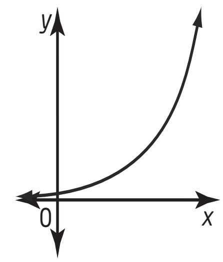
Chapter 3 Exponential Functions

3.1 The Nature of Exponential Growth

KEY CONCEPTS

- Exponential growth functions have these properties:
  - As the independent variable increases by a constant amount, the dependent variable increases by a common factor.
  - The graph increases at an increasing rate.
  - The finite differences exhibit a repeating pattern: the ratio of successive finite differences is constant.



- Any non-zero real number raised to the exponent zero is equal to 1: $b^0 = 1$ for $b \in \mathbb{R}, b \neq 0$.

Example

A flu virus is spreading among the students at Mathematica High School. Initially, on day 0, three students become ill. Each of these students passes the virus to two other students who become ill on day 1. Each of these six students passes the virus to two more students who fall ill on day 2, and so on. Assume that no one can become sick from the virus more than once.

a) How many students become ill
  i) on day 3?
  ii) on day 5?
b) Write the equation that represents this situation.
c) If the student population at Mathematica High School is 1852, how long will it take for the virus to spread throughout the entire school?

Solution

a) Three students become ill on day 0. Because each of these three students passes the virus to two other students, six students become ill on day 1. Each of these six then passes the virus to two more students, so that 12 students become ill on day 2, and so on.
Organize the information in a table. From the table, it can be seen that
  i) 24 students fall ill on day 3
  ii) 96 students fall ill on day 5

Day	Number of Students Falling Ill
0	3
1	6
2	12
3	24
4	48
5	96

b) The equation relating the number of students, s, who become ill on any day, n, is $s(n) = 3 \times 2^n$.

c) Because the number of students falling ill doubles each day, the virus will have spread throughout the entire school population on the day that the number of students becoming sick is one half of the total population. Use this number in the equation to solve for n.

$1852 = 3 \times 2^n$ Divide each side by 3.

$617.67 = 2^n$ Use trial and error to solve for n.

For $n = 9$, $2^9 = 512$.

For $n = 10$, $2^{10} = 1024$.

Since 617.67 is between 512 and 1024, n is between 9 and 10.

Try $n = 9.5$. $2^{9.5} \doteq 724$, which is higher than 617.67. So n is less than 9.5.

Try $n = 9.3$, $2^{9.3} \doteq 630$, which is a bit high.

Try $n = 9.27$, $2^{9.27} \doteq 617.37$, which is very close to 617.33.

Therefore, it will take approximately 9.27 days for the virus to spread throughout the entire school.

A Practise

1. An ant colony has an initial population of 25. The number of ants triples every day.

a) Copy and complete the table.

Day	Population	First Differences	Second Differences
0	25		
1	75		
2			
3			
4			
5			

b) Is the relationship between the ant population and the number of days an exponential relation? Explain how you can tell.

c) Examine the finite differences. Describe how the first differences and second differences are related.

d) Will the pattern of first and second differences observed in part c) continue with the third and fourth differences? Write down your conjecture.

e) Calculate the third and fourth differences. Was your conjecture in part d) correct? Explain.

2. Consider these three functions:

$y = 5x$ $y = x^5$ $y = 5^x$

a) How do the equations differ? How do the graphs of these functions differ?

b) Describe the domain and range of each function.

3. What is the value of 11^0? Use patterns and numerical reasoning to justify your answer.

4. a) Rewrite the expression $\frac{a^5}{a^5}$ by expanding both powers.

b) Divide out common factors in the numerator and denominator. What is the simplified value of this expression?

c) Write the expression $\frac{a^5}{a^5}$ as a single power by applying the quotient law.

d) Write a statement that explains how the results of parts b) and c) are related.

5. An insect colony has an initial population of 15. The number of insects quadruples every day.

   a) Which function models this exponential growth?

      A $p(n) = 15 \times 4n$
      B $p(n) = 15 \times 4^n$
      C $p(n) = 60 \times 4^n$

   b) For the correct model, explain what each part of the equation means.

6. Evaluate.

   a) 12^0

   b) $(-7)^0$

   c) $\left(\frac{5}{9}\right)^0$

   d) m^0

   e) $(2x)^0$

   f) -7^0

B Connect and Apply

☆7. Identify each function as linear, quadratic, exponential, or none of these. Justify your choice.

   a) $f(x) = 8^x$

   b) $f(x) = 11 - 9x$

   c) $f(x) = \sqrt{x}$

   d) $f(x) = x^2 - 3x + 1$

8. Identify the type of function that each table of values represents. Justify your answer.

a)

x	y
−4	16
−3	8
−2	4
−1	2
0	1
1	0.5
2	0.25

b)

x	y
−4	13
−3	10
−2	7
−1	4
0	1
1	−2
2	−5

c)

x	y
−4	−18
−3	−11
−2	−6
−1	−3
0	−2
1	−3
2	−6

d)

x	y
−4	625
−3	125
−2	25
−1	5
0	1
1	0.2
2	0.04

e)

x	y
−4	2.2
−3	0.8
−2	−0.2
−1	−0.8
0	−1
1	−0.8
2	−0.2

9. a) Use linking cubes, colour tiles, or the tools of your choice to design a growing pattern that can be described by $t(n) = 4^n$, where n is the term number and t is the total number of items in that term.

b) Draw diagrams to illustrate the first 4 terms in your pattern, where $t(0)$ is the first term.

c) How many items would you need to build

i) the 6th term?

ii) the 12th term?

d) Suppose that you have 1000 items in total to use when constructing a model of this pattern. What is the greatest number of terms you can build at the same time?

e) Suppose that you have 5000 items in total. What is the greatest number of terms you can build at the same time?

f) How does your answer to part e) differ from your answer to part d)? Explain this result.

☆**10.** A bacterial colony has an initial population of 32. The population doubles every half-hour.

a) Write an equation that relates the population, p, to time, t, measured in 30-min intervals.

b) Sketch a graph of this relationship for 2 h.

c) Use the graph to determine the approximate population after 1 h. Verify your answer using the equation. Which tool do you prefer to use? Explain why.

d) Determine the approximate population after 3.5 h. Which tool do you prefer to use for this, the equation or the graph? Explain why.

11. Adam deposits \$200 into a savings account that pays interest at 4.5% per year, compounded annually. After each compounding period, interest earned is added to the principal (the initial deposit amount) and that sum is used to calculate interest in the following period. The amount, A, in dollars, in the account can be determined using the formula $A = P(1 + i)^n$, where P is the principal, i is the annual interest rate (expressed as a decimal), and n is the number of compounding periods.

a) Write an equation that represents the amount in the account after n years.

b) Copy and complete the following table. Round values to two decimal places.

Number of Compounding Periods (years)	Amount (\$)
0	200
1	209
2	
3	
4	
5	
6	

$A(1) = 200(1 + 0.045)^1$
$= 200(1.045)^1$
$= 209$

c) Calculate the first and second differences. Is the function linear, quadratic, or neither? Explain.

d) Use the finite differences to determine what type of function this represents.

e) How does the equation confirm your result in part d)?

f) If interest is paid only at the end of each compounding period, do the points between the values in the table have meaning? Explain why or why not.

12. Refer to the formula in question 11 for calculating compound interest. Cassie deposits an inheritance of \$1500 into an account that earns interest of 3.5% per year, compounded annually. How much is in the account at the end of

a) 3 years?

b) 8 years?

13. A rural community has an emergency telephone notification plan. Phone calls are set up in equal time intervals. Initially, in interval 0, four residents know about an emergency. In the next interval, each of them calls five other residents. In the following interval, each of those residents calls five other residents, and so on for the following intervals. No resident is called more than once.

a) How many residents will be called
i) in the 3rd interval?
ii) in the 5th interval?

b) Write an equation to represent this situation.

c) If 23 698 people live in the community, how long will it take for everyone to receive a call?

d) Is this an example of exponential growth? Explain your reasoning.

14. Justin deposits \$700 into an account that earns interest of 5.5% per year, compounded annually.

a) How long will it take for Justin's investment to double in value? Explain how you solved this.

b) How much longer would it take if the account paid simple interest, at the same rate? Simple interest does not get added to the principal after each compounding period. Use the formula $I = Prt$, where I is the interest, in dollars, P is the principal, in dollars, r is interest rate (as a decimal), and t is the time, in years.

