What does raising 9 to the exponent $\frac{1}{2}$ mean? (Students may say it is the same as taking the square root of 9.)
- What is the connection between a power with the exponent $\frac{1}{2}$ and its square root? (Students may say that multiplying a value by itself results in the square of the value and that the inverse is taking a square root. Or, if a number times itself has a value of 9, then its value must be the square root of 9.)

For #3, have students express $8^{\frac{1}{3}}$ in statements similar to those in #1. As you circulate, you might have students verbalize the connection between cube roots and powers with the fractional exponent $\frac{1}{3}$.

Ask what raising 8 to the exponent $\frac{1}{3}$ means.

(Students may say that it is the same as taking the cube root of 8.)

Give students sufficient time to complete #4 either individually or with a partner. As a class, have students discuss their responses.

Meeting Student Needs

- Allow students to work in pairs or small groups. Provide coaching as needed to help students practise with both notations.

ELL

- Clarify that *pitch* refers to how low or how high a note sounds.

Common Errors

- Some students may erroneously calculate powers with rational exponents as the product of the base and the exponent (e.g., $4^{\frac{1}{2}}$ as $(4)\left(\frac{1}{2}\right) = 2$), which would appear to be correct in this case).

Rx Remind students that in a power such as 5^2, the exponent 2 indicates that the base 5 is multiplied by itself 2 times, as in $5^2 = (5)(5)$
$= 25$

Answers

Investigate Rational Exponents

1. $9^{\frac{1}{2}} = 3$ because $(3)(3) = 9$

2. $4^{\frac{1}{2}} = 2$; $16^{\frac{1}{2}} = 4$; $36^{\frac{1}{2}} = 6$; $49^{\frac{1}{2}} = 7$

3. $\sqrt[3]{8} = 2$. Example: $(2)(2)(2) = 8$. An exponent to one half results in a square root, so an exponent to one third should result in a cube root.

4. a) Example: A fractional exponent is a root.

b) $2^{\frac{1}{12}} = 1.059463...$

c) $1.059463^{12} = 1.999998$. Example: It is not because 1.059463 is rounded that the answer is close to 2. If the full calculator screen is used (with the extra registers still full), then the answer is 2 on the screen. The twelfth root and the twelfth power are inverse operations.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Encourage students to verbalize the connection between the definition for square root and a power to the exponent $\frac{1}{2}$. Similarly, have students verbalize the connection between the definition for cube root and a power to the exponent $\frac{1}{3}$.

Link the Ideas

Have students recall what they know about rational numbers. Explain that they will work with rational exponents. Discuss that a rational exponent is an exponent that is a rational number. It can be expressed as $x^{\frac{a}{b}}$, where a and b are integers, and $b \neq 0$. Use some examples to reinforce that rational exponents can be expressed in decimal or fractional form.

Students will be familiar with the exponent laws from working with integral exponents in section 4.2. Use their earlier work to make the connections to applying the exponent laws using rational numbers as bases and exponents.

As a class, have students recall that a power with a negative exponent is equal to the reciprocal of the base raised to the positive of the exponent. Remind students that the exponent 0.2 in the example is rational because it can be expressed as a ratio of two integers. Ask the class what this fraction is in lowest terms.

Model expanding the given example:

$$3^{-0.2} = \left(\frac{3^{-0.2}}{1}\right)\left(\frac{3^{0.2}}{3^{0.2}}\right)$$
$$= \frac{1}{3^{0.2}}$$

Then, walk through the following example and show that a negative exponent in the denominator of a fraction is equal to the reciprocal of the base raised to the positive of the exponent.

If $3^{-0.2} = \frac{1}{3^{0.2}}$, then $(3^{-0.2})(3^{0.2}) = \left(\frac{1}{3^{0.2}}\right)(3^{0.2})$

$$= \frac{3^{0.2}}{3^{0.2}}$$
$$= 1$$

Then, $\frac{1}{3^{-0.2}} = \left(\frac{1}{3^{-0.2}}\right)\left(\frac{3^{0.2}}{3^{0.2}}\right)$

$$= \frac{3^{0.2}}{1}$$
$$= 3^{0.2}$$

Have students summarize how the sign of an exponent can be changed from a negative to a positive value.

Example 1

In this Example, students multiply or divide powers with a common base and rational exponents.

Since students have multiplied or divided powers with integral exponents, consider having them work in pairs or small groups to answer the questions using their own strategies.

For part c), ask students what they need to do to the exponents before solving. Coach students through converting exponents to fractional or decimal form, if necessary. For parts b) and c), you might have students write one product or quotient using exponents in fractional form and the other one in decimal form.

