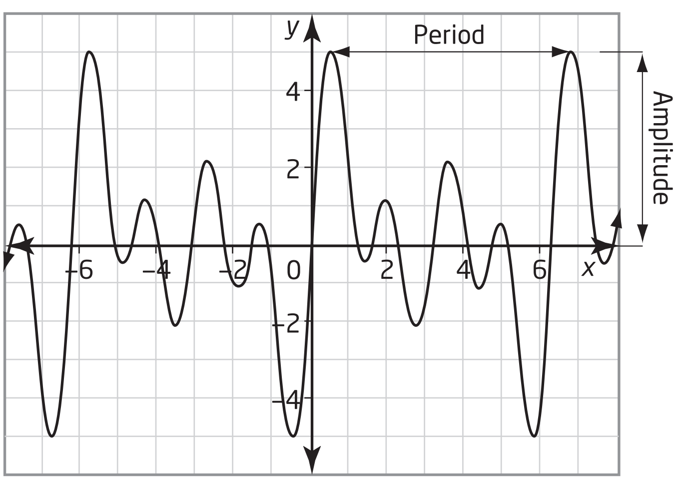
Chapter 5 Trigonometric Functions

5.1 Modelling Periodic Behaviour

KEY CONCEPTS

- A pattern that repeats itself regularly is periodic.
- A periodic pattern can be modelled using a periodic function.
- One repetition of a periodic pattern is called a cycle.
- The horizontal length of a cycle on a graph is called the period. The period may be in units of time or other units of measurement.
- A function is periodic if there is a positive number, p, such that $f(x + p) = f(x)$ for every x in the domain of $f(x)$. The least value of p that works is the period of the function.
- $f(x + np) = f(x)$, where p is the period and n is any integer.
- The amplitude of a periodic function is half the difference between the maximum value and the minimum value in a cycle.



Example

David is on a competitive swim team. As part of his daily practice at the community pool, he is required to swim the breaststroke for 10 consecutive laps of the pool. The pool is 200 m long, and he swims from the shallow end to the deep end at a constant speed of 50 m/min.

a) Graph David's distance, in metres, from his starting point in the shallow end, versus time, for 10 laps of the pool.

b) Determine the period and amplitude of the function.

c) State the domain and range.

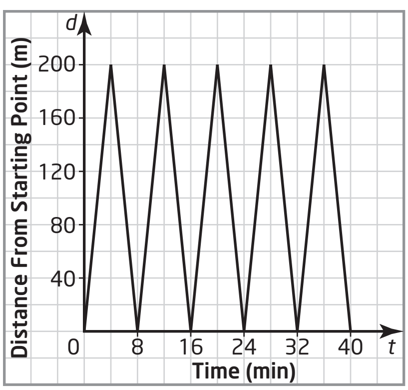
Solution

a) Recall that speed = distance ÷ time. So, time = distance ÷ speed.

$$\text{time} = 200 \div 50$$
$$= 4$$

It takes David 4 min to swim 200 m.

So, after 4 min he is 200 m away from his starting point. After another 4 min, or 8 min after starting, he is back at his starting point. Then, three points on the graph of distance versus time are (0, 0), (4, 200), and (8, 0). These points are joined with a straight line because David swims at constant speed of 50 km/h. The pattern continues until David has completed 10 laps of the pool.



b) The period of the function is 8 min, because the pattern repeats itself in 8-min intervals.

$$\text{Amplitude} = \frac{200 - 0}{2}$$
$$= 100$$

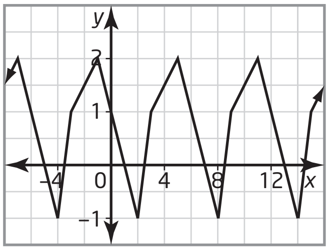
The amplitude is 100 m.

c) Let t represent the time, in minutes, and let d represent the distance from the starting point, in metres. The domain is $\{t \in \mathbb{R}, 0 \leq t \leq 40\}$. The range is $\{d \in \mathbb{R}, 0 \leq d \leq 200\}$.

A Practise

1. Classify each graph as periodic or not periodic. Justify your answers.

a)



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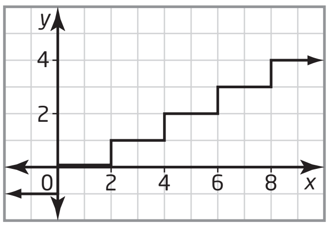

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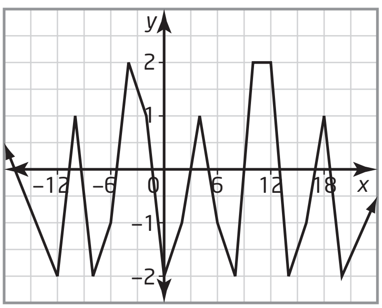