C Extend

15. Suppose you win a lottery and are given a choice of either of two prizes. Which prize would you select? Justify your answer.
Prize A: 1 cent in the first week, 2 cents in the second week, 4 cents in the third week, and so on, for 26 weeks
Prize B: \$10 000 per week for 26 weeks

16. Bacteria A has an initial population of 100 and doubles every day. Bacteria B has an initial population of 2 and quadruples daily.

a) After how long will the population of Bacteria B exceed the population of Bacteria A? What will the populations be at this point?

b) How much sooner would Bacteria B's population overtake Bacteria A's population if the doubling period for Bacteria A were twice as long?

17. The amount of a drug that remains in the human body often follows an exponential model. Suppose that a new drug follows the model $M = M_0(0.82)^{\frac{h}{4}}$, where M represents the mass of drug remaining in the body, M_0 represents the mass of the dose taken, in milligrams, and h is the time, in hours, since the dose was taken.

a) A standard dose is 200 mg. Sketch a graph showing the mass remaining in the body up to 48 h.

b) Use your graph to estimate the half-life of the drug in the body. This is the time it takes for the mass of the dosage to decrease to half of its starting value.

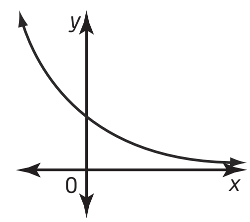
c) Check your estimate in part b) using the equation.

d) Use Technology Once the mass remaining drops to less than 1 mg, the standard test can no longer detect the drug. How long will it take before the drug is no longer detectable? Use a graphing calculator.

3.2 Exponential Decay: Connecting to Negative Exponents

KEY CONCEPTS

- Exponential decay functions have the following properties:
  - As the independent variable increases by a constant amount, the dependent variable decreases by a common factor.
  - The graph decreases at a decreasing rate.
  - They have a repeating exponential pattern of finite differences: the ratio of successive finite differences is constant.



- A power involving a negative exponent can be expressed using a positive exponent:
  $b^{-n} = \frac{1}{b^n}$ for $b \in \mathbb{R}$, $b \neq 0$, and $n \in \mathbb{N}$.
- The exponent rules hold for powers involving negative exponents.
- Rational expressions raised to a negative exponent can be simplified:
  $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$ for $a, b \in \mathbb{R}$, $a, b \neq 0$, and $n \in \mathbb{N}$.

Example

What amount of money will grow to \$2000 in 5 years if it is invested in an account that pays interest at the rate 5.5% per annum, compounded annually? The amount, A, in dollars, in an account earning interest, compounded annually, with a single deposit can be calculated using the formula $A = P(1 + i)^n$, where P is the principal, i is the annual interest rate (expressed as a decimal), and n is the number of years the principal has been earning interest.

Solution

Substitute $n = 5$, $A = 2000$, and $i = 0.055$ in the formula $A = P(1 + i)^n$. Solve for P.

$$A = P(1 + i)^n$$
$$2000 = P(1 + 0.055)^5$$
$$2000 = P(1.055)^5$$
Divide each side by $(1.055)^5$.
$$\frac{2000}{(1.055)^5} = P$$
$$P \doteq 1530.27$$

Therefore, \$1530.27 is the required amount.

A Practise

1. Write each as a power with a positive exponent.

   a) 7^{-1} **b)** 10^{-2}

   c) a^{-4} **d)** $(mn)^{-1}$

   e) -4^{-1} **f)** $(2b)^{-1}$

2. Write each as a power with a negative exponent.

   a) $\frac{1}{a^3}$ **b)** $\frac{2}{x^5}$

   c) $-\frac{1}{x^9}$ **d)** $\frac{2}{5b^6}$

3. Evaluate. Express as a fraction in simplest terms.

   a) 5^{-2}

   b) 3^{-4}

   c) 10^{-3}

   d) $(-4)^{-2}$

   e) -4^{-2}

   f) $(-4)^{-3}$

   g) $3^{-3} + 9^{-2}$

   h) $4^{-2} - 2^{-3} - 6^{-1}$

4. Apply the exponent rules to evaluate. Do not use decimals.

   a) $(5^{-2})^{-1}$

   b) $(2^{-3})^3$

   c) $\frac{9}{9^{-2}}$

   d) $6^{-4} \div 6^{-2}$

   e) $3^2 \times 3^{-4} \times 63 \times 6^0$

   f) $\frac{4^2 (4^6)}{4^5}$

5. Simplify. Express your answers using only positive exponents.

   a) $a^{-4} \times a^8$

   b) $(4v^{-5})(-3v^{-2})$

   c) $a^6 \div a^{-4}$

   d) $\frac{12m^{-5}}{4m^{-3}}$

6. Evaluate. Do not use decimals.

   a) $\left(\frac{1}{6}\right)^{-1}$

   b) $\left(\frac{4}{7}\right)^{-2}$

   c) $\left(\frac{3}{4}\right)^{-3}$

   d) $-\left(\frac{5}{3}\right)^{-4}$

7. Simplify.

   a) $\left(\frac{1}{xy}\right)^{-3}$

   b) $\left(\frac{1}{11b}\right)^{-2}$

   c) $\left(\frac{a^3}{b^2}\right)^{-4}$

   d) $\left(\frac{5m^2}{2n^4}\right)^{-3}$

B Connect and Apply

☆8. Polonium-210 is a radioactive isotope that has a half-life of 20 days. Suppose you start with a 40-mg sample.

   a) Create a table of values that gives the amount of polonium-210 remaining at the end of five intervals of 20 days each.

   b) Write an equation, in the form $f(x) = ab^x$, that relates the amount of polonium-210 remaining and time. Identify what each variable in the equation represents and give the appropriate unit for each variable.

   c) Sketch a graph of the relation. Describe the shape of the curve.

   d) How much polonium-210 will remain after 10 weeks?

   e) How long will it take for the amount of polonium-210 to decay to 8% of its initial mass? Describe the tools and strategies you used to solve this.

   f) Write two different functions to model the same situation. Explain why the two functions are equivalent.

9. Daniel is very excited about his new motorcycle! Although the motorcycle costs \$13 500, its resale value will depreciate (decline) by 20% of its current value every year. The equation relating the motorcycle's depreciated value, v, in dollars, to the time, t, in years, since the purchase is $v(t) = 13\ 500(0.8^t)$.

a) Explain the significance of each part of this equation.

b) How much will Daniel's motorcycle be worth in
i) 1 year?
ii) 6 years?

c) Explain why this is an example of exponential decay.

d) How long will it take for Daniel's motorcycle to depreciate to 50% of its original cost?

10. The amount, A, in dollars, of an investment can be determined using the formula $A = P(1 + i)^n$, where P is the principal, i is the annual interest rate (expressed as a decimal), and n is the number of compounding periods. What amount of money will grow to \$3500 in 4 years if it is invested in an account that pays interest of 6.5% per annum compounded annually?

☆11. The formula $P = A(1 + i)^{-n}$ allows you to calculate the principal (the initial amount invested), P, in an account that has been accumulating annually compounded interest, where A is the current amount in the account, i is the annual interest rate (expressed as a decimal), and n is the number of years that the principal has been earning interest.
Five years ago, Denise deposited an amount into an account that pays 7.5% per year, compounded annually. Today the account balance is \$4200.

a) What was the amount of Denise's initial deposit?

b) How much was in the account 2 years ago?

c) How much will be in the account 2 years from now? What is the total interest earned up to this point?

C Extend

12. In 1947, an investor bought Van Gogh's painting *Irises* for \$84 000. Forty years later, the same investor sold the artwork for \$49 million. If the initial investment had been deposited in a savings account paying interest compounded annually, what would be the annual interest rate for the amount to grow to \$49 million? Round your answer to one decimal place.

13. A cup of coffee contains approximately 100 mg of caffeine. When you drink the coffee, the caffeine is absorbed into the bloodstream and eventually metabolized. The half-life of caffeine in the bloodstream is approximately 5 h.

a) Write an equation that expresses the amount of caffeine, c, in milligrams, in the bloodstream t hours after drinking a cup of coffee.

b) How many hours does it take for the amount of caffeine in the bloodstream to be reduced to
i) 10 mg?
ii) 1 mg?

