8.2 Modelling and Solving Linear Systems

***Mathematics 10*, pages 432–445**

Suggested Timing

100–120 min

Materials

- map of Canada
- ruler
- grid paper or computer with graphing software

Blackline Masters

BLM 8–3 Chapter 8 Warm-Up
BLM 8–4 Chapter 8 Unit 4 Project
BLM 8–6 Section 8.2 Extra Practice

Mathematical Processes

✓ Communication (C)
✓ Connections (CN)
✓ Mental Math and Estimation (ME)
✓ Problem Solving (PS)
✓ Reasoning (R)
✓ Technology (T)
✓ Visualization (V)

Specific Outcomes

RF1 Interpret and explain the relationships among data, graphs and situations.

RF7 Determine the equation of a linear relation, given:
- a graph
- a point and the slope
- two points
- a point and the equation of a parallel or perpendicular line

to solve problems.

RF9 Solve problems that involve systems of linear equations in two variables, graphically and algebraically.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 3–5, 8, 9, 12, 23
Typical	#1 or 2, 4, 6, 7, 9, 10 or 11, 12, 13, 15 or 16, 22, 23
Extension/Enrichment	#9, 11, 12, 14, 17 or 18, 19–22, 24

Unit Project Note that #9 and 12 are Unit 4 project questions.

Planning Notes

Have students complete the warm-up questions on **BLM 8–3 Chapter 8 Warm-Up** to reinforce prerequisite skills needed for this section.

As a class, read and discuss the opening text about travelling and modes of transportation. You may wish to brainstorm some modes of travel and then rate their costs, travel times, and effects on the environment. Challenge students to research and find out how accurate their predicted ratings were. Ask students to list any other factors that travellers should consider when selecting a mode of travel. Invite students who have travelled on *The Canadian* or another train to share their experiences.

Investigate Creating a System of Linear Equations

You may wish to begin the Investigate by having a class discussion about distances to local places and then expanding to talk about the distances from students' city or town to major Canadian cities.

As students work through the Investigate, ensure they have the correct system of equations before they proceed with graphing. If students have difficulty obtaining the equations for the linear system, you can prompt them by asking the following questions:
- What is the dependent variable? the independent variable?
- What is the relationship between distance, speed, and time?
- How can you find the distance if you know the time and speed?
- How far did the train travel before the car started driving?

As students consider the reasonableness of their answers and the changes to the times of departure, you may wish to have them consider the types of answers that are acceptable in this situation:
- What number system(s) can the answers be elements of? Why?
- Is this true for all systems of linear equations?

Meeting Student Needs

- Some students may have difficulty locating cities on a map and determining the distances between them. Allow students to use distances that seem reasonable even if they are not accurate. Ensure that students use units when stating the distance measurements.
- Allow students to use a clock as necessary to calculate times.
- Encourage students to complete the Investigate with any method available to them. If students have difficulty representing the situation with a system of linear equations, allow them to solve the problem numerically.

Answers

Investigate Creating a System of Linear Equations

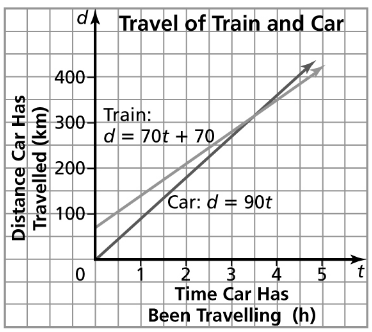
1. Example: Edmonton, AB, to Winnipeg, MB, is 1357 km.

2. Example: Car: about 15.1 h or 15 h 5 min
Train: about 19.4 h or 19 h 24 min

3. a) Let d be the distance the car has travelled, in kilometres.
b) Let t be the time the car has been travelling, in hours.

4. a) $d = 90t$ **b)** $d = 70t + 70$

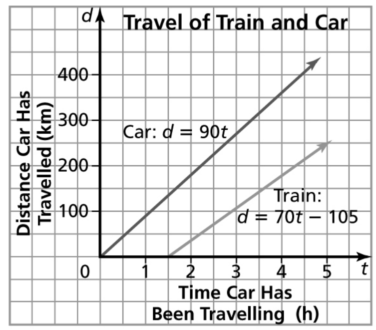
5.