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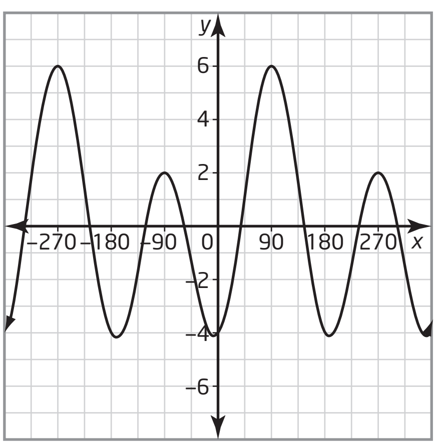

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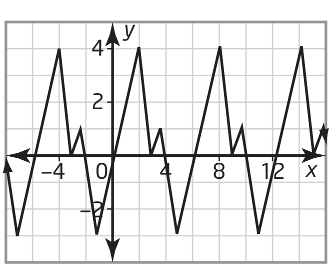

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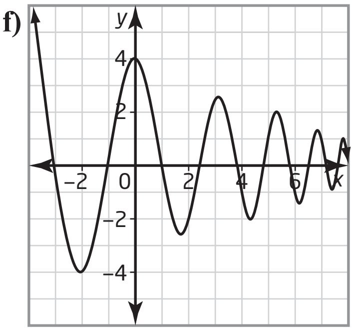

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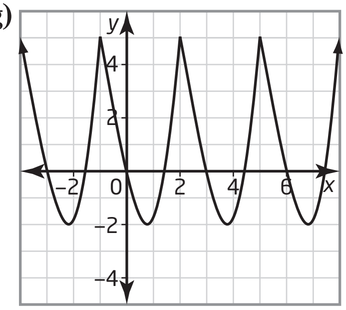

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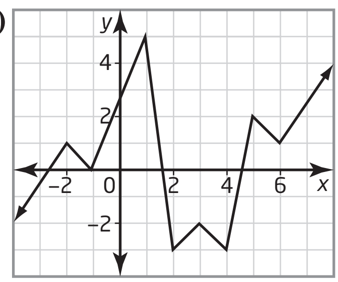


2. Determine the amplitude and period for any graph in question 1 that is periodic.

3. Sketch four cycles of a periodic function with an amplitude of 5 and a period of 2.

4. Sketch five cycles of a periodic function with an amplitude of 3 and a period of 4.

5. Do your graphs in questions 3 and 4 match those of your classmates? Explain why or why not.

6. A periodic function $f(x)$ has a period of 7. The values of $f(-2)$, $f(0)$, and $f(3)$ are −4, 6, and 2, respectively. Predict the value of each of the following. If a prediction is not possible, explain why.
   a) $f(70)$
   b) $f(26)$
   c) $f(-32)$
   d) $f(80)$

7. **a)** Sketch the graph of a periodic function, $f(x)$, with a maximum value of 6, a minimum of −2, and a period of 4.

   b) Select a value a for x, and determine $f(a)$.

   c) Determine two other values, b and c, such that $f(a) = f(b) = f(c)$.

B Connect and Apply

☆8. Sunita draws a periodic function so that $f(3) = f(9)$. Can you conclude that the period of the function is 6? Justify your answer, including a diagram.

☆9. A search light on a rocky point on a lake flashes 2 s on, and 1 s off. After 2 flashes, the light stays off for an extra 2 s.

   a) Let a 2 represent "on" and a 0 represent "off." Sketch a graph with time on the horizontal axis to represent the flashing of the light. Include two cycles.

   b) Explain why this pattern may be considered periodic.

   c) What is the period of the pattern?

   d) What is the amplitude?

10. Which of the following values do you expect to follow a periodic pattern? Justify your answer for each case.

   a) a basketball that is thrown and bounces 6 times on the ground

   b) the intensity of a signal emitted by a rotating radar antenna in a submarine

   c) Sandria's height above the ground as she skips rope

11. **Use Technology** The cycle of ocean tides represents periodic behaviour and can be modelled with a periodic function. Each day, at various locations around the world, the height of the tide above the mean low-water level is recorded. Use the Internet to find data that represents the time and height of low tide and high tide for a particular ocean location that you may have visited or would like to visit. Graph your data and explain why they represent a periodic function.

12. Anoop is on the school track team. Each day he practises by running back and forth on a straight track that is 800 m long. The beginning of the track is 25 m from the back entrance of the school. He runs at a constant speed of 8 km/h.

   a) Graph Anoop's distance, in metres, from the back entrance of the school, versus time, for 6 lengths of the track.

   b) Determine the period and amplitude of the pattern.

   c) State the domain and range.

C Extend

13. Describe the relationship between the period and the domain of a periodic function whose number of cycles is a whole number. Use an example to support your answer.

14. Describe the relationship between the range and the amplitude of a periodic function. Use an example to support your answer.