14. A pot of water on a stove is brought to a boil and then removed from the heating element. Every 5 min thereafter, the difference between the temperature of the water and the room temperature (20 °C) is reduced by 50%.

a) Write an equation to express the water temperature, T, in degrees Celsius, as a function of time, t, in minutes, since the pot of water was removed from the heating element.

b) How many minutes does it take for the temperature of the water to fall to 30 °C?

3.3 Rational Exponents

KEY CONCEPTS

- A power involving a rational exponent with numerator 1 and denominator n can be interpreted as the nth root of the base:
  For $b \in \mathbb{R}$ and $n \in \mathbb{N}$, $b^{\frac{1}{n}} = \sqrt[n]{b}$. If n is even, b must be greater than or equal to 0.
- You can evaluate a power involving a rational exponent with numerator m and denominator n by taking the nth root of the base raised to the exponent m:
  For $b \in \mathbb{R}$, $b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$ for $m \in \mathbb{Z}$, $n \in \mathbb{N}$. If n is even, b must be greater than or equal to 0.
- The exponent rules hold for powers involving rational exponents.

Example

Express each radical as a power with a rational exponent.

a) $\left(\sqrt[4]{16}\right)^3$

b) $\left(\sqrt[6]{729}\right)^3$

c) $\left(\sqrt[3]{-343}\right)^2$

Solution

First write each radical using a rational exponent. Then apply the power of a power law of exponents.

a) $\left(\sqrt[4]{16}\right)^3 = \left(16^{\frac{1}{4}}\right)^3$
$= 16^{\frac{3}{4}}$

b) $\left(\sqrt[6]{729}\right)^3 = \left(729^{\frac{1}{6}}\right)^3$
$= 729^{\frac{3}{6}}$
$= 729^{\frac{1}{2}}$

c) $\left(\sqrt[3]{-343}\right)^2 = \left((-343)^{\frac{1}{3}}\right)^2$
$= (-343)^{\frac{2}{3}}$

A Practise

1. Evaluate each cube root.

a) $\sqrt[3]{216}$

b) $(-1331)^{\frac{1}{3}}$

c) $\sqrt[3]{\frac{125}{343}}$

d) $\left(\frac{64}{729}\right)^{\frac{1}{3}}$

2. Evaluate each root.

a) $625^{\frac{1}{4}}$

b) $\sqrt[4]{\frac{1296}{81}}$

c) $729^{\frac{1}{6}}$

d) $\sqrt[5]{-1024}$

☆**3.** Express each power as a radical, and then evaluate.

a) $32^{\frac{3}{5}}$

b) $(-64)^{\frac{2}{3}}$

c) $64^{\frac{5}{6}}$

d) $6561^{\frac{5}{8}}$

☆**4.** Evaluate.

a) $1728^{-\frac{1}{3}}$

b) $36^{-\frac{3}{2}}$

c) $\left(-\frac{8}{125}\right)^{-\frac{5}{3}}$

d) $\left(\frac{1024}{243}\right)^{-\frac{3}{5}}$

☆**5.** Write as a single power, and then evaluate.

a) $8^{\frac{1}{3}} \times 8^{\frac{2}{3}}$

b) $16^{\frac{1}{4}} \div 16^{\frac{1}{2}} \times 16^{\frac{3}{4}}$

c) $64^{\frac{1}{3}} \times 64^{\frac{1}{6}} \div 64^{\frac{2}{3}}$

d) $3^{\frac{2}{3}} \times 27^{\frac{4}{9}}$

6. Simplify. Express your answers using only positive exponents.

a) $x^{\frac{1}{3}} \times x^{\frac{1}{3}}$

b) $\left(a^{\frac{1}{2}}\right)\left(a^{\frac{2}{3}}\right)$

c) $\dfrac{a^3 b^{\frac{1}{2}}}{a^{\frac{1}{3}} b^2}$

d) $\left(z^{\frac{2}{3}}\right)^{\frac{4}{5}}$

7. Simplify. Express your answers using only positive exponents.

a) $b^{\frac{4}{5}} \div b^{\frac{2}{3}}$

b) $\dfrac{a^{-\frac{3}{4}}}{a^{\frac{1}{3}}}$

c) $\left(w^{-\frac{6}{11}}\right)^{-\frac{2}{3}}$

d) $(16a)^{\frac{3}{4}}(265a)^{-\frac{1}{4}}$

8. The volume of a sphere can be expressed in terms of its surface area using the formula $V(S) = 3^{-1}(4\pi)^{-\frac{1}{2}} S^{\frac{3}{2}}$, where $V(S)$ is the volume of a sphere with surface area S. Determine the volume of an exercise ball whose surface area is 676 cm^2. Round your answer to one decimal place.

B Connect and Apply

9. a) Write a formula that expresses the circumference, C, of a circle in terms of its area, A.

b) Express your formula in part a) in an equivalent way.

c) Use your formula from part a) to determine the circumference of each circle with the given area. Round answers to one decimal place.

i) 314 cm^2

ii) 928 cm^2

iii) 1475 cm^2

d) Verify your answers in part c) using the equivalent formula from part b).

10. Yeast cells are microscopic plants that ferment sugars and starches. Yeast is used to make bread because the gases produced by the fermentation process make dough rise. The cells duplicate themselves about every half-hour, so the doubling period is 0.5 h. If the initial number of yeast cells is N_0, then the formula for the number of cells, N, after t hours is given by $N = N_0\left(2^{\frac{t}{0.5}}\right)$.

a) Determine the number of yeast cells after 1.5 hours if the initial number of cells is 50.

b) If the initial number of cells is 25, how long will it take for there to be 1000 cells?

c) Compare the formula $N = N_0(2^{2t})$ with the formula above. Are they equivalent? Explain.

d) Use the formula in part c) to verify your answer to part b).

11. Refer to question 10. Some cells double at rates different from that for yeast. In general, if an initial number of cells, N_0, has a doubling period d, then the number of cells, N, after time, t, is given by the formula $N = N_0\left(2^{\frac{t}{d}}\right)$. Note that t and d must be expressed in the same unit of time. Suppose that a bacterial strain doubles in number every 3 min and that there are 1000 bacteria initially.

a) Write the formula that corresponds with this situation.

b) How many bacteria will there be after 0.25 h?

c) When will there be 60 000 bacteria?

12. The formulas $h = 241m^{-\frac{1}{4}}$ and $r = \frac{107}{2}m^{-\frac{1}{4}}$ give the heartbeat frequency, h, in beats per minute, and respiratory frequency, r, in breaths per minute, for a resting animal with mass m, in kilograms.

a) Determine the heartbeat frequency for each animal.
i) elephant: 11 450 kg
ii) cow: 980 kg
iii) cat: 5.3 kg

b) Determine the respiratory frequency for each animal in part a).

c) Use the formula $B = \frac{1}{100}m^{\frac{2}{3}}$ to determine the brain mass, B, for each animal in part a).

d) Describe the relationships between the size of the animal and
i) the heartbeat frequency
ii) the respiratory frequency
iii) the brain mass

e) Determine the heartbeat frequency, respiratory frequency, and brain mass for your pet or an animal of your choice.

C Extend

13. Determine the inverse for each function. Then evaluate the inverse at the given x-value.

a) $f(x) = x^{\frac{3}{2}}, x = 64$

b) $f(x) = x^{\frac{4}{5}} - 2, x = 79$

c) $f(x) = \sqrt[3]{x^4} - 4, x = 1728$

14. Simplify. Express your answers using positive exponents.

a) $x^{-\frac{1}{2}} \times x^{\frac{4}{5}} \div x^{-\frac{2}{3}}$

b) $\dfrac{\sqrt[4]{x^3} \times \sqrt[3]{x^5}}{\sqrt[3]{x^6} \times \sqrt{x^7}}$

c) $\dfrac{\sqrt[4]{81m^8}}{\sqrt[3]{343m^{12}}} \div \dfrac{\sqrt[5]{32m^{10}}}{\sqrt{36m^6}}$

d) $\dfrac{\sqrt[3]{64b^{3-6x}}}{\sqrt[4]{1296b^{12x+24}}} \times \dfrac{\sqrt[5]{243b^{10-5x}}}{\sqrt{144b^{6x-4}}}$

e) $\dfrac{\left(a^{-\frac{x}{4}}\right)^{-\frac{1}{2}}\left(b^{-\frac{x}{3}}\right)^2}{\left(a^{-\frac{x}{2}}\right)\left(b^{\frac{2x}{3}}\right)^{-1}}$

15. In favourable breeding conditions, the population of a swarm of desert locusts can multiply 10-fold in 20 days.

a) Write an equation that represents the population, P, of a swarm of locusts with initial population P_0, at any time t, in days.

b) Use your answer from part a) to compare the population of a swarm of locusts after 30 days with its population after 20 days. How many times greater is the population after 30 days?