The solution is (3.5, 315). The car will catch the train 3.5 h after the car starts travelling and they will be 315 km away from their start.

6. a) $d = 90t$ and $d = 70t - 105$

b)



The lines do not actually intersect on the graph. They appear to intersect if the lines are extended to negative values, but time cannot be negative so the graph is restricted by the domain.

7. a) Example: I think the first system was easier to write because the difference in start times was already given in hours.
b) Example: Both of their distances depend upon the product of their speed and time travelling. Their speeds are different and the length of time travelled changes.
c) Example: Known information creates constants in the equations and varying information becomes variables. The graphs are dependent on what the results of speed × time give for distances. The speeds are slopes of the lines and the later start of one vehicle results in a different d-intercept.

8. Example: There are two different variables, d and t, so two equations can be written. The vehicles both travel at constant speeds so they are linear equations.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about writing a system of linear equations.	• You may wish to have students complete #1 to 5 independently or with a partner and have some groups model their graphs on the board or share with another group. • As a class, discuss the questions in #5. It may help clarify what students have graphed and assist students in moving forward with #6. • Some students may need to be prompted to understand the significance of the different starting times for the two modes of travel. Ask students what it means in terms of distance. You may wish to talk in terms of a *head start* for the car. Ask what formula they can use to determine the distance travelled. • Question #7c) is a key question. Listen for students' level of understanding and ability to tie the responses to what they learned in section 8.1.

Link the Ideas

You may wish to follow up on the discussion in this section. For instance, you could have students work in pairs, where one student writes a simple expression or equation and the partner writes several situations that could be represented by that expression or equation. Alternatively, you could have one student from each pair write a phrase in words and have the partner write the expression or equation that represents the phrase.

Example 1

In this example, students compare two rental charges. Ensure that students make the connection between the equations and the slope and intercept of each line. For Option A, the y-intercept is 30 and the slope is 8. For Option B, the slope is 14 and the y-intercept is 0. To help students interpret the information in the graph, you could ask, "Why does knowing the solution to this linear system not give enough information for you to choose a rental option?" Of course, the intent of this question is to help students understand that one rental option costs less for rentals less than 5 h, the other option costs less for rentals longer than 5 h, and for exactly 5 h, the cost is the same.

Students may not see the importance of assigning variables at the beginning of a solution. You may wish to emphasize to them that this is a crucial step to their success. Some students may need to be reminded to avoid assigning letters for variables that might be confusing, such as S or l.

The solution provides two methods of graphing. You may wish to have students compare and contrast the two methods. You may wish to ask the following questions:

- Is one method likely to be faster than the other? Explain.
- Is one method likely to be more precise than the other? Explain.
- Is one method likely to be more convenient than the other? Explain.

Example 2

This example involves two grain bins that are being emptied, so the volume decreases over time. Unlike the Investigate where one traveller starts before the other, in Example 2, both bins start emptying at the same time. The bins contain different amounts of grain when they start emptying.

As students verify their answers, you may want to ask the following questions:

- Why is the term containing *time* negative in each equation? (Note that this is one of the margin questions in the student resource.)
- What does each of the equations in the linear system represent?
- What does the point of intersection represent?
- How would you determine the volume of grain that has been removed from each bin?

When completing the Your Turn, encourage students to consider these questions:

- How can you determine reasonable scales and values for the axes of your graphs?
- What does the slope of each line mean in the context of the problem?
- What do the intercepts of each line represent?
- Would you prefer to solve this system by hand or using technology? Why?
- What does the intersection point mean in the context of this situation?
- How did you verify your solution to the system of linear equations? Is there any doubt that your answer is correct?

Example 3

Students may find this example challenging to grasp, because one unknown is not dependent on the other, and the points on the lines, other than the intersection point, have no meaning in the context. You may want to suggest that students work with smaller numbers and use patterning. Ask them the following questions:

- What two different quantities are given in this problem?
- Since there are two quantities, does it make sense that each would lead to one equation in the system?
- What would the revenue be if 5 adult tickets and 10 student/senior tickets were sold?
- What if 8 adult and 12 student/senior tickets were sold?
- What if n adult tickets and k student/senior tickets were sold?

As students consider the reasonableness of their answers and the changes to the linear system if the ticket prices change, encourage them to consider the types of answers that are acceptable in this situation. Ask the following questions:

- What number system(s) can the answers be elements of? Why?
- Is this true for all systems of equations?

