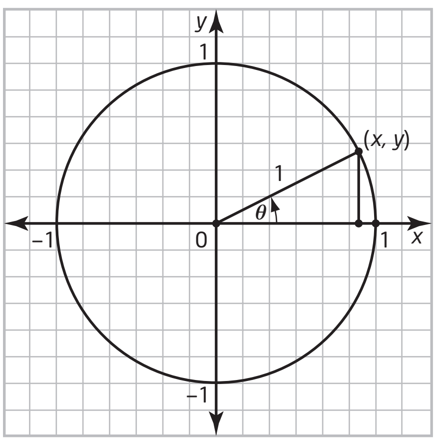
Chapter 4 Trigonometry

4.1 Special Angles

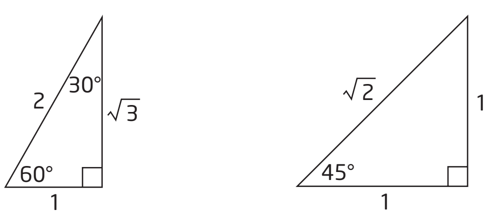
KEY CONCEPTS

- Using a unit circle is one way to find the trigonometric ratios for angles greater than 90°.
- Any point on a unit circle can be joined to the origin to form the terminal arm of an angle. The angle θ is measured starting from the initial arm along the positive x-axis, proceeding counterclockwise to the terminal arm.



The coordinates of the point (x, y) on a unit circle are related to θ such that $x = \cos\theta$ and $y = \sin\theta$.

- $\tan\theta = \frac{y}{x}$
- Exact trigonometric ratios for special angles can be determined using special triangles.



- The exact trigonometric ratios for 45° are $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, and $\tan 45° = 1$.

Example

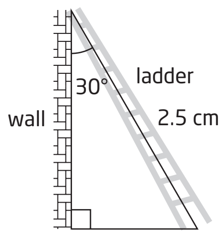
A ladder that is 2.5 m long is placed against a vertical wall so that the top of the ladder makes an angle of 30° with the wall.

a) Draw a diagram to represent this situation.

b) How far up the wall is the top of the ladder? Give an exact answer.

Solution

a)



b) Let h represent the height, in metres, from the top of the ladder to the ground. We know the length of the hypotenuse and we want to find the length of the side adjacent to 30°. Use the cosine ratio.

$\cos 30° = \frac{h}{2.5}$

Substitute $\cos 30° = \frac{\sqrt{3}}{2}$.

$\frac{\sqrt{3}}{2} = \frac{h}{2.5}$

Solve for h.

$h = \frac{2.5\sqrt{3}}{2}$

$= 1.25\sqrt{3}$

Substitute $1.25 = \frac{5}{4}$.

$h = \frac{5\sqrt{3}}{4}$

The top of the ladder is $\frac{5\sqrt{3}}{4}$ m up the wall.

A Practise

1. Complete the table. Use exact values only.

θ	sin θ	cos θ	tan θ
0°			
30°			
45°			
60°			
90°			
180°			
270°			
360°			

2. **Use Technology** Use a calculator to complete the table. Express answers to four decimal places.

θ	sin θ	cos θ	tan θ
0°			
30°			
45°			
60°			
90°			
180°			
270°			
360°			

3. **a)** What reference angle on the unit circle should you use to find the trigonometric ratios for 225°?

   b) State two other angles that have the same reference angle on the unit circle.

   c) Use a unit circle to find exact values for the three trigonometric ratios for 225°.

   d) State the exact values for the three trigonometric ratios for the angles in part b).

4. **a)** What reference angle on the unit circle would you use to find the trigonometric ratios for 150°?

   b) State two other angles that have the same reference angle on the unit circle.

   c) Use a unit circle to find exact values for the three trigonometric ratios for 150°.

   d) State the exact values for the three trigonometric ratios for the angles in part b).

5. **a)** What reference angle on the unit circle would you use to find the trigonometric ratios for 300°?

   b) State two other angles that have the same reference angle on the unit circle.

   c) Use a unit circle to find exact values for the three trigonometric ratios for 300°.

   d) State the exact values for the three trigonometric ratios for the angles in part b).

6. Use a unit circle to find the primary trigonometric ratios for 70°. Measure any side lengths needed. Compare your answers to those generated by a calculator.

7. Use a unit circle to find the primary trigonometric ratios for 220°. Measure any side lengths needed. Compare your answers to those generated by a calculator.

B Connect and Apply

☆**8.** **a)** Describe the CAST rule.

   b) Use the CAST rule to identify the two quadrants where each trigonometric ratio is positive and the two quadrants where each ratio is negative. Copy and complete the table to organize your results.

Trigonometric Ratio	Positive Quadrants	Negative Quadrants
Sine		
Cosine		
Tangent		

9. A ladder that is 3 m long is placed against a vertical wall so that the top of the ladder makes an angle of 60° with the wall.

   a) Draw a diagram to represent this situation.

   b) Find an exact expression for the distance between the bottom of the ladder and the wall.

   c) How far up the wall is the top of the ladder?

10. A sports car is 14 km south of an intersection. A van is 14 km west of the same intersection.

   a) Use trigonometry to find an exact expression for the distance between the two vehicles.

   b) Describe an alternate method that can be used to solve this problem. Check your answer using that method.

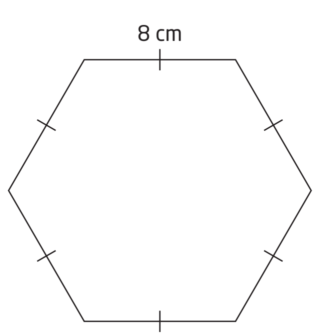
☆11. A hydro pole is stabilized at its top by two guy wires of equal length, each of which makes an angle of 60° with the ground. The wires are secured to the ground at points that are 10 m apart and on opposite sides of the pole.

   a) Draw a diagram to represent this situation.

   b) How tall is the hydro pole? Express your answer using exact values.

   c) What is the length of each wire? Express your answer using exact values.

12. A floor tile in the shape of a regular hexagon has side lengths of 8 cm. Determine the area of the tile.



13. Determine the exact value of each expression.

   a) $\cos 45° \times \sin 225° + \cos 210°$

   b) $\tan 330° \times \cos 240° - 2 \cos 270°$

   c) $\tan 60° \times 3 \sin 90° - \sin 315°$

14. Prove that $(\sin 30°)^2 + (\cos 30°)^2 = (\sin 315°)^2 + (\cos 315°)^2$.

C Extend

15. Determine all the possible measures of θ, where $0° \leq \theta \leq 360°$, that satisfy each equation.

   a) $\sin \theta = \frac{\sqrt{3}}{2}$

   b) $(\cos \theta)^2 = \frac{1}{2}$

   c) $\sqrt{3} \tan \theta + 1 = 0$

16. The angle of elevation from point A on the ground to the top of a water tower is 30°. From point B, which is 10 m closer to the tower than point A, the angle of elevation is 45°. Determine the height of the water tower.

17. Each trigonometric ratio has a reciprocal ratio. The reciprocal of the tangent ratio is the cotangent ratio.

   $\cot \theta = \frac{1}{\tan \theta}$

   a) Show that the formula for the area, A, of a regular polygon with n sides in terms of its side length, s, is $A = \frac{ns^2}{4} \cot\left(\frac{180°}{n}\right)$.

   b) Use the trigonometric ratios of special triangles and the formula in part a) to derive a formula for the area of each regular polygon.

   i) square

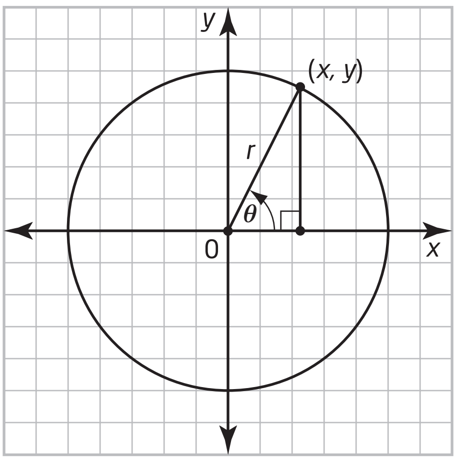
   ii) hexagon

   iii) equilateral triangle

4.2 Co-terminal and Related Angles

KEY CONCEPTS

- The primary trigonometric ratios for the angle θ in standard position that has a point (x, y) on its terminal arm can be calculated as $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, and $\tan \theta = \frac{y}{x}$, where $r = \sqrt{x^2 + y^2}$.