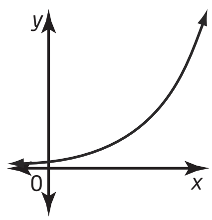
16. a) Is $(a + b)^{\frac{1}{2}} = a^{\frac{1}{2}} + b^{\frac{1}{2}}$, where a and b are positive integers? Justify your answer.

b) In general, is $(a + b)^{\frac{1}{n}} = a^{\frac{1}{n}} + b^{\frac{1}{n}}$, where a, b, and n are positive integers? Justify your answer.

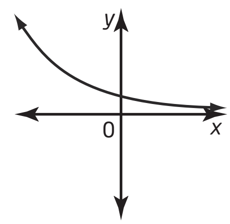
3.4 Properties of Exponential Functions

KEY CONCEPTS

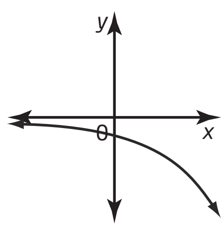
- The graph of an exponential function of the form $y = ab^x$ is
  - increasing if $a > 0$ and $b > 1$



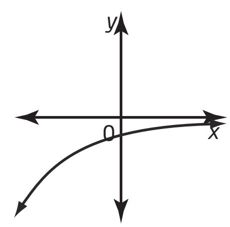
  - decreasing if $a > 0$ and $0 < b < 1$



  - decreasing if $a < 0$ and $b > 1$



  - increasing if $a < 0$ and $0 < b < 1$



- The graph of an exponential function of the form $y = ab^x$, where $a > 0$ and $b > 0$, has
  - domain $\{x \in \mathbb{R}\}$
  - range $\{y \in \mathbb{R}, y > 0\}$
  - a horizontal asymptote at $y = 0$
  - a y-intercept of a
- The graph of an exponential function of the form $y = ab^x$, where $a < 0$ and $b > 0$, has
  - domain $\{x \in \mathbb{R}\}$
  - range $\{y \in \mathbb{R}, y < 0\}$
  - a horizontal asymptote at $y = 0$
  - a y-intercept of a
- You can write an equation to model an exponential function if you are given enough information about its graph or properties.
- Sometimes it makes sense to restrict the domain of an exponential model based on the situation it represents.

Example

A radioactive substance decays exponentially. The percent, P, of the substance that remains after t years is represented by the function $P(t) = 100\left(\frac{6}{5}\right)^{-t}$.

a) Express the function using a positive exponent.

b) Which part of the equation from part a) indicates that the function represents exponential decay? Explain.

c) Graph the function. How does the graph support your answer to part b)?

d) State the domain and range of the function.

e) What does the horizontal asymptote indicate for this situation?

f) Determine the percent of the substance that remains after 12 years by using

i) the graph

ii) the equation

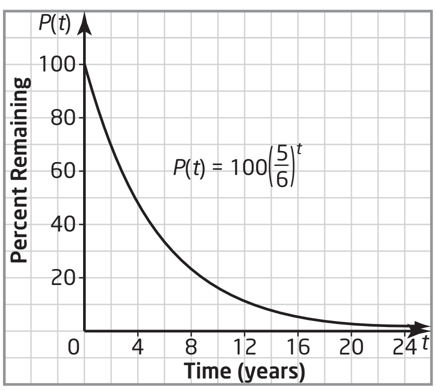
Solution

a) To express the function using a positive exponent, take the reciprocal of the fraction $\frac{6}{5}$.

$P(t) = 100\left(\frac{5}{6}\right)^t$

b) The common ratio $\frac{5}{6}$ is a proper fraction (value is between 0 and 1) that when multiplied by 100 will make the value smaller.

c)



The graph falls from left to right, which means that as time passes the percent of radioactive substance remaining is decreasing.

d) domain $\{t \in \mathbb{R}, t \geq 0\}$; range $\{P \in \mathbb{R}, 0 \leq P < 100\}$

e) The horizontal asymptote indicates that as time passes the percent remaining of the substance decreases, becoming closer to 0 but never actually falling to 0. This means that there is always some amount of the radioactive substance that remains.

f) **i)** From the graph, it can be estimated that at 12 years approximately 11% of the initial amount of the substance remains.

ii) Substitute $t = 12$ in the equation and solve for P.

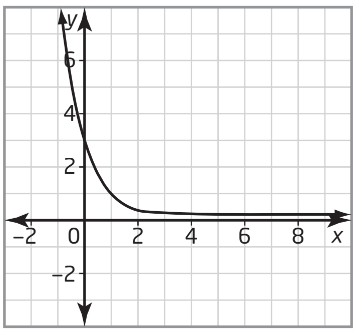
$$P(12) = 100\left(\frac{6}{5}\right)^{-12}$$
$$\doteq 11.2$$

Therefore, approximately 11.2% of the substance remains.

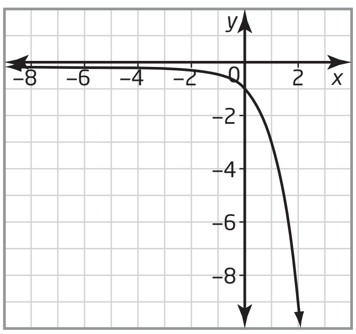
A Practise

1. Match each graph with its corresponding equation.

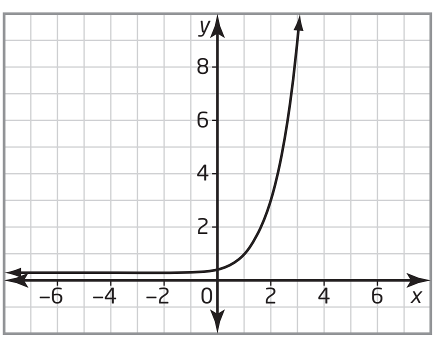
a)



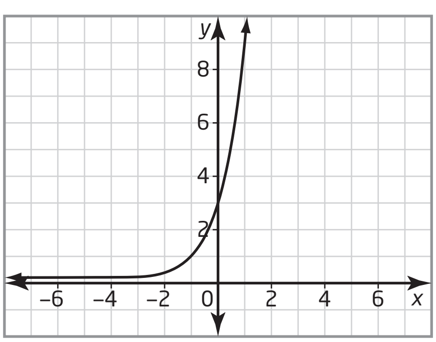
b)



c)



d)

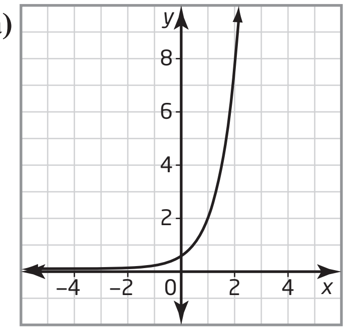


A $y = 3(3^x)$ **B** $y = 3\left(\frac{1}{3}\right)^x$

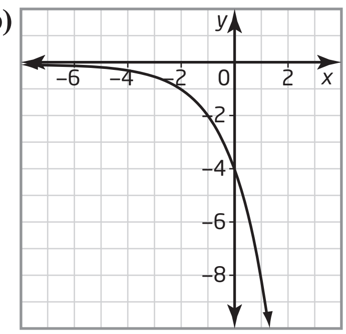
C $\frac{1}{3}(3^x)$ **D** $y = -3^x$

2. Match each equation with its corresponding graph. Justify your choice.

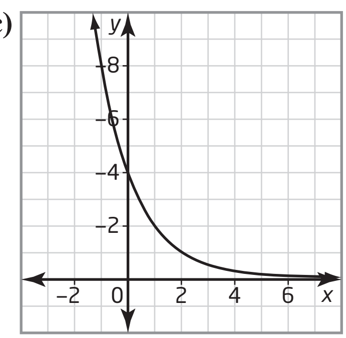
a)