If students need assistance completing the Your Turn, you can ask them to begin by defining the variables they will use in the system. Asking questions like the following should help students develop the equations they need.

- How does the total time of 5.25 h lead to one equation?
- How does the total distance of 440 km lead to one equation?
- How do you determine the distance a person has driven? What quantities do you need to use?

Key Ideas

Have students read the Key Ideas with a partner. The last bullet contains two key points. Have students find or create an example to clarify each of the five points. When they are satisfied with the examples they have chosen, they can work with another team and compare examples. Students may wish to include the Key Ideas and the examples they have added in their Foldable or math journal.

Meeting Student Needs

- This section has many word problems dealing with money. You may wish to bring in money manipulatives to demonstrate an activity.
- You may wish to remind students of the following strategy for translating words to mathematical expressions and equations: Read line by line and see if a mathematical statement can be written from the information given. Use the given variables or assign variables then.
- Write common operations on the whiteboard; then, have students suggest other words or phrases that mean the same thing. For example, for *subtraction*, students might write *difference* or *is less than*. Emphasize that students need to pay close attention when writing subtraction statements.
- Students will likely benefit from a discussion of the window of a graph, whether they are graphing by hand or using technology.
- Discuss with students what expenses they have when they rent a movie and watch it at home. Compare them to the costs involved in going to a movie in a theatre.
- In Example 2, allow students to explore the equations by asking them to generate a table of values for the amount of grain remaining after the first several minutes. Have students look for a pattern in how they found the values.

ELL

- Some students may not be familiar with the concept of rent. Explain the meaning of rent to students and emphasize that rent is based on a specified unit of time. You may wish to brainstorm examples of things that are rented on an hourly, daily, weekly, monthly, and yearly basis.

Enrichment

- Ask students to create a scenario that can be modelled by two simultaneous equations. (Students' questions will vary. Check to ensure their equations and solutions are accurate. Example:

  Ten years ago, Madge was $\frac{3}{2}$ times as old as Fran.

  In ten years, Fran will be $\frac{5}{6}$ times as old as Madge.

  How old are they now?

Time	Fran	Madge
10 years ago	$F - 10$	$M - 10$
This year	F	M
10 years from now	$F + 10$	$M + 10$

  $M - 10 = \frac{3}{2}(F - 10)$ and $F + 10 = \frac{5}{6}(M + 10)$

  Madge is 30 and Fran is 23 and 4 months.)

Gifted

- Challenge students with this question: The numerals of a two-digit number add to seven. If the number is written with the numerals reversed, the new number is nine greater than the original number. What is the original number? (34) Encourage students to create their own number question.

Answers

Example 1: Your Turn

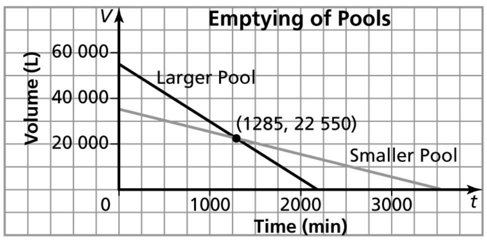
a) Let m represent the duration of the main act, in minutes.
Let a represent the duration of the opening act, in minutes.
$2a - m = 3$ and $m + a = 132$

b) The solution to the linear system is (87, 45). It represents the duration of each act. The main act was on stage for 87 min and the opening act was on stage for 45 min.

Example 2: Your Turn

a) Let V represent the volume of water remaining in each pool, in litres. Let t represent the time, in minutes. $V = 54\,675 - 25t$ and $V = 35\,400 - 10t$

b)



Example: The graph shows that the volume of water remaining in each pool decreases over time. Both lines stop at the horizontal axis because the volume of water left is 0 L. The two lines intersect at (1285, 22 550). This is the only point when the two pools contain the same volume of water at the same time. At 1285 min, each pool has 22 550 L of water remaining. Before 1285 min, the larger pool contains more water. After 1285 min, the smaller pool contains more water.