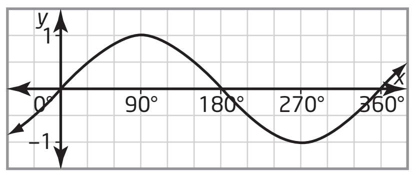
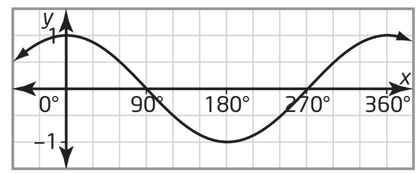
15. Is the following piecewise function periodic? Justify your answer.

$$f(x) = \begin{cases} x - 5, & 0 \le x < 10 \\ x - 15, & 10 \le x < 20 \\ x - 25, & 20 \le x < 30 \end{cases}$$

5.2 The Sine Function and the Cosine Function

KEY CONCEPTS

- The sine and cosine ratios, along with the unit circle, can be used to construct sine and cosine functions.
- Both the sine and cosine functions have a wave-like appearance, with a period of 360°.

Properties	$y = \sin x$	$y = \cos x$
sketch of graph		
maximum value	1	1
minimum value	−1	−1
amplitude	1	1
domain	$\{x \in \mathbb{R}\}$	$\{x \in \mathbb{R}\}$
range	$\{y \in \mathbb{R}, -1 \le y \le 1\}$	$\{y \in \mathbb{R}, -1 \le y \le 1\}$
x-intercepts	0°, 180°, and 360° over one cycle	90° and 270° over one cycle
y-intercept	0	1
intervals of increase (over one cycle)	$\{x \in \mathbb{R}, 0° \le x \le 90°, 270° \le x \le 360°\}$	$\{x \in \mathbb{R}, 180° \le x \le 360°\}$
intervals of decrease (over one cycle)	$\{x \in \mathbb{R}, 90° \le x \le 270°\}$	$\{x \in \mathbb{R}, 0° \le x \le 180°\}$

Example

Consider the function $y = \sin x - \cos x$.

a) Predict the y-intercept of the function.

b) Predict the x-intercepts from 0° to 720°. Justify your answer.

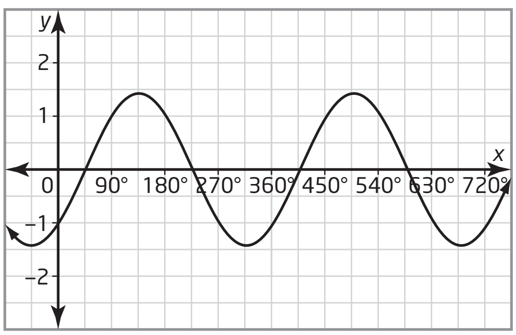
c) Use a graph or a graphing calculator to verify your answers to parts a) and b).

Solution

a) The y-intercept of the function $y = \sin x$ is 0, and the y-intercept of the function $y = \cos x$ is 1. So, the y-intercept of $y = \sin x - \cos x$ is $0 - 1$, or -1.

b) The x-intercepts occur when $y = 0$.
Solve $\sin x - \cos x = 0$, that is, where $\sin x = \cos x$.
This is true for x-values 45°, 225°, 405°, and 585°.

c)



From the graph, the y-intercept is -1 and the x-intercepts are 45°, 225°, 405°, and 585°.

B Connect and Apply

☆**1.** You are in a car of a Ferris wheel. The wheel has a radius of 10 m and turns counterclockwise. Let the origin be at the centre of the wheel. Begin each sketch in parts a) and b) when the radius from the centre of the wheel to your car is along the positive x-axis.

a) Sketch a graph of your horizontal displacement versus the angle through which you turn for one rotation of the wheel. Which function models the horizontal displacement? Justify your choice.

b) Sketch a graph of your vertical displacement versus the angle through which you turn for one rotation of the wheel. Which function models the vertical displacement? Justify your choice.

2. Refer to question 1. Suppose each sketch in parts a) and b) begins when the radius from the centre of the wheel to your car is along the negative y-axis. Sketch a graph of your horizontal displacement versus the angle through which you turn for one rotation of the wheel. Which function models the horizontal displacement? Justify your choice.

3. The hour hand on a clock has a length of 14 cm. Let the origin be at the centre of the clock.

   a) Sketch a graph of the vertical position of the tip of the hour hand versus the angle through which the hand turns for a time period of 48 h, assuming the hour hand starts at 9.

   b) Sketch a graph of the horizontal position of the tip of the hour hand versus the angle through which the hand turns for a time period of 48 h, assuming the hour hand starts at 3.

   c) How many cycles appear in the graph of part a)?

   d) How many cycles will appear in the graph of part a) if you use the minute hand rather than the hour hand? Explain your prediction.