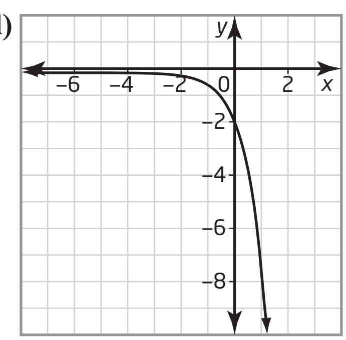
b)



c)



d)



A $y = -4(2^x)$ **B** $y = -2(4^x)$

C $y = \frac{1}{2}(4^x)$ **D** $y = 4\left(\frac{1}{2}\right)^x$

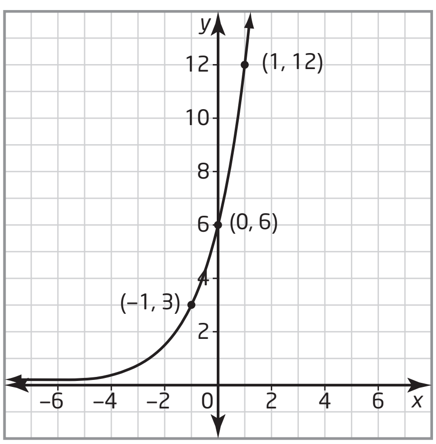
3. **a)** Sketch the graph of an exponential function that satisfies all of these conditions:
   - domain $\{\mathbb{R}\}$
   - range $\{y \in \mathbb{R}, y > 0\}$
   - y-intercept is 2
   - function is increasing

   b) Is this the only possible curve? Explain.

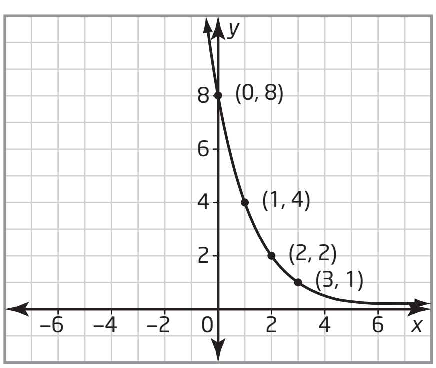
4. **a)** Sketch the graph of an exponential function that satisfies all of these conditions:
   - domain $\{\mathbb{R}\}$
   - range $\{y \in \mathbb{R}, y < 0\}$
   - y-intercept is −3
   - function is decreasing

   b) Is this the only possible curve? Explain.

5. Write an exponential equation to match the graph.



6. Write an exponential equation to match the graph.



7. A radioactive sample, with an initial mass of 32 mg, has a half-life of 4 days.

   a) Which of the following equations models this exponential decay?

   A $A = 32\left(2^{\frac{t}{4}}\right)$

   B $A = 32\left(\frac{1}{2}\right)^{4t}$

   C $A = 32\left(\frac{1}{2}\right)^{\frac{t}{4}}$

   D $A = 2\left(25^{\frac{t}{4}}\right)$

   b) What is the amount of radioactive material remaining after 1 week?

B Connect and Apply

☆8. Graph each function and identify the
   i) domain
   ii) range
   iii) x- and y-intercepts, if they exist
   iv) intervals of increase/decrease
   v) asymptote

   a) $f(x) = \left(\frac{1}{4}\right)^x$

   b) $y = 3 \times 2.5^x$

   c) $y = -\left(\frac{1}{5}\right)^x$

☆9. **a)** Graph each function, with or without graphing technology.

   i) $f(x) = 3^x$

   ii) $g(x) = \frac{3}{x}$

   b) In what way are the graphs

   i) alike?
   ii) different?

   c) Compare the asymptotes of these functions. What do you observe?

10. **a)** Graph each function, with or without graphing technology.

   i) $f(x) = \left(\frac{1}{3}\right)^x$

   ii) $g(x) = \frac{3}{x}$

   b) In what way are the graphs

   i) alike?
   ii) different?

   c) Compare the asymptotes of these functions. What do you observe?

11. a) Predict how the graphs of the following functions are related.

i) $f(x) = 4^{-x}$

ii) $g(x) = \left(\frac{1}{4}\right)^x$

b) Use Technology Graph both functions using graphing technology and check your prediction from part a).

c) Use algebraic reasoning to explain this relationship.

12. A radioactive substance decays exponentially. The percent, P, of the substance that remains after t years is represented by the function $P(t) = 100(2.3)^{-t}$.

a) Express the function using a positive exponent and no decimal.

b) Which part of the equation from part a) indicates that the function represents exponential decay? Explain.

c) Graph the function. How does the graph support your answer in part b)?

d) State the domain and range of the function.

e) What does the horizontal asymptote indicate for this situation?

f) Determine the percent of the substance that remains after 4 years by using

i) the graph

ii) the equation

13. The percent, P, of surface light present under water is expressed as a function of depth, d, in metres, by the equation $P = 100(0.975)^d$.

a) Graph the function.

b) At what depth is only 50% of the surface light present?

14. The formula $V(S) = \frac{(4\pi)^{-\frac{1}{2}}}{3} \times S^{\frac{3}{2}}$ relates the volume, V, and surface area, S, of a sphere.

a) Find the volume, to the nearest cubic metre, of a spherical tank with surface area 250 m^2.

b) Rewrite the above formula to express S in terms of V.

c) Use your formula from part b) to find the surface area, to the nearest square metre, of a spherical tank with volume 150 m^3.

C Extend

15. In a laboratory, 320 mg of iodine-131 is stored for 40 days. At the end of this period only 10 mg of the element remain.

a) Determine the half-life of iodine-131.

b) State two different equations to model this situation.

c) Graph the function. State the domain and range.

d) After how many days did 60 mg of iodine-131 remain? Explain how you found this answer.

16. Suppose a cone has a fixed height of 25 m.

a) Write an equation, using rational exponents where appropriate, to express the radius of the base of the cone in terms of its volume.

b) How should you limit the domain of this function so that the mathematical model fits the situation?

c) What impact does doubling the volume have on the radius of the base? Explain.

3.5 Transformations of Exponential Functions

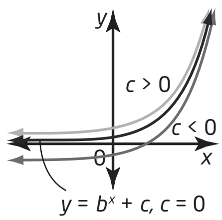
KEY CONCEPTS

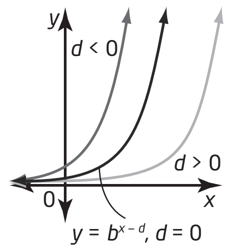
- Exponential functions can be transformed in the same way as other functions.
- The graph of $y = ab^{k(x-d)} + c$ can be found by performing the following transformations on the graph of the base $y = b^x$:

Horizontal and Vertical Translations

– If $d > 0$, translate right d units; if $d < 0$, translate left.

– If $c > 0$, translate up c units; if $c < 0$, translate down.



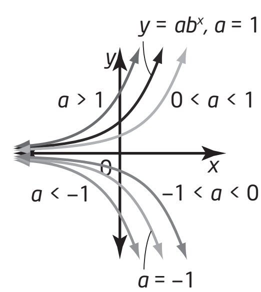


Vertical Stretches, Compressions, and Reflections

– If $a > 1$, stretch vertically by a factor of a.

– If $0 < a < 1$, compress vertically by a factor of a.

– If $a < 0$, reflect in the x-axis and stretch or compress.

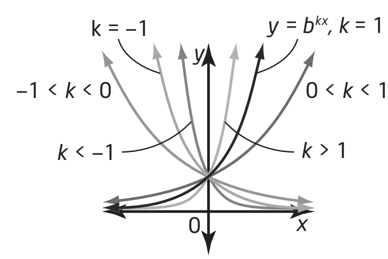


Horizontal Stretches, Compressions, and Reflections

– If $k > 1$, compress horizontally by a factor of $\frac{1}{k}$.

– If $0 < k < 1$, stretch horizontally by a factor of $\frac{1}{k}$.