Example 3: Your Turn

Let f represent the time that Jamie's father drives, in hours.
Let c represent the time that Jamie's cousin drives, in hours.
$f + c = 5.25$ and $90f + 80c = 440$
Jamie's father drives for 2 h and her cousin drives for 3 h 15 min.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Some students may require remediation in translating the math into algebraic expressions. Prompt students to verbalize what variables they will choose and the meaning of *less than* and *twice the time*. • Have students identify how they will label the axes. • You may suggest that students compare their responses to part b). Have students explain their thinking about what the solution represents. Discuss this important point as a class to clarify any misinterpretations.
Example 2 Have students do the Your Turn related to Example 2.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • If students are having difficulty, ask them what parallels exist between this question and Example 2. Have them verbalize what variables could be used in setting up the system. • Encourage students to use a table similar to the one in the example. This will assist them in setting up their equations and the system. • You may wish to discuss, as a class, how students will decide on an appropriate scale for their graph as the values are quite large. • Prompt students to verbalize how they would label the axes. Ask what the intersection point would mean, given their labels. • Remind students of the importance of verifying their solution.
Example 3 Have students do the Your Turn related to Example 3.	• Encourage students to verbalize their thinking. • You may wish to have students work with a partner. • Students may find this Your Turn challenging because one unknown is not dependent on the other. The intersection point will be a solution to both systems but requires students to interpret it in terms of the labels given to the axes and in terms of the context. • Students may require some coaching in identifying the significance of the values given in the problem (440 km, 90 km/h, 80 km/h, 5.25 h). • Prompt students to verbally identify variables that could be used in the problem. • Before students graph the system, ensure that they have written a correct slope-intercept form.

Check Your Understanding

Practise

For #2, students translate words into equations. This question may be slightly more challenging for students than #1, because some calculations are required. Students need to be more mindful of assigning variables in this question.

Question #4 has a monetary context. Some students may find this type of question difficult. A table of values has been provided to help students develop the system of equations.

Apply

Question #5 is similar to Example 2. If students have difficulty with this question, you can refer them to the example. A solution to the system is not required, but students need to be able to interpret the solution.

In #6, students write and graph a system of equations, which they do not need to solve. Students explain what the solution represents within the given context.

In #7, students are asked to solve a system of equations. This system involves a linear rate, so you can assist students by asking them to work through this question using the slope-intercept form of the equation of a line.

Question #8 is similar to Example 1. Students need to pay attention to units, because values are stated in cents and in dollars and cents at different points in the question.

Question #9, which involves retrofitting, is related to the unit project. It is likely that students have some experience with the idea of water conservation. You may want to allow a brief discussion at this point, to emphasize the link to the project. For example, students could be invited to check the flow rate of shower heads in their home, if possible. Students may have knowledge of other considerations or costs when retrofitting. Prompt students to discuss whether the solution to the linear system is reasonable based on the length of a typical shower. If other flow rates are available, students could write a new linear system using those data and compare the solution to that system to the solution to the original question.

Question #11 asks students to go a step beyond finding a solution to a linear system and decide whether the solution fits within the context of the question. It is not enough to find the intersection point. Students need to decide whether the domain of the system includes that point, though the question is not posed in those terms. You may want to have students identify any assumptions they are making, for example, that both cyclists can continue at their current speeds.

The topic of #12 is wildlife, which links to the unit project. If students need assistance writing the system of linear equations, refer them to Example 1. Many students will likely have prior knowledge of wetlands, which may provide an interesting topic for a brief group discussion.

For #13, students use the distance-speed-time relationships to write and solve the system of equations. In particular, the equation distance = speed × time is needed to write the correct system of linear equations.

In #14, students use rates, specifically the rate at which two people can paint a fence, in linear feet per hour. It may not be obvious to students that each person will paint for the same amount of time to complete the task. Students may also be initially surprised that one of the equations in the system is a constant function.

In #15 and 16, students use the endpoints of the given intervals to help determine the lines that comprise each system of equations. Students need to solve these linear systems, and they may choose to create the graphs directly without first determining the equations.

Extend

For #19, students form a linear system by translating words into equations. This question requires careful reading by students, because the question involves subtraction.

You can assist students with #20 by asking them, “What two factors affect the net speed of a person swimming when a current is present?” Further, you can ask how those two factors combine to produce the net speed when the person is swimming with and against the current.

Question #21 is commonly described as a *mixing problem*. You may want to ask students how they convert percents to decimals.

Create Connections

Question #23 offers students two options for modelling the situation. Students may choose to use a table of values or a diagram.