C Extend

4. **a)** Predict what the graph of $y = -\sin x$ looks like. Use a table of values or technology to draw a sketch of the graph. Is your prediction correct?

   b) Sketch the graph of $y = \sin x$ on the same set of axes as the graph of $y = -\sin x$ from part a).

   c) Describe the similarities and differences between graph of $y = -\sin x$ and the graph of $y = \sin x$.

5. Repeat question 4 for the function $y = -\cos x$.

☆ 6. Consider the function $y = (\sin x)^2$.

   a) Predict the y-intercept of the function. Justify your prediction.

   b) Predict the x-intercepts from 0° to 720°. Justify your predictions.

   c) Predict the maximum value and the minimum value of the function. Justify your predictions.

   d) Predict the range and the amplitude of the function. Justify your predictions.

   e) Use a graph or a graphing calculator to verify your answers to parts a) to d).

   f) Describe the similarities and differences between the graph of $y = (\sin x)^2$ and the graph of $y = \sin x$.

7. Consider the function $y = (\cos x)^2$.

   a) Predict the y-intercept of the function.

   b) Predict the x-intercepts from 0° to 720°.

   c) Predict the maximum value and the minimum value of the function.

   d) Predict the range and the amplitude of the function.

   e) Use a graph or a graphing calculator to verify your answers to parts a) to d).

   f) Describe the similarities and differences between the graph of $y = (\cos x)^2$ and the graph of $y = \cos x$.

8. **a)** Predict how the graphs of $y = \frac{\sin x}{\cos x}$ and $y = \tan x$ are related. Justify your prediction.

   b) Graph the functions in part a) to verify your predictions.

9. Repeat question 8 for the functions $y = \frac{\cos x}{\sin x}$ and $y = \cot x$.

5.3 Investigate Transformations of Sine and Cosine Functions

KEY CONCEPTS

- The sine function may be transformed by introducing factors a, k, d, and c: $y = a \sin [k(x - d)] + c$
  - a determines the amplitude of the function. The amplitude is $|a|$.
  - k determines the period, p, of the function according to the relation $p = \frac{360°}{|k|}$.
  - d determines the horizontal translation, or phase shift, of the function. If d is positive, the shift is to the right. If d is negative, the shift is to the left.
  - c determines the vertical translation, or vertical shift, of the function. If c is positive, the shift is up. If c is negative, the shift is down.
- The cosine function may be transformed in the same way: $y = a \cos [k(x - d)] + c$. You will work through examples involving the cosine function in the exercises.

Example

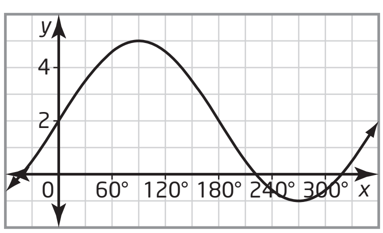
a) Write an equation in the form $y = a \sin [k(x - d)] + c$ that models each graph with the following properties.

i) The graph of $y = \sin x$ is reflected in the x-axis, has a phase shift of 20° to the left, and a period of 360°. The range of the graph is $\{y \in \mathbb{R}, 1 \le y \le 5\}$.

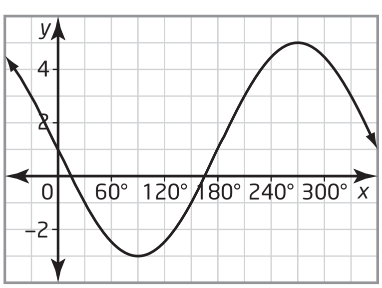
ii) The maximum value of the graph is 1, the minimum value is −7, the period is 180°, and the phase shift is 0.

b) Write an equation in the form $y = a \sin x + c$ that models each graph.

i)



ii)



Solution

a) i) The range of the graph is used to find the amplitude and the vertical translation. Since the y-values extend from 1 to 5, the amplitude is $\frac{5-1}{2}$, or 2. Since the graph is reflected in the x-axis, $a = -2$.

The graph of $y = -2 \sin x$ extends from -2 to 2, but this graph extends from 1 to 5. Translate the graph of $y = -2 \sin x$ up 3 units. Therefore, $c = 3$.

The phase shift is 20° to the left, so $d = -20°$.

The period is 360°, so $k = 1$.

An equation that models this graph is $y = -2 \sin (x + 20°) + 3$.

ii) The minimum and maximum values indicate the range of the graph. Use this information to find the amplitude and the vertical shift.

The amplitude is $\frac{1-(-7)}{2}$, or 4. So, $a = 4$.

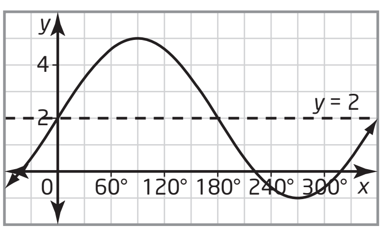
The graph of $y = 4 \sin x$ extends from -4 to 4. Since the maximum value of the given graph is 1, it must be shifted down 3 units. So, $c = -3$.