- Exactly two angles between 0° and 360° have the same sine ratio.
- Exactly two angles between 0° and 360° have the same cosine ratio.
- Exactly two angles between 0° and 360° have the same tangent ratio.
- Pairs of related angles can be found using the coordinates of the endpoints of their terminal arms. Use a reference angle in the first quadrant.
- Co-terminal angles are angles with the same terminal arm. They can be positive or negative.

Example

Solve the equation $\sin \theta = -\frac{15}{17}$ for $0° \leq \theta \leq 360°$.

Solution

Note that the sign of $\sin \theta = -\frac{15}{17}$ is negative. From the CAST rule we know that the sine function is negative in the third and fourth quadrants, so there are two angles that satisfy this equation.

Use a calculator to determine the smallest positive angle such that $\sin \theta = \frac{15}{17}$.

$\sin^{-1}\left(\frac{15}{17}\right) = 62°$

In the third quadrant, the angle is $180° + 62° = 242°$.

In the fourth quadrant, the angle is $360° - 62° = 298°$.

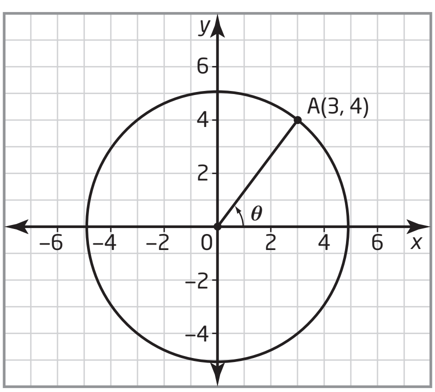
Therefore, the two solutions to $\sin \theta = -\frac{15}{17}$ are $\theta = 242°$ and $\theta = 298°$.

A Practise

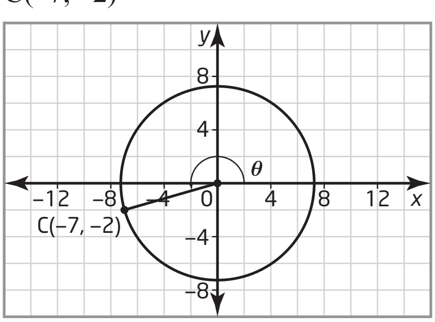
Unless specified otherwise, all angles are between 0° and 360°.

1. The coordinates of a point on the terminal arm of an angle θ are shown. Determine the exact trigonometric ratios of θ.

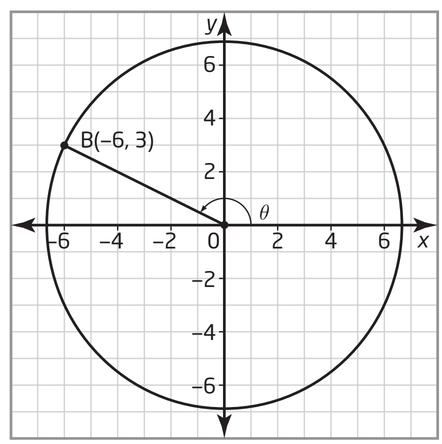
a) A(3, 4)



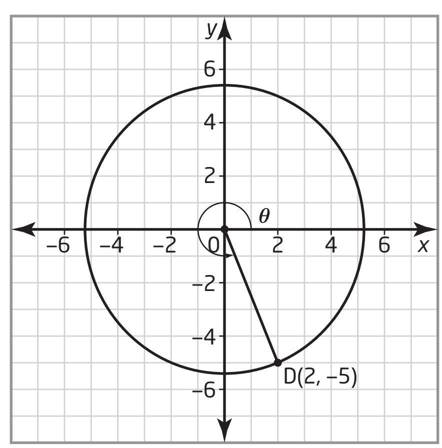
b) C(−7, −2)



c) B(−6, 3)



d) D(2, −5)



2. The coordinates of a point on the terminal arm of an angle θ are given. Determine the exact trigonometric ratios of θ.

a) G(−3, 5)

b) H(−15, 8)

c) I(−3, −4)

d) J(−5, 12)

e) K(7, 3)

f) L(1, −9)

3. One of the primary trigonometric ratios of an angle is given, as well as the quadrant in which the terminal arm lies. Find the other two primary trigonometric ratios.

a) $\cos A = -\frac{8}{17}$, second quadrant

b) $\sin B = -\frac{4}{5}$, third quadrant

c) $\tan C = -\frac{12}{5}$, fourth quadrant

d) $\sin D = \frac{7}{\sqrt{85}}$, first quadrant

e) $\cos E = -\frac{3}{13}$, second quadrant

f) $\tan F = -\frac{8}{13}$, fourth quadrant

4. Determine two other angles that have the same trigonometric ratio as each given angle. Draw a sketch with all three angles labelled.

a) sin 60°

b) cos 210°

c) tan 315°

d) sin 140°

e) cos 285°

f) tan 190°

☆5. **a)** Determine any three positive angles that are co-terminal with 205°.

b) Determine any three negative angles that are co-terminal with 310°.

6. Which pairs of angles are co-terminal? Justify your answer.

a) 40° and 280°

b) 80° and 440°

c) 110° and 830°

d) 170° and 510°

e) 50° and −310°

f) 200° and −520°

g) 100° and −200°

h) 320° and 680°

7. Determine the exact primary trigonometric ratios of each angle. You may wish to use the unit circle to help you.

a) ∠A = −30°

b) ∠B = −240°

c) ∠C = −180°

d) ∠D = 405°

e) ∠E = 750°

f) ∠F = 570°

B Connect and Apply

☆8. Without using a calculator, determine two angles between 0° and 360° that have a sine of $-\frac{1}{\sqrt{2}}$.

9. Two angles between 0° and 360° have a tangent of 1. Without using a calculator, determine the angles.

10. The sine of each of two angles between 0° and 360° is $-\frac{\sqrt{3}}{2}$. Without using a calculator, determine the angles.

11. Three angles between 0° and 360° have a tangent that is 0. What are the angles? Which other trigonometric ratio is 0 for these angles?

12. The point P(5, −7) is on the terminal arm of ∠A.

a) Determine the primary trigonometric ratios for ∠A and ∠B such that ∠B has the same sine as ∠A.

b) Use a calculator and a diagram to determine the measures of ∠A and ∠B, to the nearest degree.

13. The point Q(−6, −1) is on the terminal arm of ∠C.

a) Determine the primary trigonometric ratios for ∠C and ∠D such that ∠D has the same cosine as ∠C.

b) Use a calculator and a diagram to determine the measures of ∠C and ∠D, to the nearest degree.

14. The point R(−2, 3) is on the terminal arm of ∠E.

a) Determine the primary trigonometric ratios for ∠E and ∠F such that ∠F has the same tangent as ∠E.

b) Use a calculator and a diagram to determine the measures of ∠E and ∠F, to the nearest degree.

15. Solve each equation for $0° \leq \theta \leq 360°$.

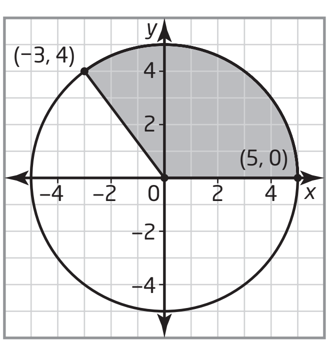
a) $\sin \theta = -\frac{3}{4}$

b) $\cos \theta = \frac{2}{3}$

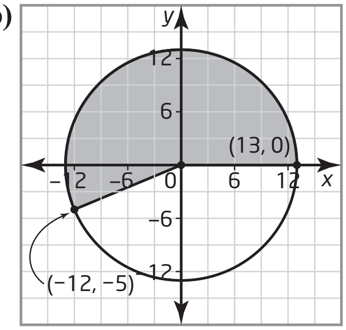
c) $\tan \theta = -\frac{5}{7}$

16. Determine the area of the sector of the circle shown.

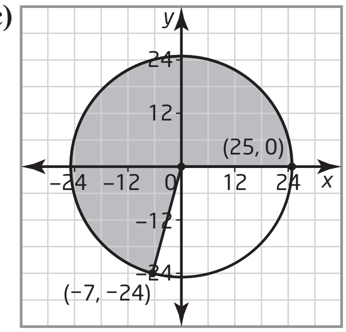
a)



b)



c)



C Extend

17. An angle θ is in standard position on a coordinate grid. The terminal arm of θ is in the second quadrant on the line with equation $3y + 2x = 0$. Determine the measure of angle θ.