For part d), ask students how they can use powers of 2 to convert to the same base.

As a class, walk through the given solutions. Have students compare their solutions to the given solutions. You might have students work in small groups to discuss the given solutions and explain why and how the methods shown work. Ask questions such as the following:

- What methods do you use to multiply or divide powers with the same base?
- Which method do you prefer? Why?

Have students work with a partner and challenge them to use different methods to answer the Your Turn questions. Have partners discuss their solutions and work together to resolve any differences in the solutions.

Example 2

In this Example, students simplify and then evaluate powers with rational exponents.

Work through the solutions as a class. Alternatively, since students have simplified powers with integral exponents, consider having them work in pairs or small groups to answer the questions using their own strategies before working through the given solutions as a class.

For part a), emphasize the importance of raising the coefficient of the power, 4, as well as the variable base to the outside exponent. You may need to remind students of the connection between 0.5 and $\frac{1}{2}$.

For parts b) and c), show both methods. Ask students which method they prefer and why.

For part b), Method 1, have students use mental math to add $3 + \frac{3}{2}$.

For part c), ask the following questions:

- How can you use prime factorization to convert to the same base?
- Why do you convert the exponent from –0.75 to $-\frac{3}{4}$?

Discuss that in general, it may be helpful, but is not necessary, that all rational exponents be in the same form after being simplified. Discuss situations when it would be better to leave rational exponents in fractional form. For example, $\frac{2}{3}$ does not convert into an exact decimal, so it should be left in fractional form, for better accuracy.

Have students work with a partner and challenge them to use different methods to solve the Your Turn questions. Have partners discuss their solutions and work together to resolve any differences in the final solutions.

As a class, have students discuss the method they used to simplify part b). Ask the following questions:

- Did you simplify within the brackets first or did you raise each power to the exponent 9 first?
- Which method do you prefer? Why?

Example 3

In this Example, students solve a population growth problem involving powers with rational exponents.

As a class, read the problem and walk through the solution. For part a), ask students how they know from the values in the formula that it represents a growth model. (The base is greater than 1.)

Discuss the fractional exponent and reinforce that the doubling of the bacteria occurs every 42 h. Ask students what exponents represent 84 h and 126 h.

For parts c) and d), ask students if it is reasonable to round up. Why? (Fractions of bacteria do not make sense.)

Before assigning the Your Turn, you might explain that a fund is an investment that pools money from many individuals and invests it according to the fund's objectives. Explain that depending on factors such as the risk of the investment, mutual funds can increase or decrease in value. If needed, clarify that *quarterly* means every three months. You might ask students about the connection between 12.6% and 1.126 in the formula. Point out that the base of the power in the formula is equal to 112.6%. Then, break this percent into the sum: 100% + 1.26%. Have students do the Your Turn and then explain their solution to a classmate.

Key Ideas

The Key Ideas reinforce the principle that a power with a negative exponent can be written as a power with a positive exponent. Ask students to develop their own example and record it in section 4.3 of their Foldable.

Direct students to the table summarizing the exponent laws as they apply to powers with rational exponents. Have students verify the expressions by converting each exponent to either decimal or fractional form and then simplifying. Have students use the back of the insert for section 4.3 to provide their own example of a rational exponent for each exponent law. Have them include an example of a power with a rational exponent that is written in both decimal and fractional form. Encourage students to exchange their examples with those of a partner and check the solutions. Check that students apply the exponent laws to coefficients of powers that are raised to another exponent.

Meeting Student Needs

- For the Link the Ideas, consider modelling an example of each exponent law to help reinforce students' understanding about rational exponents. Alternatively, have students work with a partner to verify each example.
  - Product of Powers: $(3^{\frac{1}{2}})(3^{\frac{1}{2}}) = 3$
  - Quotient of Powers: $\dfrac{5^{0.5}}{5^{0.25}} = 5^{0.25}$
  - Power of a Power: $(3^{\frac{1}{5}})^{\frac{1}{2}} = 3^{\frac{1}{10}}$
  - Power of a Product: $(4y)^{0.6} = (4^{0.6})(y^{0.6})$
  - Power of a Quotient: $\left(\dfrac{2}{5}\right)^{\frac{2}{3}} = \dfrac{2^{\frac{2}{3}}}{5^{\frac{2}{3}}}$
  - Zero Exponent: $(-5)^0 = 1$
- For the Link the Ideas, some students may benefit from restating the principle about powers with negative exponents in their own words. Refer them to the notes in their Foldable to help them. For example, a negative exponent in the numerator can be written as a positive exponent in the denominator. Also, a negative exponent in the denominator can be written as a positive exponent.
- Encourage students to refer to the classroom posters illustrating the exponent laws. Reinforce that the same laws apply to rational exponents.
- For Example 1, have students restate the exponent laws for multiplying or dividing powers with the same bases:
  - To multiply powers with the same base, add the rational exponents.
  - To divide powers with the same base, subtract the rational exponents.
- Encourage students to perform the calculations involving fractions without technology. Allow students access to calculators when simplifying questions involving rational exponents.
- Reinforce that when simplifying rational exponents that are in fractional and decimal form, the exponents need to be converted so all are fractions or decimals. Be prepared to help students review how to change from one form to the other.
- For Example 3, ask students how the base or the exponent of the power indicates a growth model or a decay model.