Since the period is 180°, then $\frac{360°}{k}$ and $k = 2$.

The phase shift is 0, so $d = 0$.

An equation that models this graph is $y = 4 \sin 2x - 3$.

b) i) The amplitude and vertical shift can be determined by finding the horizontal reference line that cuts the graph in half. This represents the middle or rest position of the graph. It occurs at $y = 2$.

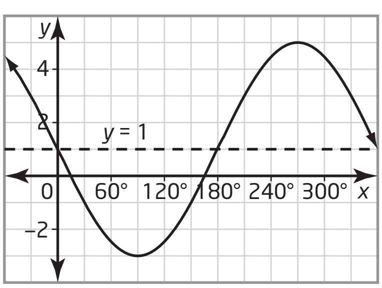


This means that the vertical shift is 2 units up, so $c = 2$.

The maximum value is 3 units above this horizontal line and the minimum value is 3 units below this horizontal line. So, the amplitude of the graph is $a = 3$.

An equation that models this graph is $y = 3 \sin x + 2$.

ii) The horizontal reference line is $y = 1$, as shown.



This means that the vertical shift is 1 unit up, so $c = 1$.

The maximum value is 4 units above this horizontal line and the minimum value is 4 units below this horizontal line. So, the amplitude of the graph is 4. Since the graph is reflected in the x-axis, $a = -4$.

An equation that models this graph is $y = -4 \sin x + 1$.

A Practise

1. Sketch one cycle for each function. Include an appropriate scale on each axis. State the vertical stretch or compression and amplitude of the function.

a) $y = 5 \sin x$

b) $y = \frac{4}{5} \sin x$

c) $y = -4 \sin x$

d) $y = -\frac{4}{3} \sin x$

2. Sketch one cycle for each function. Include an appropriate scale on each axis. State the vertical stretch or compression and the amplitude of the function.

a) $y = 6 \cos x$

b) $y = \frac{3}{2} \cos x$

c) $y = -3 \cos x$

d) $y = -\frac{1}{2} \cos x$

3. Determine the horizontal stretch or compression and the period of each function.

a) $y = 3 \sin 2x$

b) $y = 4 \sin \frac{1}{2}x$

c) $y = 2 \sin \frac{2}{3}x$

d) $y = -5 \sin \frac{1}{4}x$

e) $y = 2 \cos 3x$

f) $y = -6 \cos 5x$

g) $y = -\frac{3}{4} \cos \frac{3}{4}x$

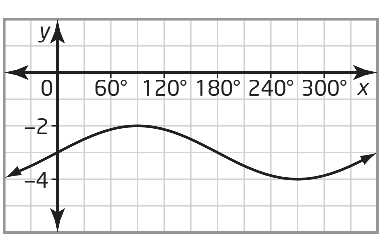
h) $y = \frac{7}{4} \cos \frac{3}{2}x$

☆**4.** Match each graph with its corresponding equation. Give reasons for your choices.

A $y = \sin x + 2.5$

B $y = \sin x - 3$

a)



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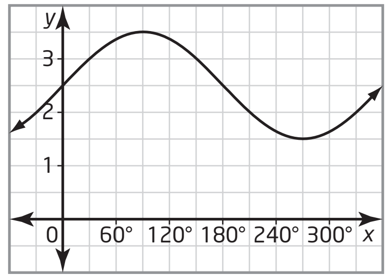
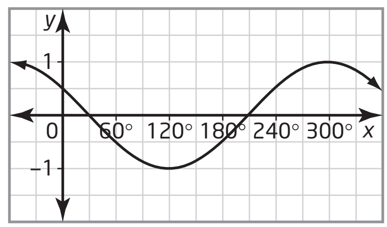


5. Match each graph with its corresponding equation. Give reasons for your choices.

A $y = \cos (x - 30°)$

B $y = \cos (x + 60°)$

a)



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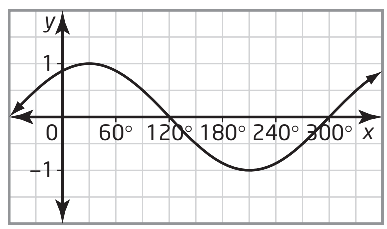


6. Determine the phase shift and the vertical shift with respect to $y = \sin x$ for each function.

a) $y = \sin (x - 40°) + 2$

b) $y = 2 \sin (x + 60°) - 3$

c) $y = -3 \sin (x - 38°) + 5$

d) $y = 4 \sin [3(x + 30°)] - 6$

7. Determine the phase shift and the vertical shift with respect to $y = \cos x$ for each function.

a) $y = \cos (x + 70°)$

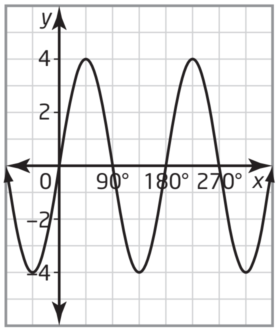
b) $y = 7 \cos (x - 82°) + 8$

c) $y = -5 \cos (x + 100°) - 1$

d) $y = 10 \cos [3(x - 120°)] + 9$

8. Write two equations, one in the form $y = a \sin kx$ and one in the form $y = a \cos [k(x - d)]$, to match each graph.