– If $k < 0$, reflect in the y-axis and stretch or compress.



- Some exponential functions can easily be written using different bases. For example, $y = 2^{4x}$ is equivalent to $y = 16^x$.

Example

a) Write two equivalent equations for $y = -\left(\frac{1}{9}\right)^{3x} + 1$.

b) For each of the three equations, state the base function and the parameters. Then describe the corresponding transformations.

Solution

a) $y = -\left(\frac{1}{9}\right)^{3x} + 1$

Express $\frac{1}{9}$ as 9^{-1} or as 3^{-2}. Two equivalent equations are

i) $y = -(9^{-1})^{3x} + 1$
$= -9^{-3x} + 1$

ii) $y = -(3^{-2})^{2x} + 1$
$= -3^{-6x} + 1$

b) i) For the function $y = -\left(\frac{1}{9}\right)^{3x} + 1$, the base function is $y = \left(\frac{1}{9}\right)^{x}$ and the parameters are $a = -1$, $k = 3$, $d = 0$, $c = 1$.

- Since $a = -1$, the graph of $y = \left(\frac{1}{9}\right)^{x}$ is reflected in the x-axis.
- Since $k = 3$, the graph is compressed horizontally by a factor of $\frac{1}{3}$.
- Since $c = 1$, the graph is then translated up 1 unit.

ii) For the function $y = -(9^{-1})^{3x} + 1$, the base function is $y = 9^x$ and the parameters are $a = -1$, $k = -3$, $d = 0$, $c = 1$.

- Since $a = -1$, the graph of $y = 9^x$ is reflected in the x-axis.
- Since $k = -3$, the graph of $y = 9^x$ is reflected in the y-axis and compressed horizontally by a factor of $\frac{1}{3}$.
- Since $c = 1$, the graph is then shifted up 1 unit.

iii) For the function $y = -(3^{-2})^{2x} + 1$, the base function is $y = 3^x$ and the parameters are $a = 1$, $k = -6$, $d = 0$, $c = 1$.

- Since $a = -1$, the graph of $y = 3^x$ is reflected in the x-axis.
- Since $k = -6$, the graph of $y = 3^x$ is reflected in the y-axis, compressed horizontally by a factor of $\frac{1}{6}$.
- Since $c = 1$, the graph is then translated up 1 unit.

A Practise

1. Complete the second column of the table by describing the transformation associated with the parameter described in the first column.

The Roles of the Parameters *a*, *k*, *d*, and *c* in Exponential Functions of the Form $y = ab^{k(x-d)} + c$ $(b > 0, b \neq 1)$	
Role of *c*	**Transformation on Graph of $y = b^x$**
$c > 0$	
$c < 0$	
Role of *d*	
$d > 0$	
$d < 0$	
Role of *a*	
$a > 1$	
$0 < a < 1$	
$-1 < a < 0$	
$a < -1$	
Role of *k*	
$k > 1$	
$0 < k < 1$	
$-1 < k < 0$	
$k < -1$	
Domain and Range of $y = ab^{k(x-d)} + c$	
The domain is always _______.	i) When the graph is below its horizontal asymptote the range is _______. ii) When the graph is above its horizontal asymptote the range is _______.

2. Describe the transformation that maps the function $y = 5^x$ onto each function.

a) $y = 5^x + 3$
b) $y = 5^{x-2}$
c) $y = 5^{x+1}$
d) $y = 5^{x-4} - 6$

3. Sketch a graph of each function in question 2. Use the graph of $y = 5^x$ as the base.

4. Describe one or more transformations that map the function $y = 7^x$ onto each function.

a) $y = \left(\frac{1}{3}\right)7^x$
b) $y = 7^{2x}$
c) $y = -7^x$
d) $y = 7^{-\frac{1}{3}x}$

5. Sketch a graph of each function in question 4. Use the graph of $y = 7^x$ as the base. Be sure to choose an appropriate scale for your axes.

6. Write an equation for the function that results from each transformation applied to the base $y = 11^x$.

a) reflection in the *y*-axis
b) stretch vertically by a factor of 4
c) stretch horizontally by a factor of 3
d) reflect in the *x*-axis and compress horizontally by a factor of $\frac{1}{5}$

B Connect and Apply

☆**7.** The graph of $y = 4^x$ is transformed to obtain the graph of $y = -3[4^{2(x+1)}] + 5$.

a) State the parameters. Describe the corresponding transformations and use these to complete the table. Then use the points to graph $y = -3[4^{2(x+1)}] + 5$.

$y = 4^x$	$y = 4^{2x}$	$y = -3[4^{2x}]$	$y = -3[4^{2(x+1)}] + 5$
(−1, 0.25)			
(0, 1)			
(1, 4)			
(2, 16)			
(3, 64)			

b) State the domain, range, and equation of the horizontal asymptote for this function.

8. Sketch a graph of $y = \left(-\frac{1}{5}\right)5^{x+4} - 2$ by using $y = 5^x$ as the base and applying transformations.

☆9. a) Describe the transformations that must be applied to the graph of $f(x) = 3^x$ to obtain the transformed function $y = -f(4x) - 7$. Then write the corresponding equation.
b) State the domain, range, and equation of the horizontal asymptote.

10. a) Write the equation of the function that represents $f(x) = 2^x$ after it is reflected in the x-axis, stretched horizontally by a factor of 5, reflected in the y-axis, and then translated down 3 units and right 1 unit.
b) State the domain and range, and the equation of its horizontal asymptote.

11. The temperature, in degrees Celsius, of a cooling metal bar is given by the function $T = 18 + 100(0.5)^{0.3t}$, where t is the time, in minutes.
a) Sketch a graph of this relation.
b) What is the asymptote of this function? What does it represent?
c) How long will it take for the temperature to be within 0.1 °C of the value of the asymptote?

12. a) Rewrite the function $y = \left(\frac{1}{16}\right)^x$ in three different ways, using a different base in each case.
b) State the base function for each equation and describe how the base function is transformed.

13. a) Write an equation for a function whose asymptote is $y = -5$, with a y-intercept of 3.
b) Is the function you produced in part a) the only possible answer? Use transformations to help explain your answer.

C Extend

14. A 250-g sample of one type of radioactive substance has a half-life of 138 days. A 175-g sample of another type of radioactive substance has a half-life of 16 days.
a) For each substance, write an equation that represents the amount A, in grams, of the radioactive sample that remains after t days.
b) What is the base function for each equation?
c) Describe the transformations that must be applied to the base function to obtain each equation.
d) Use a negative exponent to write an equivalent equation for each of the equations in part a).
i) What is the base function?
ii) Which transformations that must be applied to this base function are the same as those in part c)?
iii) Which new transformation is required?
e) For each substance, determine the mass that remains after 20 weeks.
f) How long does it take for each sample to decay to 10% of its original amount?

15. Refer to question 14. Suppose there is an initial amount of A_0 grams of a radioactive substance that has a half-life of h days.
a) Write a general equation to represent the amount A, in grams, that remains after t days.
b) Describe the role of A_0 and h in the equation in terms of transformations.
c) Rewrite the equation in part a) so that it includes a reflection in the y-axis.

3.6 Making Connections: Tools and Strategies for Applying Exponential Models

KEY CONCEPTS

- You can use a variety of tools to construct algebraic and graphical models, including
  - a graphing calculator
  - dynamic statistics software such as *Fathom*™
  - a spreadsheet
- Various types of regression (e.g., linear, quadratic, exponential) can be used to model a relationship. The best choice will effectively describe the trend between and beyond the known data values.
- Exponential functions are useful in modelling situations involving continuous growth or decay/depreciation.

Example

In 1940, the population of the town of Mathville was 3000. Since then, the population has doubled every 6 years.

a) Copy and complete the table.

Year	6-Year Interval	Population
1940	0	
1946	1	
1952	2	
1958	3	
1964	4	
1970	5	
1976	6	
1982	7	
1988	8	
1994	9	
2000	10	

b) Graph the data.

c) Use your graph to estimate the population in 1991.

d) Use your graph to estimate when the population will reach 5 million.