18. △ABO has vertices A(−3, 0), B(−3, −7), and O(0, 0). A circle, of radius 1 unit and centre O(0, 0), intersects OB at point P. Point C has coordinates C(1, 0), and ∠COP = θ.

a) Determine the coordinates of P.

b) Use the results of part a) to determine $\sin \theta$ and $\cos \theta$.

c) Compare the values of $\sin \theta$ and $\cos \theta$ with those you would obtain by using the lengths of the sides of △ABO. What do you notice?

4.3 Reciprocal Trigonometric Ratios

KEY CONCEPTS

- The reciprocal trigonometric ratios are defined as follows:

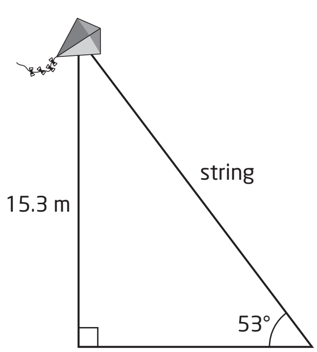
$$\csc\theta = \frac{\text{hypotenuse}}{\text{opposite}}$$
$$= \frac{1}{\sin\theta}$$
$$\sec\theta = \frac{\text{hypotenuse}}{\text{adjacent}}$$
$$= \frac{1}{\cos\theta}$$
$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$
$$= \frac{1}{\tan\theta}$$

Example

A kite on a string is flying at a height of 15.3 m above the ground. The angle between the kite and the ground is 53°. Use a reciprocal trigonometric ratio to determine the length of the string, to the nearest centimetre.

Solution

Draw a diagram to represent this situation.



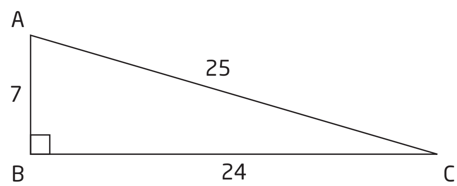
Let s represent the length of the string, in metres. Since we know the side opposite the angle and we need to find the hypotenuse, use the cosecant ratio.

$$\csc 53^\circ = \frac{s}{15.3}$$
$$s = 15.3 \csc 53^\circ$$
$$\doteq 19.2$$

The string is 19.2 m long.

A Practise

1. Given ΔABC, determine the six trigonometric ratios for $\angle$C.



2. Repeat question 1 for $\angle$A.

3. State the reciprocal trigonometric ratio that corresponds to each primary trigonometric ratio.
   a) $\sin\theta = \frac{3}{5}$
   b) $\cos\theta = \frac{1}{\sqrt{2}}$
   c) $\tan\theta = \frac{7}{3}$
   d) $\cos\theta = -\frac{6}{\sqrt{61}}$
   e) $\tan\theta = -5$
   f) $\sin\theta = -\frac{12}{13}$
   g) $\cos\theta = 0$
   h) $\sin\theta = 1$

4. State the primary trigonometric ratio that corresponds to each reciprocal trigonometric ratio.
   a) $\sec\theta = \frac{8}{3}$
   b) $\csc\theta = \frac{5}{4}$
   c) $\cot\theta = -\frac{1}{\sqrt{3}}$
   d) $\sec\theta = -\frac{17}{15}$
   e) $\csc\theta = \sqrt{2}$
   f) $\cot\theta = -\frac{9}{4}$
   g) $\sec\theta = -1$
   h) $\csc\theta =$ undefined

5. Use a calculator to determine the six trigonometric ratios of each angle, to three decimal places.
   a) 40°
   b) 36°
   c) 88°
   d) 110°
   e) 237°
   f) 319°
   g) 95°
   h) 67°
   i) 124°

6. Use a calculator to find each value of θ, to the nearest degree, where $0° \leq \theta \leq 90°$.
   a) $\csc\theta = 1.624$
   b) $\cot\theta = 0.675$
   c) $\sec\theta = 1.058$
   d) $\cot\theta = 0.554$
   e) $\sec\theta = 1.325$
   f) $\csc\theta = 1.305$
   g) $\cot\theta = 3.732$
   h) $\sec\theta = 3.628$

7. Determine exact expressions for the six trigonometric ratios for 210°. Hint: Draw a diagram of the angle in standard position. Then, use special triangles to determine exact values.

8. Determine exact expressions for the six trigonometric ratios for 225°.

9. Determine exact expressions for the six trigonometric ratios for 90°.

10. Determine two angles between 0° and 360° that have a cosecant of $-\frac{2}{\sqrt{3}}$. Use the unit circle to help you. Do not use a calculator.

11. Find the measure, to the nearest degree, of an angle in the first quadrant that satisfies each ratio.

a) $\sin A = \frac{6}{11}$

b) $\cos B = \frac{4}{5}$

c) $\tan C = \frac{13}{5}$

d) $\csc D = \frac{10}{7}$

e) $\sec E = \frac{8}{5}$

f) $\cot F = \frac{14}{9}$

g) $\sec G = 3$

h) $\csc H = \frac{7}{4}$

12. Determine two angles between 0° and 360° that have a cotangent of 1. Use the unit circle to help you. Do not use a calculator.

13. Each point lies on the terminal arm of an angle in standard position. Determine exact expressions for the six trigonometric ratios for the angle.

a) P(−3, 4)

b) Q(−12, −5)

c) R(−8, 15)

d) S(6, 2)

e) T(2, −3)

f) U(−7, −12)

g) V(1, 5)

h) W(6, 11)

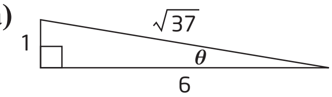
i) X(−5, 5)

j) Y(8, −3)

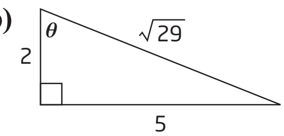
B Connect and Apply

14. Determine the three reciprocal trigonometric ratios for each triangle. Then, use one of the ratios to find the measure of angle θ, to the nearest degree.

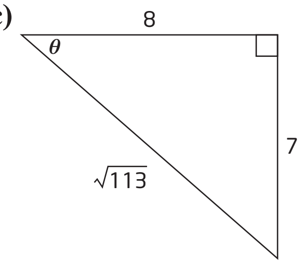
a)



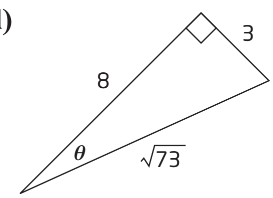
b)



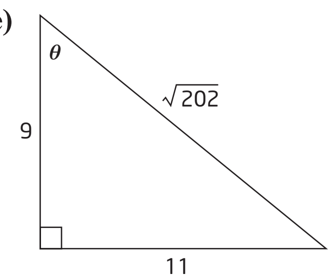
c)



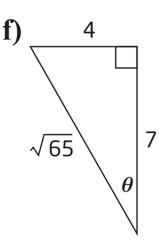
d)



e)



f)



15. Determine two angles between 0° and 360° that have a cosecant of −3.5. Round answers to the nearest degree.

16. Determine two angles between 0° and 360° that have a secant of −6.3.

17. Determine two angles between 0° and 360° that have a cotangent of 4.

18. Find the value of the other five trigonometric ratios for angle θ if $\tan \theta = -\frac{24}{7}$ and the terminal arm lies in the fourth quadrant.

19. Find the value of the other five trigonometric ratios for angle θ if $\csc \theta = \frac{5}{4}$ and the terminal arm lies in the second quadrant.

20. Find the value of the other five trigonometric ratios for angle θ if $\cos \theta = -\frac{1}{3}$ and the terminal arm lies in the third quadrant.

21. Find the value of the other five trigonometric ratios for angle θ if $\sec \theta = \frac{7}{4}$ and the terminal arm lies in the first quadrant.