Question #24 gives students the solution to a system of linear equations, and has them determine the equations in the system.

Unit Project

You may wish to have a group discussion about recent renovations in students' homes. Prompt the discussion with the following questions:

- What types of renovations have taken place?
- What was the motivation for the changes?
- Did you install energy-efficient furnaces? on-demand hot water? low-flow shower heads? Emphasize that these relationships can be illustrated graphically to show the cost saving for homeowners.

Meeting Student Needs

- Provide **BLM 8–6 Section 8.2 Extra Practice** to students who would benefit from more practice.
- Have students research the purpose and meaning of the Métis sash and try to have a student bring in a sash. Most Métis families have a sash. Students can create a mini sash using yarn and braiding the threads.
- Questions #1 and 2 allow students opportunities to develop equations without the pressure of also needing to solve them. Students should discuss appropriate variables for each question in addition to the writing of the equations.
- You may wish to display posters listing the steps involved in problem solving. (Read it, Plan it, Solve it, Look back)
- You may wish to have students complete two or three questions from the Apply section. Encourage students to list the step-by-step process they followed when working through each question. Ask the following questions:
  - What was difficult? What was easy?
  - Was there a section that was confusing?
  - What linear system did they use to solve the equations?

  Students could share this information with other students at the end of the section.

ELL

- In #9, to perform a *graphical analysis* means to explain your answer by creating a graph and referring to it to support your answer.

Common Errors

- In #16, students may forget to divide by 4 for the number of weeks in the period.

R_x Remind students to carefully consider the information in the question and to have a clear idea of the problem they are solving.

Web Link

For information about the Test of Metal mountain bike race referred to in #11, go to www.mhrmath10.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 3–5, and 8. Students who have no problems with these questions can go on to the remaining questions.	• You may wish to have students work with a partner. • For #1, students can refer back to the examples they have completed. The translation required in the question is at an entry level. Remind students that assigning a variable that is appropriate for the context of the question makes it easier to work with and remember. • If students are having difficulty with #3, prompt them to consider what the two ways are for Molly to get points (goals and assists). Ask what variables could represent these values. Ask how these variables and the 32 points could make up one of the equations. Finally, ask them how Molly gets most of her points and how this could be used to write a second equation. • You may wish to have students add another row to the table in #4 for totals. Ask them what total values should appear in the Number of Coins column and in the Value of Coins column. • Some students may need coaching in changing the value of the coins to cents. • You may wish to have students review Example 2 for #5. Suggest they set up a table of values for the problem to assist them in writing their system. • Students should find #8 to be a straightforward question. Refer them to Example 1 if they require a pattern. Ensure that they can understand and verbalize the meaning of $19 ($y$-intercept). Encourage students to fully explain their reasoning for their choice of plan. Some students may need to be explicitly asked to address both the plans up to the point of intersection, and after this value as well.
Unit 4 Project If students complete #9 and 12, which are related to the Unit 4 project, take the opportunity to assess how their understanding of chapter outcomes is progressing.	• Some students may require prompting for #9a). Have them identify the cost of heating 1 L of water. Ask how they can determine the cost for 170 L of water. Once students have completed part a), ensure they understand how this information can be used to set up their system. Have them verbalize their process or have them work with a partner to complete part b). • It may be beneficial to take a few minutes to discuss the value in conserving water by using a shower head that uses less water. Talk about whether it is a cost-saving decision. • For #12, it is important to coach students to use appropriate variables. Suggest they use a table, as in Example 1, to set up their system. • Some students may require coaching to interpret the meaning of *critical* in the question and how it plays a role in this situation. Remind students that *justify* means to support their explanation mathematically. • You may wish to provide students with **BLM 8-4 Chapter 8 Unit 4 Project**, and have them finalize their answers.
Assessment *as* Learning	
Create Connections Have all students complete #23. If they have no difficulty with this question, have them complete #24.	• Students may find it helpful to refer back to the Investigate to assist them with #23. For students who could benefit from a more scaffolded approach, encourage them to use a table. • Note that #23 asks students to describe how they would determine whether James will catch up to Gavin. You may wish to have them follow through and actually complete and solve it. • In #24, student to write a system from a given graph. Assign this question only if students have a good understanding of systems and can translate their understanding of slopes and y-intercepts into a context.