- For Example 3: Your Turn, check students' understanding of the concept of interest and how it is earned. For example, students living in Northern communities may not have access to banks. You may wish to explain simple interest and compound interest and how the number of compounding periods affects the final amount of the investment.

Common Errors

- Some students may have difficulty evaluating powers with rational exponents when using their calculator.

Rx Coach students to use proper key sequencing on their particular model. Check that they place brackets around the fractional exponents.

- Students may struggle with fraction operations when applying the exponent laws to powers with rational exponents.

Rx Help students review fraction operations and how to convert between fractions and decimals.

Web Link

For a tutorial on rational exponents, go to www.mhrmath10.ca and follow the links.

Answers

Example 1: Your Turn

a) x^5 **b)** $p^{\frac{-3}{4}}$ **c)** $4^0 = 1$ **d)** $1.5^{\frac{7}{6}}$ or $\left(\frac{3}{2}\right)^{\frac{7}{6}}$

Example 2: Your Turn

a) $27^{\frac{2}{3}}x^{\frac{12}{3}} = 9x^4$ **b)** $(t^{12})(t^3) = t^{15}$ **c)** -27 **d)** $\left(\frac{64}{x^3}\right)^{\frac{2}{3}} = \left(\frac{4}{x}\right)^2 = \frac{16}{x^2}$

Example 3: Your Turn

a) Example: 12.6% = 0.126; 1 = full investment, so 1.126 = full investment plus interest

b) The value is $5465.42 after the third quarter.

c) The value is $7138.14 after three years.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Provide similar questions before having students try the Your Turn. • Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • For each question, ask students to verbalize the exponent law they would use. • For part c), ask students whether they would change both exponents to fractional form or decimal form. Consider using some examples to help students generalize that powers with rational exponents that convert to non-terminating decimals should be kept in fractional form.
Example 2 Have students do the Your Turn related to Example 2.	• Provide similar questions before having students try the Your Turn. • Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Some students may benefit from coaching to decide which exponent laws to apply and how to simplify a question such as $(27x^6)^{\frac{2}{3}}$. Ask them to indentify the exponent of 27.
Example 3 Have students do the Your Turn related to Example 3.	• Have students identify and record the meaning of each variable in the formula. • Provide a similar problem to students who would benefit from more practice. Allow them to work with a partner and talk through their thinking.

Check Your Understanding

Practise

For #1 and 2, encourage students to practise working with exponents in both decimal and fractional form. Remind students to raise any coefficients to the exponent.

For #1f), you might model converting to a positive base. Using an example, ask students to explain how to convert –8 to a positive:

$$\begin{aligned}(-8)^{\frac{2}{3}} &= [(-1)(8)]^{\frac{2}{3}}\\ &= (-1^{\frac{2}{3}})(8^{\frac{2}{3}})\\ &= (-1)^{\frac{2}{3}}(2)^{(3)\left(\frac{2}{3}\right)}\\ &= (1)(2^2)\\ &= 4\end{aligned}$$

For #5, remind students to simplify using the exponent laws before using a calculator to evaluate.

For #6, ask students how the base of 1.1 in the formula relates to the growth rate. Refer students to the Did You Know? about species of fish that are stocked in BC.

Apply

Allow students some choice in the questions they need to do. Encourage students to solve the problems using a method of their choice. You might have students compare their method for some problems with that of a classmate who solved the problems in a different way.

For #7, students perform error analysis. For each problem, encourage them to confirm that the initial expression and the final expression are not equivalent by evaluating the expressions using a whole number in place of the variable.

For #9c), students must remember that the exponent is negative.

For #10c), discuss why exponential growth results in overcrowding and the subsequent decline of the population as the fish compete for limited resources. Ask students to provide some limiting factors (e.g., space, food) to exponential growth.

For #11, some students may not understand why the base is 0.88 (100% – 12%). A base less than 1 implies that depreciation is occurring.