22. A guy wire supporting a telephone pole is secured to the ground at a point 16.7 m from the base of the pole. The wire makes an angle of 48° with the ground.
    a) Use a reciprocal trigonometric ratio to write an equation that can be used to determine the length of the wire.
    b) Use your equation in part a) to find the length of the wire, to the nearest centimetre.

23. A wheelchair ramp to the front porch of a house is to be built so that it has an angle of inclination of 14.5° and a height of 1.3 m.
    a) Use a reciprocal trigonometric ratio to write an equation that can be used to determine the length of the ramp.
    b) Use your equation in part a) to find the length of the ramp, to the nearest centimetre.

24. Determine the value of each expression if $\cos A = \frac{5}{14}$ and $\sin B = \frac{4}{13}$.
    a) $\frac{\cos A \tan A}{\csc A}$
    b) $\frac{\sin B \sec B}{\cot B}$
    c) $\frac{\sin A \cot A}{\sec A}$
    d) $\frac{\cos B \csc B}{\tan B}$

C Extend

25. a) Use expressions for the primary trigonometric ratios in terms of x, y, and r to show that $\sin^2 \theta + \cos^2 \theta = 1$, regardless of the value of θ. This equation is known as the Pythagorean identity. Why is this name appropriate?
    b) Write an equivalent equation in terms of the reciprocal trigonometric ratios.

26. Consider an angle θ in the third quadrant such that $\sec \theta = -\frac{a}{b}$. Determine expressions in terms of a and b for the other five trigonometric ratios for θ. State any restrictions on a and b.

27. Given that $\cot \theta = \frac{2x}{x + 1}$, $x \neq -1$, and θ is in the first quadrant, determine the other two reciprocal trigonometric ratios.

4.4 Problems in Two Dimensions

KEY CONCEPTS

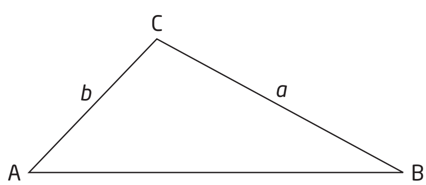
- Primary trigonometric ratios are used to solve triangles that contain a right angle.
- The sine law is used to solve oblique triangles when two angles and a side are given. In the case when two sides and an opposite angle are given, there may be two possible solutions, one solution, or no solution. This is known as the ambiguous case.
- The cosine law is used to solve oblique triangles when two sides and a contained angle or three sides and no angles are given.

Example

Given ΔABC with known side lengths a and b and known angle A, state the conditions for no solution, one solution, and two solutions for the length of the third side, c.

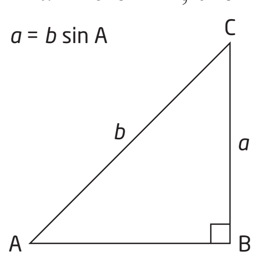
Solution

Case 1: $a \geq b$
In this case, CB $\geq$ CA. Therefore, only one triangle can be constructed.

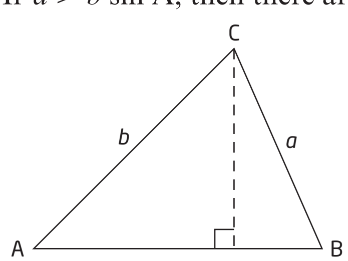


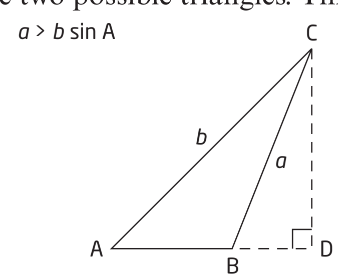
Case 2: $a < b$
In this case, there are three possibilities.

i) If $a < b \sin A$, then no triangle can be drawn.
ii) If $a = b \sin A$, then there is one right triangle and one solution.



iii) If $a > b \sin A$, then there are two possible triangles. This is the ambiguous case.





A Practise

1. For each triangle, select the most appropriate method among primary trigonometric ratios, the sine law, and the cosine law. Justify your choice.

a) In $\triangle ABC$, $\angle C = 90°$, $\angle B = 28°$, and $c = 12$ cm. Determine a.

b) In $\triangle PQR$, $\angle P = 51°$, $\angle R = 39°$, and $p = 9$ m. Determine r.

c) In $\triangle ABC$, $a = 10$ cm, $b = 7$ cm, and $c = 8$ cm. Determine $\angle B$.

d) In $\triangle DEF$, $\angle D = 130°$, $e = 10$ cm, and $f = 8$ cm. Determine d.

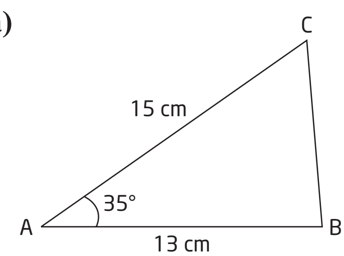
e) In $\triangle XYZ$, $\angle X = 90°$, $y = 15.6$ km, and $z = 12.3$ km. Determine $\angle Y$.

2. Determine the indicated quantity for each triangle in question 1. Round each answer to one decimal place.

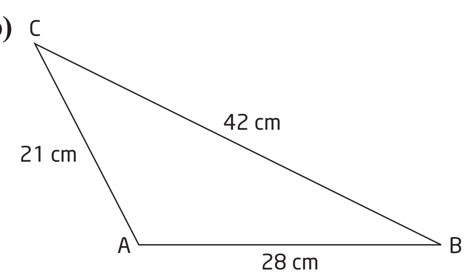
3. The shadow of a tree that is 18.5 m tall measures 10.2 m in length. Determine the angle of elevation of the sun.

4. Solve each triangle by finding all unknown values.

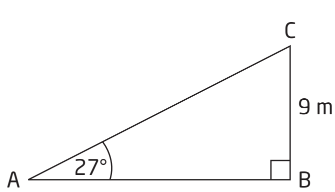
a)



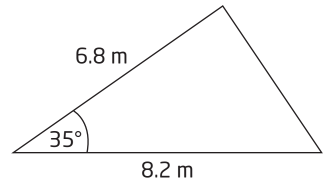
b)



c)



5. A roof truss spans a width of 8.2 m. One piece of the truss is 6.8 m long and set at an angle of 35°, as shown. How long is the third piece of the truss?



6. A radar station at point A is tracking two ships. Ship B is 4.5 km away in a direction 42° east of north. Ship C is 3.3 km away in a direction 58° west of north. How far apart are the two ships, to the nearest tenth of a kilometre?

7. A tunnel is to be built through a hill to connect the towns of Mathville and Trigville with a straight road. A straight road between Mathville and Functionville is 2 km long. A straight road from Trigville to Functionville is 5 km long. The angle between the two roads is 63°.

a) Determine the length of the road between Mathville and Trigville, to the nearest tenth of a kilometre.

b) Determine the angle between the road connecting Functionville and Mathville and the road from Mathville to Trigville.

8. For each of the following, draw possible diagrams that match the given measurements. Then, calculate the length of the unknown side.

a) In $\triangle ABC$, $c = 9$ cm, $b = 11$ cm, and $\angle B = 48°$.

b) In $\triangle ABC$, $a = 5.9$ m, $b = 7.8$ m, and $\angle A = 36°$.

9. For each triangle, two sides and a non-included angle are given. Determine two possible measures for $\angle C$ and for the length of side b.

a) In $\triangle ABC$, $a = 2.4$ cm, $c = 3.2$ cm, and $\angle A = 28°$.

b) In $\triangle DEF$, $d = 3$ cm, $e = 5$ cm, and $\angle D = 30°$.

B Connect and Apply

☆**10.** In ΔABC, $a = 1.2$ cm, $b = 2.7$ cm, and $\angle A = 32°$.

a) Calculate $b \sin A$.

b) How many solutions occur for ΔABC?

11. In ΔABC, $a = 2.4$ cm, $b = 4.8$ cm, and $\angle A = 30°$.

a) Calculate $b \sin A$.

b) How many solutions occur for ΔABC?

12. In ΔABC, $a = 6.1$ cm, $b = 8.1$ cm, and $\angle A = 32°$.

a) Calculate $b \sin A$.