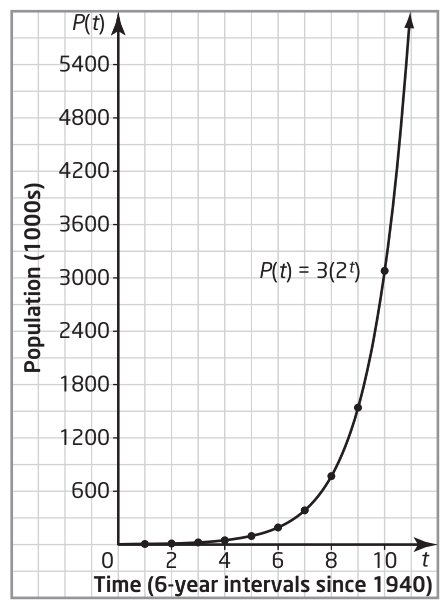
e) Write an equation to model this situation, where P represents the population of Mathville, in thousands, and t represents the time, in 6-year intervals, since 1940.

Solution

a)

Year	6-Year Interval	Population
1940	0	3 000
1946	1	6 000
1952	2	12 000
1958	3	24 000
1964	4	48 000
1970	5	96 000
1976	6	192 000
1982	7	384 000
1988	8	768 000
1994	9	1 536 000
2000	10	3 072 000

b) Graph the data using an appropriate scale for each axis. In this solution, each square represents one 6-year interval on the horizontal axis and a population of 300 000 on the vertical axis.



c) 1991 is in the middle of the 6-year interval from 1988 to 1994. This point is at 8.5 along the horizontal axis. For this point, the corresponding value on the vertical axis is about 1 100 000, which is the estimated population in 1991.

d) To determine when the population will reach 5 million, find that value on the vertical axis and then find its corresponding value on the horizontal axis.

The horizontal value is approximately 10 and $\frac{2}{3}$ intervals.

This corresponds to $(10 + 6) + \left(\frac{2}{3} \times 6\right)$,

or 64 years since 1940. Therefore, the population of Mathville reached 5 000 000 in 2004.

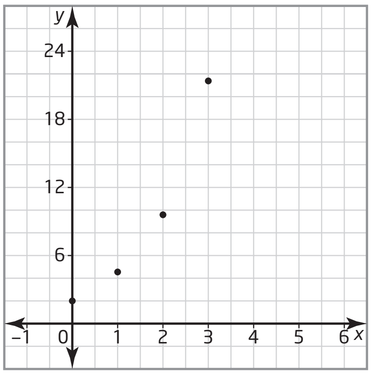
e) Since the population is doubling, the equation will be of the form $P = P_0(2^t)$, where P_0 is the initial population, in thousands, and 2 represents the doubling.

The initial population is $P_0 = 3$ (in thousands), so the equation that models this situation is $P(t) = 3(2^t)$.

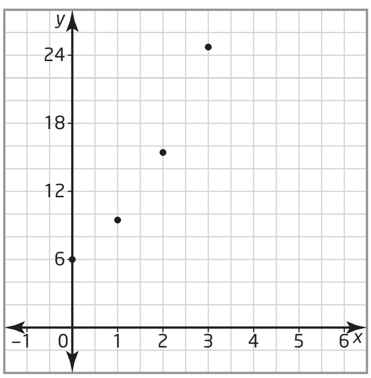
A Practise

1. For each exponential scatter plot, select the corresponding equation of its curve of best fit.

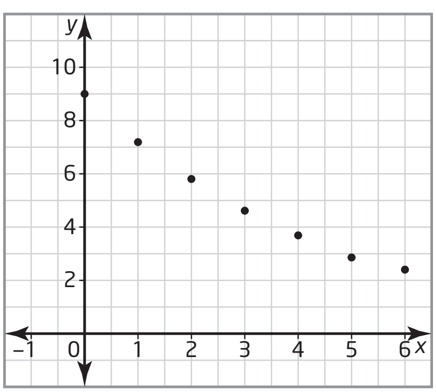
Graph A



Graph B



Graph C

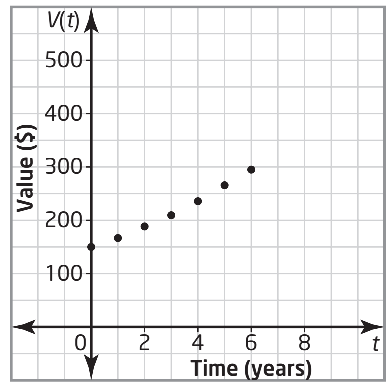


Equations

$y = 20 \times 0.85^x$
$y = 6 \times 1.6^x$
$y = 2 \times 0.7^x$
$y = 2 \times 2.2^x$
$y = 9 \times 0.8^x$
$y = 9 \times 1.2^x$

2. Pick one of the unmatched equations from question 1. Sketch a scatter plot that the equation could fit.

3. Toni has invested some money in a mutual fund. The scatter plot shows the value of her investment after the first few years.



a) Do the data appear to have an exponential trend? Explain your reasoning.

b) Estimate values of a and b to develop an exponential model for the data of the form $V(n) = a \times b^n$. Explain how you arrived at your estimated values.

c) Use the tool of your choice to find an exponential model of these data.

d) Use the exponential model you produced in part c) to predict the value of Toni's investment after 12 years.

e) Approximately how long it will take for Toni's investment to grow to $1000?

4. A fully charged cell phone battery loses 2% of its charge every day.

a) Determine an equation that models the percent charge, C, that remains in the battery after t days.

b) Determine the percent charge remaining in the battery after
i) 25 days
ii) 50 days
iii) 75 days
iv) 100 days

c) Determine the half-life of the battery.

B Connect and Apply

☆**5.** *Best Electronics* has had constant growth since 1995. Its profits are modelled by the equation $P(t) = 7.4(1.59)^t$, where P is the yearly profit, in thousands of dollars, and t is the number of years since 1995.

a) Make a table of values to show annual profits for year 0 (1995) and each of the following 10 years. Approximate values to one decimal place.

b) Graph the data in the table. Does this represent exponential decay or exponential growth? Explain.

c) What was the company's profit in 1995?

d) Predict the company's yearly profit in 2015.

e) According to the model, when does the yearly profit reach $500 000 000?

6. Air pressure decreases as altitude increases. The following table gives the air pressure, $p(a)$, measured in kilopascals (kPa), at an altitude of a km above sea level.

Altitude (km)	Pressure (kPa)
0	100.0
4	70.0
8	49.0
12	34.3
16	24.0
20	16.8
24	11.8

a) Construct a scatter plot and the curve of best fit for the data.

b) Determine the equation of the exponential function that best represents the data.

c) Determine the air pressure at the summit of each mountain.
i) Mount Logan, altitude 6050 m
ii) Mount Everest, altitude 8848 m

d) What altitude corresponds to an air pressure of 20 kPa?

7. Coffee, tea, cola, and chocolate contain caffeine. When you consume caffeine, the percent, P, of caffeine remaining in your body over time is represented by the function $P = 100(0.87)^t$, where t is the elapsed time, in hours.

a) Make a table of values for the percent of caffeine remaining for a 24-hour period, in 2-h intervals. Approximate values to one decimal place.

b) Does the function represent exponential growth or exponential decay? Justify your answer graphically.

c) Determine the percent of caffeine remaining in your body after
i) 1 h **ii)** 9 h **iii)** 15 h

d) How long will it take for the percent of caffeine to drop by 50%?

C Extend

8. In a steel mill, red-hot slabs of steel are pressed many times between heavy rollers. The width of the slab remains the same on every pass; however, the length increases by 20% and the thickness decreases by 17%. Consider a slab that is p metres long and q metres thick.

a) Write an equation to represent the length, l, in metres, of the slab after n passes.

b) Write an equation to represent the thickness, t, in metres, of the slab after n passes.

c) Use your results from parts a) and b) to write equations for the length and thickness of a slab that is 2.00 m long and 0.50 m thick.

d) How many passes are needed until the length is at least 20 m? How thick is the slab at this point?

e) How many passes are needed until the thickness is about 1 mm? How long is the slab at this point?

Chapter 3 Review

3.1 The Nature of Exponential Growth

1. A bacteria colony that has an initial population of 85 triples every hour.

   a) Which function models this exponential growth?