a)



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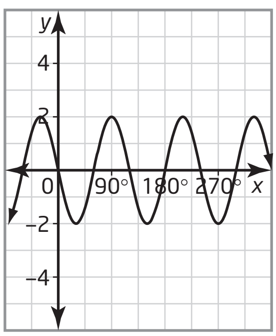

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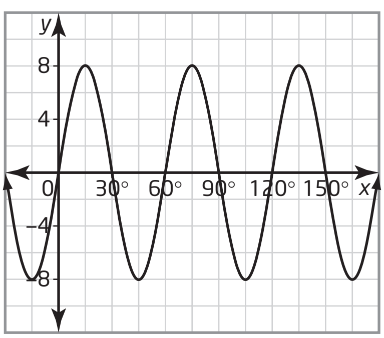
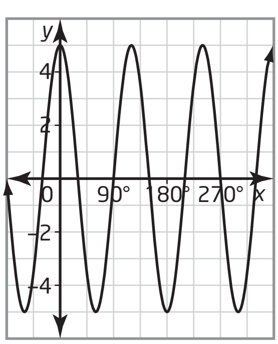


9. Write two equations, one in the form $y = a \cos kx$ and one in the form $y = a \sin [k(x - d)]$, to match each graph.

a)



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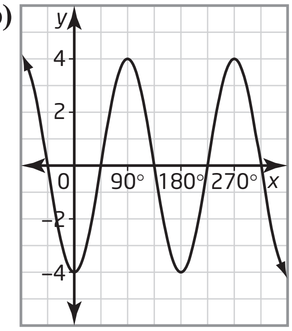

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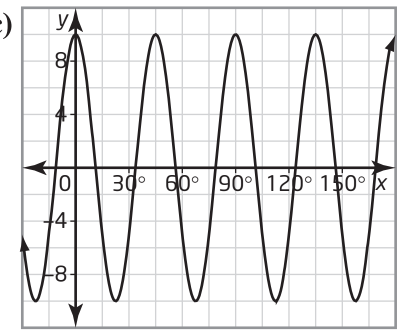


10. a) State the phase shift and the vertical shift of each sinusoidal function.

i) $y = \sin (x + 140°) + 5$

ii) $y = 3 \sin x + 2$

iii) $y = \sin (x + 55°) - 8$

iv) $y = 5 \sin (x - 90°) + 7$

b) Sketch one cycle of the graph of each function. Include an appropriate scale on each axis.

11. a) State the vertical shift and the amplitude of each sinusoidal function.

i) $y = \cos (x - 80°)$

ii) $y = 3 \cos x - 2$

iii) $y = \cos (x + 25°) + 1$

iv) $y = 5 \cos (x - 135°) - 4$

b) Sketch one cycle of the graph of each function. Include an appropriate scale on each axis.

B Connect and Apply

☆ **12.** The vertical position, y, in centimetres, of a point on the rim of the wheel of a stationary spinning bicycle after time, t, in seconds, can be modelled by the equation $y = 30 \sin 540t + 45$.

a) What is the lowest vertical position that the point reaches? Justify your answer.

b) What is the highest vertical position that the point reaches? Justify your answer.

c) What is the period of rotation of the wheel, in seconds?

d) Suppose that the period of the rotation of the wheel doubles. How does the equation change? Justify your answer.

13. What transformations must be applied to the graph of $y = \sin x$ to obtain the graph of each of the following? Justify your answer.

a) $y = \sin (x + 90°) - 2$

b) $y = -\sin (x - 60°) + 4$

☆ **14.** Write an equation in the form $y = a \sin (x - d) + c$ for each of the following. Justify your answer.

a) The graph of $y = \sin x$ is shifted 25° to the left and 5 units up.

b) The graph of $y = \sin x$ is reflected in the x-axis, and shifted 42° to the right and 2 units down.

15. Determine the amplitude, the period, the phase shift, and the vertical shift of each function.

a) $y = 3 \sin [2(x - 30°)] + 1$

b) $y = \frac{1}{2} \cos [3x + 360°] - 6$

c) $y = 4 \sin \left[\frac{1}{4}(x + 45°)\right] - 3$

d) $y = 0.6 \cos [1.2(x - 90°)] + 2$

C Extend

16. The siren of an alarm system is set up so that the frequency of the sound fluctuates with time. In a 12-s interval, the maximum frequency of 1100 Hz is heard four times, including at 0 s and 12 s. The minimum frequency of 120 Hz is heard three times. Determine a cosine function that models the frequency, f, in Hertz, of the siren in terms of time, t, in seconds.