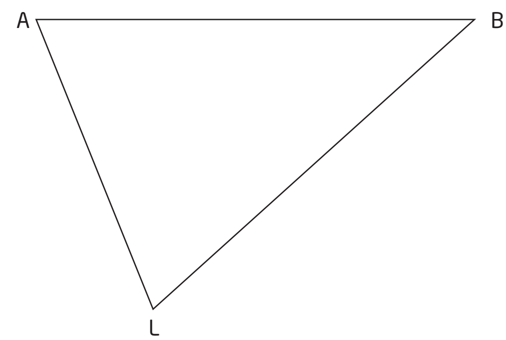
b) How many solutions occur for ΔABC?

13. Two forest fire stations, A and B, are 20 km apart. A ranger at station A sees a fire at point C, 15 km away. The angle between the line AB and the line AC to the fire is 21°. How far, to the nearest tenth of a kilometre, is station B from the fire?

14. Two ships, S and T, are 120 km apart when they pick up a distress call from a yacht. Ship T estimates that the yacht is 70 km away and that the angle between the line from T to S and the line from S to the yacht is 28°. What are two possible distances, to the nearest tenth of a kilometre, from ship S to the yacht?

15. A pedestrian bridge is built over a river. The angle of depression from one end of the bridge to a large rock beside the river is 37°. The distance from that end of the bridge to the rock is 112 m, while the distance from the rock to the other end of the bridge is 75 m. Determine the length of the bridge.

16. Two small islands, Aqua and Belli, are 32.0 km apart. How far is each island from a lighthouse, at point L, on the main shore if $\angle A = 68°$ and $\angle B = 42°$?



17. Kettletown is 27.0 km from Teatown and 32.0 km from Coffeetown. The angle from Kettletown to Coffeetown to Teatown is 29°. Determine two possible distances between Teatown and Coffeetown.

18. A parallelogram has vertices A(4, 4), B(9, 1), C(0, −5), and D(−3, −2). Determine the length of each diagonal.

19. Solve ΔABC, given BC = 6.0 cm, AC = 4.0 cm, and $\angle B = 30°$.

C Extend

20. In ΔABC, AC = 2, AB = 3, and BC = 4. Prove each statement.

a) $\sin B = \frac{1}{2} \sin A$

b) $\sin C = \frac{3}{4} \sin A$

☆**21. a)** Determine the possible range of values for side a so that ΔABC has two solutions if $\angle A = 40°$ and $b = 50.0$.

b) Determine the possible range of values for side a so that ΔABC has no solution if $\angle A = 56°$ and $b = 125.7$.

c) Determine the possible range of values for side a so that ΔABC has exactly one solution if $\angle A = 57°$ and $b = 73.7$.

4.5 Problems in Three Dimensions

KEY CONCEPTS

- Three-dimensional problems involving triangles can be solved using one or more of the following: the Pythagorean theorem, the six trigonometric ratios, the sine law, and the cosine law.

- The method chosen to solve a triangle depends on the known information.

Example

Two stunt pilots are flying to an air show to join an aerial formation at noon. At 9:30 a.m., the first pilot passes over a navigational beacon, A, at a speed of 200 km/h and sets a course 40° east of north to go directly to the formation point for a noon arrival. At 10:00 a.m., the second pilot reaches another beacon, B, located 150 km northwest of beacon A.

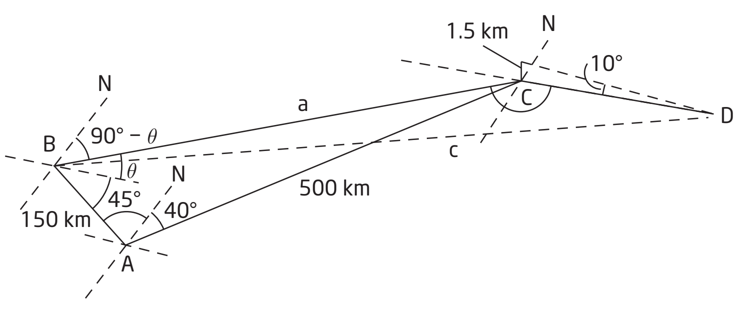
a) Determine the speed and direction that the second pilot must fly from beacon B in order to reach the formation point at noon.

b) As seen from an airfield due east of the formation point, the angle of elevation of the aircraft is 10°. If both aircraft are at an altitude of 1500 m, how far is the airfield from beacon B? Round your answer to the nearest kilometre.

Solution

Draw a diagram to represent this situation. In the illustration, A represents the location of the first navigational beacon, B is the location of the second beacon, C is the ground point directly below the aerial formation area, and D is the airfield.

The first pilot is flying at 200 km/h so that in 2.5 h (from 9:30 a.m. to noon) the pilot travels a distance of 500 km.



a) From the diagram, $\angle A = 85°$ and the angle $90° - \theta$ represents the direction of flight of the second aircraft.

Since the length of two sides and the measure of the contained angle are known, use the cosine law to solve for a.

$$a^2 = 150^2 + 500^2 - 2(150)(500)\cos 85°$$
$$\doteq 259\,426.64$$
$$a \doteq 509.3$$

The second pilot must fly a distance of 509.3 km in 2 h, so the plane must travel at a speed of approximately 255 km/h.

To find the measure of angle θ, first use the sine law to determine $\angle B$.

$$\frac{\sin B}{500} = \frac{\sin 85^\circ}{509.3}$$
$$\sin B \doteq 0.978\ 00$$
$$\angle B \doteq 78^\circ$$

$$\theta = 78^\circ - 45^\circ$$
$$= 33^\circ$$

The direction that the second aircraft must head is $90^\circ - 33^\circ$, or 57° east of north.

Therefore, in order to join the aerial formation at noon, the second pilot must fly from beacon B at a speed of 255 km/h in a direction 57° east of north.

b) From the diagram, CD represents the distance from the formation point to the airfield and is the base of a right triangle having a vertical distance of 1.5 km (the altitude of the aircraft). Use the primary trigonometric ratios to calculate the length of CD.

$$\tan 10^\circ = \frac{\text{opposite}}{\text{adjacent}}$$
$$\tan 10^\circ = \frac{1.5}{\text{CD}}$$
$$\text{CD} = \frac{1.5}{\tan 10^\circ}$$
$$\text{CD} \doteq 8.506$$

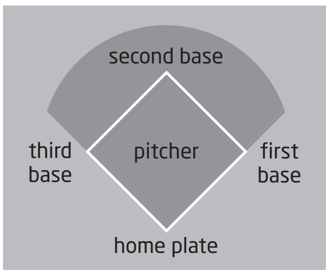
Therefore, the airfield is approximately 8.5 km east of the formation point.
As illustrated on the diagram, DB is the distance from the airfield to the second beacon. The line is one side of $\triangle$BCD. $\angle C$ equals $90^\circ + (90 - \theta)^\circ$, or 147°. The angle is contained by sides BC and CD, the lengths of which are known. Therefore, use the cosine law in $\triangle$BCD to determine the distance, c, from the airfield to the second beacon:

$$c^2 = (509.3)^2 + (8.5)^2 - 2(509.3)(8.5)\cos 147^\circ$$
$$c^2 = 259\ 386.49 + 72.25 - (8658.1)\cos 147^\circ$$
$$c^2 \doteq 266\ 720.03$$
$$c \doteq 516.45$$

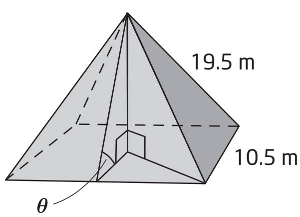
The airfield is approximately 516 km from the second beacon.

A Practise

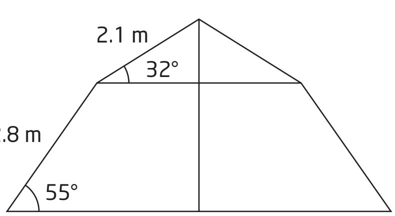
1. The bases on a baseball diamond are 27.4 m apart. The pitcher pitches, and the batter hits a fly ball straight up 18 m. What is the maximum angle of elevation of the ball, to the nearest degree, as seen by the pitcher if he is standing at the centre of the diamond?



2. A pyramid has a square base of side length 10.5 m. The slant height of the pyramid is 19.5 m.
   a) Determine the height of the pyramid, to the nearest tenth of a metre.
   b) Determine the angle θ between one face and the base, to the nearest tenth of a degree.