   A $p(n) = 85 \times 3^n$

   B $p(n) = 85 \times 2^n$

   C $p(n) = 85 \times 3^n$

   b) For the correct model, explain what each part of the equation means.

2. Evaluate.

   a) 11^0

   b) $(-4)^0$

   c) $\left(\frac{2x}{7}\right)^0$

   d) -2^0

3. a) Identify each function as linear, quadratic, or neither. Justify your choice.

   i) $f(x) = 7^x$

   ii) $f(x) = 3x + \sqrt{6}$

   iii) $f(x) = 1 - x^2$

   b) Without calculating the finite differences, describe the relationship between the finite differences and each type of function.

3.2 Exponential Decay: Connecting to Negative Exponents

4. Write each as a power with a positive exponent.

   a) x^{-3}

   b) $3b^{-2}$

5. Write each as a power with a negative exponent.

   a) $\frac{1}{w^4}$

   b) $\frac{-3}{b^8}$

6. Evaluate.

   a) 6^{-3}

   b) $5^{-2} + 5^{-3}$

   c) $(12^{-4})(12^3)$

   d) $\frac{(8^3)(8^{-4})}{8^{-2}}$

   e) $\left(\frac{6}{5}\right)^{-3}$

   f) $\left(-\frac{8}{7}\right)^{-2}$

7. Simplify. Express your answers using only positive exponents.

   a) $(3b^{-5})^{-2}$

   b) $(-2a^2b^{-2})^{-3}$

   c) $(3x^4)^9 \div (3^5x^6)^3$

   d) $\left(\frac{2c^4}{3d^2}\right)^{-5}$

   e) $\left(\frac{6^3\,a^5}{5^5\,b^9}\right)^7 \times \left(\frac{5^{11}\,b^2}{6^6\,a^3}\right)^3$

3.3 Rational Exponents

8. Evaluate.

   a) $\left(-\frac{27}{64}\right)^{-\frac{2}{3}}$

   b) $\left(\frac{16}{81}\right)^{-\frac{7}{4}}$

   c) $27^{\frac{1}{3}} + 16^{\frac{3}{4}} - 32^{\frac{4}{5}}$

   d) $\sqrt[5]{243^2}$

9. Express each radical as a power with a rational exponent.

   a) $\left(\sqrt[5]{-3125}\right)^4$

   b) $\left(\sqrt[5]{32}\right)^3$

10. Express each power as a radical. Then evaluate.

    a) $(-32)^{\frac{4}{5}}$

    b) $343^{-\frac{2}{3}}$

    c) $(-125)^{\frac{2}{3}}$

11. Express as a single power and then evaluate.

a) $16^{\frac{5}{4}} \times 16^{\frac{1}{2}} \div 16^{\frac{9}{4}}$

b) $81^{\frac{1}{2}} \div 81^{\frac{3}{4}} \times 81^{\frac{7}{4}}$

c) $256^{\frac{1}{4}} \times 256^{\frac{3}{8}} \div 256^{\frac{1}{2}}$

12. Simplify. Express your answers using only positive exponents.

a) $\frac{s^{\frac{3}{4}}}{s^{\frac{1}{3}}}$

b) $\left(m^{\frac{3}{7}}n^{\frac{4}{5}}\right)^{\frac{2}{3}}$

c) $\left(k^{\frac{3}{7}}\right)^{-\frac{1}{2}}$

d) $6v^{\frac{1}{2}}\left(32v^{-\frac{1}{3}}\right)^{-\frac{6}{5}}$

13. The area, A, of an equilateral triangle with side length s is given by $A = \frac{\sqrt{3}}{4s^{-2}}$.

a) Rearrange the formula to express the side length s in terms of the area A.

b) Use your answer in part a) to determine the length of the sides when the area is 12.6 m^2.

3.4 Properties of Exponential Functions

14. Graph each function and identify the

i) domain

ii) range

iii) x- and y-intercepts, if they exist

iv) intervals of increase/decrease

v) asymptote

a) $f(x) = \left(\frac{1}{6}\right)^x$

b) $y = 4 \times 3.5^x$

c) $y = -\left(\frac{1}{4}\right)^x$

15. A radioactive sample, with an initial mass of 28 mg, has a half-life of 5 days.

a) Write an equation to represent the amount, A, of radioactive sample that remains after t days.

b) Explain why this situation represents exponential decay.

c) Without graphing, describe the shape of the graph of this function. State the domain and range.

d) What is the amount of radioactive sample remaining after 2 weeks?

3.5 Transformations of Exponential Functions

16. Sketch a graph of $y = 4^{-1.5x + 3}$, by using $y = 4^x$ as the base and applying transformations.

17. a) Describe the transformations that must be applied to the graph of $f(x) = 0.5^x$ to obtain the transformed function $y = 2f\left[\frac{1}{3}(x - 5)\right]$. Then write the corresponding equation.

b) State the domain, range, and equation of the horizontal asymptote.

18. a) Write the equation of the function that represents $f(x) = \left(\frac{1}{4}\right)^x$ after it is compressed horizontally by a factor of $\frac{1}{2}$, compressed vertically by a factor of $\frac{1}{3}$, reflected in the x-axis, and shifted 4 units to the left and 6 units up.

b) State the domain, range, and equation of the horizontal asymptote.

3.6 Making Connections: Tools and Strategies for Applying Exponential Models

19. The population of Canada in 1981 was approximately 24 million. The population since then has increased approximately 1.4% per year.

a) Make a table of values by determining the population every 2 years for 20 years, since 1981.

b) Determine an equation that models the data in part a).

c) What was the population of Canada in 2000? Round your answer to the nearest hundred thousand.

d) Predict the population in 2012, to the nearest hundred thousand.

e) According to the model, when will Canada's population reach 40 million?

Chapter 3 Math Contest

1. Given that $\frac{a^9c^3}{b} = 8$ and $\frac{b^7c^3}{a^3} = 18$, one possible value of $(abc)^3$ is

   A 4 **B** 6

   C 144 **D** 12

2. If $f(x) = 4^{3-2x}$, then the value of $[f(2+x)][f(2-x)]$ is

   A 16 **B** $\frac{1}{16}$

   C 4 **D** $\frac{1}{4}$

3. The y-intercept of the function $y = 8\left(\frac{1}{4}\right)^{3x+2} - 1$ is

   A −1 **B** 8

   C −1 **D** $-\frac{1}{2}$

4. Consider the following system of equations.

   $x_1 + x_2 = 6$
   $x_2 + x_3 = 7$
   $x_1 + x_3 = 8$

   What is the value of $x_1 + x_2 + x_3$?

   A 21 **B** 7.5

   C 7 **D** 10.5

5. A new operation $a \otimes b$ is defined as $a \otimes b = (b-1)^{a+2}$. The value of $[(-3) \otimes 4] \otimes 9$ is

   A 128 **B** $\frac{1}{128}$

   C 512 **D** $\frac{1}{512}$

6. Without using a calculator, determine the value of $\frac{36^{\frac{4}{7}}}{\sqrt[7]{6}}$.

7. Find all the solutions to $3^x - 2x - 1 = 0$.

8. A number is between 25 and 35. When this number is added to its cube, the result is 29 822. When the same number is subtracted from its cube the result is

   A 29 672 **B** 29 758

   C 29 772 **D** 29 760

9. Consider the function $y = 18\left(\frac{1}{3}\right)^x - 2$. If the x-intercept is a and the y-intercept is b, determine the value of $6a - b$.

   A 4 **B** 12

   C −4 **D** 8

10. If $f(x) = (-2)^x$ and $f(x+2) - f(x+3) + f(x+5) = kf(x)$, what is the value of k?

11. If $3^y = 5$, then the value of $3^{4y} - 11(3^{2y})$ is

    A 350 **B** $\sqrt[4]{5} - 11\sqrt{5}$

    C $-10\sqrt[4]{5}$ **D** 550

12. Consider three circles such that the radius of the first circle is r, the radius of the second circle is $4r$, and the radius of the third circle is $8r$. What is the ratio of the area of the circles, in order from smallest to largest?

    A 1:4:8 **B** 1:2:4

    C 1:2:2 **D** 1:16:64

13. What is the range of the function $y = -5\left(\frac{1}{3}\right)^x + 2$?