17. A performer in a circus spins a lasso in a circle perpendicular to the ground. The height, h, in metres, of the knot above the ground is modelled by the function $h = -1.2 \cos (300t) + 1.5$, where t is the time, in seconds.

a) What is the maximum height of the knot?

b) What is the minimum height of the knot?

c) What is the period of this function?

d) Determine the height of the knot after 20 s.

18. A child's toy consists of a rubber kangaroo on a spring that is attached to a wooden platform. When the toy kangaroo is pulled down toward the base of the platform and then released, it bounces up and down. Its height, h, in centimetres, above the base after t seconds is modelled by the function $h(t) = 10 \sin [360t + 540] + 15$.

a) What is the maximum height of the kangaroo?

b) What is the minimum height of the kangaroo?

c) What is the period of this function?

d) When is the kangaroo 18 cm above the base of the platform?

5.4 Graphing and Modelling with $y = a \sin [k(x - d)] + c$ and $y = a \cos [k(x - d)] + c$

KEY CONCEPTS

- The amplitude, period, phase shift, and vertical shift of sinusoidal functions can be determined when the equations are given in the form $f(x) = a \sin [k(x - d)] + c$ or $f(x) = a \cos [k(x - d)] + c$.
- The domain of a sinusoidal function is $\{x \in \mathbb{R}\}$. The range extends from the minimum value to the maximum value of the function. Any cycle can be used to determine the minimum and the maximum.
- Transformations can be used to adjust the basic sine and cosine functions to match a given amplitude, period, phase shift, and vertical shift.
- The equation of a sinusoidal function can be determined given its properties.
- The equation of a sinusoidal function can be determined given its graph.

Example

a) State the values of the parameters in the transformed function $y = -\frac{4}{5} \sin \left[\frac{2}{3}(x + 60°)\right] + 1$.

b) Describe the transformations that must be applied to the graph of $y = \sin x$ to obtain the graph of $y = -\frac{4}{5} \sin \left[\frac{2}{3}(x + 60°)\right] + 1$.

c) State the period and the range of the transformed function in part a).

Solution

a) Comparing the given equation $y = -\frac{4}{5} \sin \left[\frac{2}{3}(x + 60°)\right] + 1$ to the general equation $y = a \sin [k(x - d)] + c$ gives $a = -\frac{4}{5}$, $k = \frac{2}{3}$, $d = -60°$, and $c = 1$.

b) Since $a = -\frac{4}{5}$ is negative, the graph of $y = \sin x$ is reflected in the x-axis and the amplitude is $\frac{4}{5}$.

Since $k = \frac{2}{3}$, the graph is horizontally stretched by a factor of $\frac{3}{2}$.

Since $d = -60°$ and $c = 1$, the graph is shifted left 60° and up 1 unit.

c) The period is $\frac{360°}{\frac{2}{3}}$, or 540°.

Since the graph is shifted up 1 unit and the amplitude is $\frac{4}{5}$, the range is $\left\{y \in \mathbb{R}, -\frac{1}{5} \le y \ge \frac{9}{5}\right\}$.

A Practise

1. Determine the amplitude, the period, the phase shift, and the vertical shift of each function with respect to $y = \sin x$.

   a) $y = 2 \sin [3(x - 15°)] + 4$

   b) $y = -5 \sin [12(x + 60°)] - 2$

   c) $y = 6 \sin [18(x + 45°)] + 1$

   d) $y = \frac{2}{3} \sin \left[\frac{3}{5}(x - 30°)\right] + \frac{4}{5}$

2. Determine the amplitude, the period, the phase shift, and the vertical shift of each function with respect to $y = \cos x$.

   a) $y = -\frac{4}{7} \cos [10(x + 16°)] + 9$

   b) $y = 11 \cos [36(x + 75°)] - 3$

   c) $y = 8 \cos [60(x - 12°)] + 7$

   d) $y = -\frac{5}{9} \cos \left[\frac{2}{5}(x - 85°)\right] + \frac{3}{8}$

3. **a)** Describe the transformations that must be applied to the graph of $f(x) = \sin x$ to obtain the graph of $g(x) = 5 \sin 3x - 3$. Apply each transformation, one step at a time, to sketch the graph of $g(x)$.

   b) State the domain and range of $f(x)$ and $g(x)$.

   c) Modify the equation for $g(x)$ to include a phase shift of 45° to the left. Call this function $h(x)$. Apply the phase shift to the graph of $g(x)$ and transform it to $h(x)$.

4. **a)** Transform the graph of $f(x) = \cos x$ to $g(x) = 3 \cos 4x + 1$ by applying transformations to the graph one step at a time.

   b) State the domain and range of $f(x)$ and $g(x)$.

   c) Modify the equation for $g(x)$ to include a phase shift of 45° to the right. Call this function $h(x)$. Apply the phase shift to the graph of $g(x)$ and transform it to $h(x)$.