3. A square-based tent has the cross-sectional shape shown. The side wall goes up at an angle of elevation of 55° for 2.8 m, then continues at an angle of elevation of 32° for another 2.1 m to the peak.



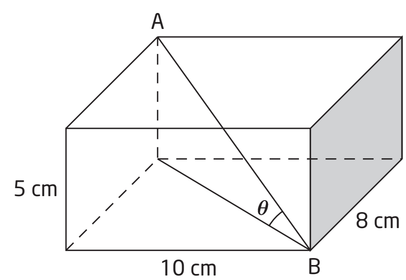
   a) Determine the height of the tent.
   b) Determine the side length of the base.
   c) Determine the length of one of the diagonals of the base.

4. A gift box for a perfume bottle is in the shape of a pyramid with a rectangular base. The dimensions of the base are 10 cm by 7 cm and the length of each side edge is 16 cm. Determine the height of the pyramid.

B Connect and Apply

5. Beni and Alessio watch a rocket as it is launched. Beni is 0.8 km closer to the launching pad than Alessio. When the rocket disappears from view, its angle of elevation for Beni is 36.5° and for Alessio is 31.9°. Determine the altitude of the rocket, to the nearest tenth of a kilometre, at the time it disappears from view.

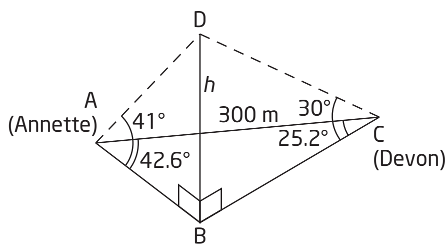
6. A rectangular prism has length 10 cm, width 8 cm, and height 5 cm.



   a) Determine the length of the diagonal AB.
   b) Determine the measure of angle θ.

☆**7.** Angela is a surveyor. To find the height, h, of an inaccessible cliff, she takes measurements at point A some distance from the cliff and at a second point, C, which is along the base of the cliff and 400 m from point A. Angela determines the angle of elevation from point C to the top of the cliff is 18°. She finds that the angle from point B, which is at the base of the cliff directly below the top, to C to A is 35°, and that the angle from B to A to C is 27°.
   a) Draw a diagram to represent this situation.
   b) What is the height of the cliff, to the nearest metre?

8. Annette and Devon want to determine how high their model rocket can fly. The rocket is launched from point B and reaches an altitude, *h*. Annette is standing at point A and measures the angle of elevation to the tip of the rocket at its highest point (D) to be 41°. Devon is standing at point C, 300 m from Annette, and measures the angle of elevation to be 30°. If ∠BAC is 42.6° and ∠BCA is 25.2°, what altitude does the rocket reach, to the nearest metre?

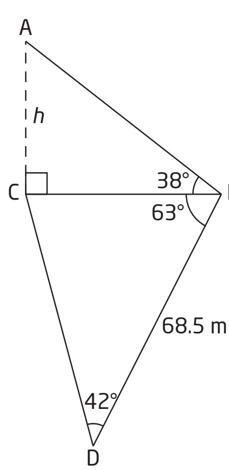


☆9. Kamira is flying in a hot-air balloon and notices a barn directly below the balloon and a farmhouse located at an angle of depression of 28°. After the balloon rises vertically a further 58 m, the angle of depression to the farmhouse is 42°.

a) How high is the balloon before it rises?

b) How far is the barn from the farmhouse?

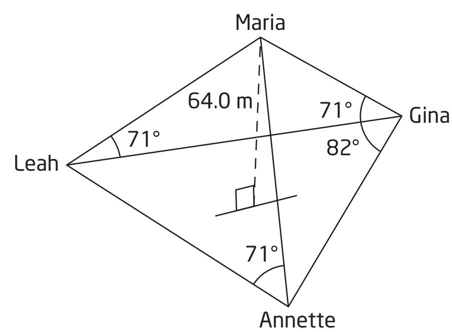
10. To determine the height, *h*, of a tree in Algonquin Park, a conservation officer records measurements in a diagram. Determine the height of the tree.



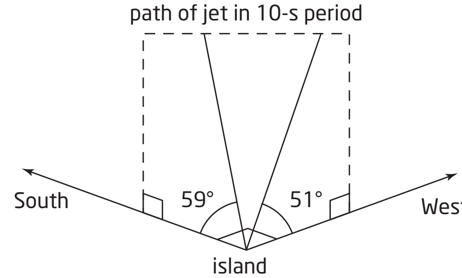
11. A hydro tower located on the side of a hill casts a 55-m-long shadow down the hill when the angle of elevation of the sun is 48.3°. Determine the height of the tower if the angle of inclination of the hill is 21.4°.

C Extend

12. Maria is flying in a hot-air balloon at a height of 64 m. Her friends, Leah, Gina, and Annette, are on the ground at different points so that the balloon is between them. The angle of elevation from each girl to the balloon is 71°. The angle formed between the lines of sight from Gina to Leah and from Gina to Annette is 82°. Determine the distance between Leah and Annette.



13. A jet is flying in a straight line at an average speed of 850 km/h at a constant altitude above level ground. At a certain time, the jet's location is due south of an island such that the angle of elevation from the island to the jet is 59°. Ten seconds later, the jet is due west of the island at an angle of elevation of 51°. Determine the altitude of the jet.



4.6 Trigonometric Identities

KEY CONCEPTS

- A trigonometric identity is a relation among trigonometric ratios that is true for all angles for which both sides are defined.
- The basic identities are the Pythagorean identity, the quotient identity, and the reciprocal identities:
- **Pythagorean Identity** $\sin^2 \theta + \cos^2 \theta = 1$
- **Quotient Identity** $\dfrac{\sin \theta}{\cos \theta} = \tan \theta$
- **Reciprocal Identities** $\csc \theta = \dfrac{1}{\sin \theta}$

  $\sec \theta = \dfrac{1}{\cos \theta}$

  $\cot \theta = \dfrac{1}{\tan \theta}$
- The basic identities can be used to prove more complex identities.
- Identities can be used to simplify solutions to problems that result in trigonometric expressions.

Example

Prove that $\sin^4 \theta + 2\cos^2 \theta - \cos^4 \theta = 1$ for all θ.

Proof

$$\begin{aligned}
\text{L.S.} &= \sin^4 \theta + 2\cos^2 \theta - \cos^4 \theta \\
&= \sin^4 \theta - \cos^4 \theta + 2\cos^2 \theta && \text{Rearrange the terms.} \\
&= (\sin^2 \theta - \cos^2 \theta)(\sin^2 \theta + \cos^2 \theta) + 2\cos^2 \theta && \text{Factor using difference of squares.} \\
&= (\sin^2 \theta - \cos^2 \theta)(1) + 2\cos^2 \theta && \text{Substitute } \sin^2 \theta + \cos^2 \theta = 1 \\
&= \sin^2 \theta - \cos^2 \theta + 2\cos^2 \theta \\
&= \sin^2 \theta + \cos^2 \theta \\
&= 1 \\
&= \text{R.S.}
\end{aligned}$$

Therefore, L.S. = R.S., and the statement is true for all θ.

A Practise

1. Using the Pythagorean identities, state an equivalent expression for each.
   a) $\sin^2 \theta$
   b) $\tan^2 \theta$
   c) $\sec^2 \theta$
   d) $1 - \sin^2 \theta$
   e) $1 - \csc^2 \theta$
   f) $\cot^2 \theta - \csc^2 \theta$
   g) $-\dfrac{1}{\csc^2 \theta}$
   h) $\dfrac{\cos^2 \theta}{\sin^2 \theta}$

2. Express each expression in a simpler form.
   a) $\cos \theta \sec \theta$
   b) $\tan \theta \cos \theta$
   c) $\tan \theta + \cot \theta$
   d) $\sqrt{1 - \cos^2 \theta}$
   e) $\tan^2 \theta - \sec^2 \theta$
   f) $\dfrac{\cos \theta}{1 + \sin \theta} + \dfrac{\cos \theta}{1 - \sin \theta}$

3. Factor to simplify each expression.
   a) $\sin^4 \theta + \sin^2 \theta \cos^2 \theta$
   b) $(\sec \theta)^2 (\sin \theta)^2 + (\sin \theta)^2$
   c) $4 \cos^2 \theta + 8 \cos \theta \sin \theta + 4 \sin^2 \theta$
   d) $\sin^4 \theta - \cos^4 \theta$

4. Use the definitions of the primary trigonometric ratios in terms of x, y, and r to prove each identity.
   a) $\tan \theta \cos \theta = \sin \theta$
   b) $\cot \theta \sec \theta = \csc \theta$
   c) $\dfrac{1 + \cot^2 \theta}{\csc^2 \theta} = 1$

5. Prove that $\cos \theta \cot \theta = \dfrac{1}{\sin \theta} - \sin \theta$.

6. Explain how technology may be used to verify an identity.

B Connect and Apply

7. Prove that $\dfrac{\cos^2 \theta}{1 - \sin \theta} = 1 + \sin \theta$.

☆8. Prove that $\dfrac{1 + \cot \theta}{\csc \theta} = \sin \theta + \cos \theta$.