For #14, you might explain that the formula is derived from Isaac Newton's Law of Cooling. The law states that the rate at which a warm object cools is approximately proportional to the difference between the temperature of the object and the temperature of its environment.

For #15, refer students to the Did You Know? about Johannes Kepler.

Extend

For #16, encourage students to try folding a sheet of paper the number of times that corresponds to the answer. Some students may be successful if you provide the hint that 100% represents the starting area and 1% represents the ending area. Have students discuss with a classmate how to use these values in the formula. Encourage them to test their answer.

Create Connections

For #18, consider having students brainstorm some real-life applications of rational exponents. Students can use the ideas as a springboard to develop their own response.

For #19, encourage students to develop an example to help with their explanation, and provide a correction. Have them discuss their correction with a partner.

Meeting Student Needs

- Allow students to work in pairs.
- Have students refer to the exponent laws in their Foldable notes and the classroom posters when working through the questions.
- For word problems, encourage students to identify the key terms (e.g., half-life, doubles) to help determine what values are needed.
- For #15, some students may be successful if you restate the meaning of the variables in the formula and prompt them about the order to use for solving the problem.
- Provide **BLM 4–8 Section 4.3 Extra Practice** to students who would benefit from more practice.

ELL

- Teach the following terms in context: *stocked annually*, *trout*, *char*, *kokanee*, *guppies*, *photographic enlarger*, *planetary system*, *orbital radius*, *planetary motion*, *consecutive folds*, and *bloodstream*.

Enrichment

- Tell students about the Rule of 72, which is used to estimate the amount of time it takes to double the value of an investment earning compound interest, depending on the interest rate. For a term deposit invested at 3%, it would take 24 years to double your money.

  $\frac{72}{3} = 24$

  For a term deposit invested at 5%, it would take 14.4 years.

  $\frac{72}{5} = 14.4$

  Have students try out the rule using different interest rates.

- Challenge students who are interested in famous mathematicians to research Sir Isaac Newton and Gottfried Leibnitz and the controversy about who invented calculus. Have them present a report on their findings. They may find the related Web Link at the end of this section helpful.
- For the Did you know? on page 183, have students research Johannes Kepler's work on planetary motion. They might present a report using a format of their choice. They may find the related Web Link at the end of this section helpful.

Gifted

- One of the major concerns about climate change involves the melting of sea ice, in both area and volume. Have students research this issue and speculate how scientists use mathematical models of melting sea ice and shrinking polar ice sheets in order to predict future trends.
- Challenge students interested in the formula related to musical pitch to explore the related Web Link on this page.

Web Link

For information about Sir Isaac Newton and Gottfried Leibnitz and the calculus controversy, go to www.mhrmath10.ca and follow the links.

For information about Johannes Kepler's work on planetary motion, go to www.mhrmath10.ca and follow the links.

For a lesson involving the formula that expresses the relationship between the pitches of adjacent musical notes, go to www.mhrmath10.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 3a) to c), 4a), c), and e), 5a) to c), 6, 7, and 9 to 11. Students who have no problems with these questions can go on to the remaining questions.	• Encourage students to verbalize their thinking. • Provide additional coaching with Example 1 for #1 and with Example 2 for #3 to 5. Coach students through correcting their errors before having them try some of the questions that were not assigned. • For #1, encourage students to refer to the notes about the exponent laws in their Foldable. • For #3, some students may benefit from reviewing the process of opposite operations. Model solving equations such as $x + 1 = 4$ and $\frac{x}{2} = 3$, and ask them to identify the opposite operations that will allow them to solve for x in each case. Encourage them to apply the same thinking to #3c). • Provide additional coaching with Example 3 for #6. Have students talk through their thinking and help them correct their errors. Check for understanding before moving on to #9 to 11. • For #7, prompt students to verbalize the strategy and the exponent law they would apply for each solution. Have them work through simplifying the initial expression and then identify the error. • For #9 and 11, have students record the meaning of each variable as part of the solution. For #11, clarify the meaning of *decreasing in value* and discuss how the wording affects what values are used in an equation.
Assessment *as* Learning	
Create Connections Have all students complete #18 and 19.	• Encourage students to verbalize their thinking. • Allow students to work with a partner to discuss the questions and then develop an individual response orally or in written form. • For #18, encourage students to generate their own problem. You might allow students to adapt a problem from the student resource. • You might encourage students who have difficulty in choosing an example for #19 to use one of the questions in #3. The solutions to these types of questions typically show errors in thinking. In particular, #3f) might be appropriate as students often erroneously multiply the base numbers 2 and 3.