9. Prove that $\dfrac{\tan^2 \theta}{1 + \tan^2 \theta} = \sin^2 \theta$.

☆10. Refer to the Example at the beginning of this section. Use an alternate method to prove the given statement.

11. Prove each identity.
   a) $\dfrac{\cos \theta \sin \theta}{\cot \theta} = 1 - \cos^2 \theta$
   b) $\dfrac{\sin^2 \theta}{1 - \cos \theta} = 1 + \cos \theta$
   c) $\dfrac{1 + \sec \theta}{\sec \theta - 1} = \dfrac{1 + \cos \theta}{1 - \cos \theta}$
   d) $\dfrac{1 + \sin \theta}{1 - \sin \theta} = \dfrac{\csc \theta - 1}{\csc \theta + 1}$

12. Simplify $\sin^2 \theta + \dfrac{1 + \cot^2 \theta}{1 + \tan^2 \theta} + \cos^2 \theta$.

13. Prove that $2 \sin (-\theta) - \cot \theta \sin \theta \cos \theta = (\sin \theta - 1)^2 - 2$.

14. a) Start with an identity that has already been proved. Make substitutions and rearrange terms until the identity is no longer recognizable.
   b) Trade the new identity you have written with that of a classmate. Use a different method to prove the identity written by your classmate.

15. Explain why the identity $\dfrac{\cos \theta}{1 + \sin \theta} + \dfrac{\cos \theta}{1 - \sin \theta} = \dfrac{2}{\cos \theta}$ is not true for $\theta = 90°$ and $\theta = 270°$.

16. Prove that
$$\frac{1 + \sin \theta + \cos \theta}{1 - \sin \theta + \cos \theta} = \frac{1 + \sin \theta}{\cos \theta}.$$

17. Prove that
$$\frac{1}{1+\sin\theta}+\frac{1}{1-\sin\theta}=\frac{2}{\cos^2\theta}.$$

18. Prove that
$$\frac{7\sin\theta+5\cos\theta}{\cos\theta\sin\theta}=7\sec\theta+5\csc\theta.$$

19. Prove that $(\csc\theta-\cot\theta)^2=\dfrac{1-\cos\theta}{1+\cos\theta}$.

20. Prove that $\sec^2\theta+\csc^2\theta=\dfrac{\csc^2\theta}{\cos^2\theta}$.

21. Prove that $\sec\theta-\dfrac{\sin\theta}{\cot\theta}=\dfrac{1}{\sec\theta}$.

22. Prove that $\tan^2\theta\cos^2\theta=1-\dfrac{\sin^2\theta}{\tan^2\theta}$.

23. Prove that $\dfrac{\tan\theta-\sin\theta}{\sin^3\theta}=\dfrac{\sec\theta}{1+\cos\theta}$.

24. Prove that $\dfrac{1+\sin\theta}{\tan\theta}=\cos\theta+\cot\theta$.

25. Prove that $(\tan\theta+\cot\theta)^2$
$=\sec^2\theta+\dfrac{1}{\sin^2\theta}$.

26. **Use Technology** For each equation, use a graphing calculator to graph each side to determine if the equation appears to be an identity.

a) $\dfrac{1}{\sin^2\theta}+\dfrac{1}{\cos^2\theta}=1$

b) $\cos\theta\,(\cos\theta-\sec\theta)=-\sin^2\theta$

c) $2\sin\theta+(1-\sin\theta)^2=2-\cos^2\theta$

d) $\dfrac{\sin^3\theta-\cos^3\theta}{\sin\theta-\cos\theta}=\sin^2\theta+\cos^2\theta$

27. Refer to your results in question 26.

a) Prove each of the statements that you found is an identity.

b) For those statements that are not identities, provide an example to show why.

28. Prove that
$$(\sin\theta+\cos\theta)^2=\frac{2+\sec\theta\csc\theta}{\sec\theta\csc\theta}.$$

29. Prove that
$$\frac{\cot\theta}{\csc\theta-1}+\frac{\cot\theta}{\csc\theta+1}=2\sec\theta.$$

30. Prove that
$$\frac{\tan\theta+\cos\theta}{\sin\theta}=\frac{1}{\cos\theta}+\frac{\cos\theta}{\sin\theta}.$$

31. Prove that
$$\frac{\cos^2\theta-\sin^2\theta}{\cos^2\theta+\sin\theta\cos\theta}=\frac{\cot\theta-1}{\cot\theta}.$$

32. Prove that $\tan\theta\sin\theta+\cos\theta-\sec\theta+1$
$=\sec^2\theta\cos^2\theta$.

33. Prove that $\dfrac{1-\tan^2\theta}{\tan\theta-\tan^2\theta}=1+\dfrac{1}{\tan\theta}$.

34. **a)** Prove that $\tan^2\theta(1+\cot^2\theta)=\sec^2\theta$.

b) Create an identity similar to the identity in part a) and prove it.

35. **a)** Prove each identity.

i) $(1-\cos^2\theta)(1+\tan^2\theta)=\tan^2\theta$

ii) $(1-\sin^2\theta)(1+\cot^2\theta)=\cot^2\theta$

b) For each identity in part a), create a similar identity and prove it.

C Extend

36. Prove that $\sin^2\theta-\cos^2\theta=\dfrac{2-\sec^2\theta}{-\sec^2\theta}$.

37. Prove that
$$\frac{\csc\theta}{1-\csc\theta}+\frac{\csc\theta}{\csc\theta+1}=-\frac{2\sin\theta}{\cos^2\theta}.$$

38. Prove that $\sec^2\theta-\dfrac{\tan\theta}{\cos\theta}=\dfrac{1}{1+\sin\theta}$.

39. Prove that $\dfrac{\tan^3\theta}{1+\tan^2\theta}+\dfrac{\cot^3\theta}{1+\cot^2\theta}$
$=\dfrac{1-2\sin^2\theta\cos^2\theta}{\sin\theta\cos\theta}$.

40. Simplify $\dfrac{1}{4\sin^2\theta\cos^2\theta}-\dfrac{(1-\tan^2\theta)^2}{4\tan^2\theta}$.

Chapter 4 Review

4.1 Special Angles

1. For each angle, use exact values to show that $\sin^2 x + \cos^2 x = 1$.

a) $x = 60°$ **b)** $x = 150°$ **c)** $x = 225°$

2. Determine the exact value of each expression.

a) $\sin 45° \times \tan 30° \times \cos 120°$

b) $\sin 240° + \tan 225° - \cos 330°$

c) $\sin 270° \times \tan 300° - \cos 180°$

3. A guy wire is fastened 6 m from the base of a flagpole and makes an angle θ with the ground. For each given angle θ, determine

i) the length of the wire

ii) how far up the pole the wire is fastened

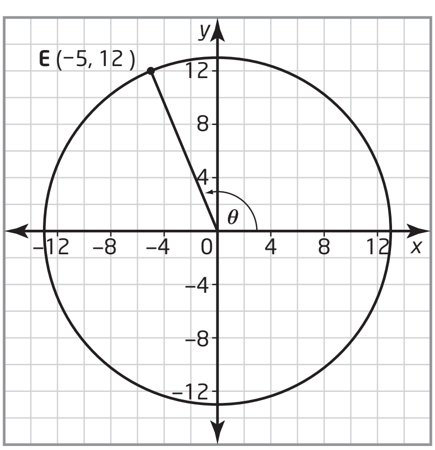
a) 45° **b)** 30° **c)** 60°

4. A tiling company develops a floor tile in the shape of a regular hexagon that has an area of 30 cm^2. Determine the exact length of the six equal sides of the tile.

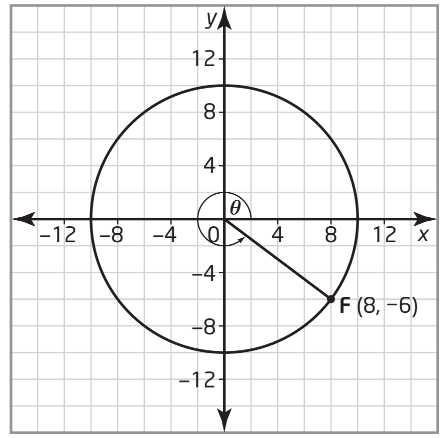
4.2 Co-terminal and Related Angles

5. The coordinates of a point on the terminal arm of an angle θ are shown. Determine the exact trigonometric ratios for θ.

a) E(−5, 12)



b) F(8, −6)



6. One of the primary trigonometric ratios for an angle is given, as well as the quadrant in which the terminal arm lies. Find the other two primary trigonometric ratios.

a) $\sin G = -\frac{5}{11}$, third quadrant

b) $\cos E = \frac{4}{7}$, first quadrant

7. Solve each equation for $0° \leq \theta \leq 360°$.

a) $\sin \theta = \frac{5}{8}$

b) $\cos \theta = -0.35$

c) $\tan \theta = \frac{4}{9}$

4.3 Reciprocal Trigonometric Ratios

8. Determine exact expressions for the six trigonometric ratios for 120°.

9. Each point lies on the terminal arm of an angle in standard position. Determine exact expressions for the six trigonometric ratios for each angle.

a) T(24, −7)

b) U(−5, −3)

c) V(4, −9)

d) W(1, 3)

4.4 Problems in Two Dimensions

10. For each triangle, determine the number of solutions then solve the triangle if possible.

a) In $\triangle ABC$, $\angle A = 71°$, $a = 12.2$ m, and $b = 11.4$ m.

b) In $\triangle ABC$, $\angle A = 55°$, $a = 7.1$ cm, and $b = 9.6$ cm.

c) In $\triangle ABC$, $\angle A = 44°$, $a = 9.3$ mm, and $b = 12.3$ mm.

d) In $\triangle DEF$, $\angle D = 42°$, $d = 8.5$ km, and $f = 7.3$ km.

e) In $\triangle DEF$, $\angle E = 38°$, $d = 16.6$ mm, and $e = 13.4$ mm.

f) In $\triangle DEF$, $\angle D = 47°$, $d = 8.1$ m, and $f = 12.2$ m.

11. A marathon swimmer starts at Island A, swims 9.2 km to Island B, then 8.6 km to Island C. The angle formed by a line from Island B to Island A and a line from Island C to Island A is 52°. How far does the swimmer have to swim to return directly to Island A?

12. A solar-heated house is 10 m wide. The south side of the roof, containing the solar collectors, rises for 8 m at an angle of elevation of 60°.

a) Determine the length of the north side of the roof.

b) At what angle of elevation does the north side of the roof rise?

4.5 Problems in Three Dimensions

13. The pilot of a hot-air balloon, flying above a bridge, measures the angles of depression to each end of the bridge to be 54° and 71°. The direct distance from the balloon to the nearer end of the bridge is 270 m. Determine the length of the bridge.

14. Two roads intersect at an angle of 34°. Car A leaves the intersection on one of the roads and travels at 80 km/h. At the same time, Car B leaves the intersection on the other road and travels at 100 km/h. Two hours later a reporter in a traffic helicopter, which is above and between the two cars, notes that the angle of depression to the slower car is 20° and that the helicopter is 1 km from that vehicle. How far is the faster car from the helicopter?

15. The port of Math Harbour is located 200 km from Trig Town Inlet in a direction 50° east of north. A yacht leaves Trig Town Inlet at 8:00 a.m. and sails in a direction 15° west of north at a speed of 15 km/h. At the same time, a fishing boat leaves Math Harbour on a course 80° west of south at a speed of 20 km/h. Determine the distance, to the nearest kilometre, between the yacht and the fishing boat at 1:00 p.m.

4.6 Trigonometric Identities

16. State an equivalent expression for each.

a) $\cos^2\theta$

b) $\csc^2\theta$

c) $\cot^2\theta$

d) $\sec^2\theta - 1$

e) $\sec^2\theta - \tan^2\theta$

f) $\dfrac{1}{\sec^2\theta}$

17. Prove each identity.

a) $\sec^2\theta + \csc^2\theta = (\tan\theta + \cot\theta)^2$

b) $\dfrac{\sin\theta + \cos\theta\cot\theta}{\cot\theta} = \sec\theta$

c) $\sin^2\theta = \cos\theta(\sec\theta - \cos\theta)^2$

d) $\dfrac{1}{1 + \cos\theta} = \csc^2\theta - \dfrac{\cot\theta}{\sin\theta}$

e) $\dfrac{1 + \csc\theta}{\cot\theta} - \sec\theta = \tan\theta$

Chapter 4 Math Contest

1. In $\triangle ABC$, $\angle B = 60°$ and $\angle C = 90°$. The bisector of $\angle B$ meets AC at D. Determine the ratio AD:DC.

2. An equilateral triangle is inscribed in a circle. Determine the exact ratio of the area of the triangle to the area of the circle.

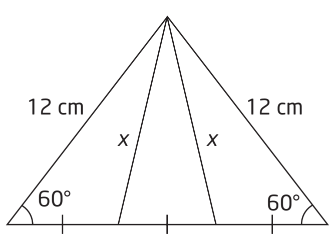
3. The perpendicular bisectors of the sides of an equilateral triangle, with side lengths 20 cm, intersect at the centroid. How far is the centroid from each side?

4. Which is the perimeter of an equilateral triangle with height $5\sqrt{3}$ cm?

   A $30\sqrt{3}$ cm **B** $10\sqrt{3}$ cm

   C $10 + 5\sqrt{3}$ **D** 30 cm

5. An equilateral triangle with side length 12 cm is divided into three triangles of equal area by two line segments of length x passing through a vertex. Which is the exact length of x?



   A $6\sqrt{7}$ cm **B** $4\sqrt{7}$ cm

   C $6\sqrt{3}$ cm **D** $4\sqrt{3}$ cm

6. The length of one diagonal of a rhombus is equal to the length of one of its sides, x. Which is the length of the other diagonal in terms of the length of its sides?

   A $x\sqrt{2}$ **B** $\sqrt{2x}$

   C $x\sqrt{3}$ **D** $\sqrt{3x}$

7. In $\triangle ABC$, $\angle A = x$, $\angle B = 3x$, and $\angle C = 4x$. Which is the value of the expression $4 \sin 4x + 2 \cos^2 2x + 6 \tan 6x$?

   A 1 **B** -1

   C $-\frac{1}{2}$ **D** $\frac{1}{2}$

8. If P, Q, and R are the angles of a triangle such that $\cos P = 0$ and $\cos R = \frac{1}{2}$, then the value of $\sin Q + \cos Q$ is

   A $\frac{1 + \sqrt{3}}{2}$ **B** 0

   C $\frac{\sqrt{2}}{2}$ **D** $\sqrt{2}$

9. Determine the degree measure of the two equal angles in an isosceles triangle whose side lengths are 4 cm, 4 cm, and $(\sqrt{24} - \sqrt{8})$ cm.

   A 45° **B** 75°

   C 55° **D** 65°

10. Given $\frac{\tan \theta}{\cos \theta} = \frac{3}{8}$, where $0° < \theta < 180°$, which is the exact value of $\sin \theta$?

    A 3 **B** $\frac{8}{3}$

    C $\frac{1}{8}$ **D** $\frac{1}{3}$

11. When simplified, the expression $1 - \frac{\sin^2 \theta}{1 + \cot \theta} - \frac{\cos^2 \theta}{1 + \tan \theta}$ is equivalent to

    A $\sin^2 \theta - \cos^2 \theta$

    B 1

    C $\sin \theta \cos \theta$

    D $\frac{\sin \theta + \cos \theta}{1 + \sec \theta}$