5. A sinusoidal function has an amplitude of 6 units, a period of 180°, and a maximum at (0, 2).

   a) Represent the function with an equation using a sine function.

   b) Represent the function with an equation using a cosine function.

6. A sinusoidal function has an amplitude of $\frac{3}{4}$ units, a period of 1080°, and a maximum at $\left(0, \frac{11}{4}\right)$.

   a) Represent the function with an equation using a sine function.

   b) Represent the function with an equation using a cosine function.

B Connect and Apply

7. Consider the function $f(x) = 8 \sin [3x + 60°] - 8$.

   a) Determine the amplitude, the period, the phase shift, and the vertical shift of the function with respect to $y = \sin x$.

   b) What are the maximum and minimum values of the function?

   c) Determine the first three x-intercepts to the right of the origin.

   d) Determine the y-intercept of the function.

8. Consider the function $g(x) = 6 \cos [5(x - 60°)]$.

   a) Determine the amplitude, the period, the phase shift, and the vertical shift of the function with respect to $y = \cos x$.

   b) What are the maximum and minimum values of the function?

   c) Determine the first three x-intercepts to the right of the origin.

   d) Determine the y-intercept of the function.

9. **Use Technology** Use a graphing calculator or graphing software to verify your answers to questions 7 and 8.

☆**10.** Write an equation in the form $y = a \sin [k(x - d)] + c$ that represents the graph of $y = \sin x$ after it is reflected in the x-axis, vertically stretched by a factor of 3, horizontally compressed by a factor of $\frac{1}{4}$, shifted 35° to the left, and translated 8 units down.

11. a) Transform the graph of $f(x) = \sin x$ to $g(x) = -3 \sin \left[\frac{1}{4}(x - 50°)\right] + 7$. Show each step in the transformation.

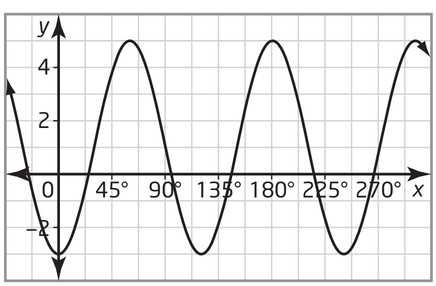
b) State the domain and range of $f(x)$ and $g(x)$.

12. a) Transform the graph of $f(x) = \cos x$ to $g(x) = \frac{3}{4} \cos [6(x + 45°)] - 2$. Show each step in the transformation.

b) State the domain and range of $f(x)$ and $g(x)$.

☆**13.** Represent the graph of $f(x) = 6 \sin [5(x - 60°)]$ with an equation using a cosine function.

14. a) Determine the equation of a sine function that represents the graph shown.



b) Determine the equation of a cosine function that represents the graph.

15. At the end of a dock, high tide of 16 m is recorded at 9:00 a.m. Low tide of 6 m is recorded at 3:00 p.m. A sinusoidal function can model the water depth versus time.

a) Construct a model for the water depth using a cosine function, where time is measured in hours past high tide.

b) Construct a model for the water depth using a sine function, where time is measured in hours past high tide.

c) Construct a model for the water depth using a sine function, where time is measured in hours past low tide.

d) Construct a model for the water depth using a cosine function, where time is measured in hours past low tide.

e) Compare your models. Which is the simplest representation if time is referenced to high tide? low tide? Explain why there is a difference.

C Extend

16. Consider the relation $y = \sqrt{\cos x}$.

a) Sketch the graph of the function $y = \cos x$ over two cycles.

b) Use the graph from part a) to sketch a prediction for the shape of the graph of $y = \sqrt{\cos x}$.

c) Use Technology Use technology to graph $y = \sqrt{\cos x}$ and to check your prediction. Resolve any differences.

d) Predict how the graph of $y = \sqrt{\cos x + 1}$ differs from the graph of $y = \sqrt{\cos x}$.

e) Graph $y = \sqrt{\cos x + 1}$ and compare it to your prediction. Resolve any differences.

17. Consider the function $y = a \sin [k(x - d)] + c$.

a) Write an expression for the maximum value of y. For what values of x does this occur?

b) Write an expression for the minimum value of y. For what values of x does this occur?

18. Repeat question 17 for $y = a \cos [k(x - d)] + c$.

5.5 Data Collecting and Modelling

KEY CONCEPTS

- Data can be collected from physical models using tools such as a motion sensor.
- Data can be downloaded from statistical sources such as Statistics Canada.
- Data can sometimes be modelled using a sinusoidal function.
- Use a graph or a table to build a model to determine the amplitude, phase shift, period, and vertical shift of a sinusoidal function.
- Predictions about the behaviour of an altered model can be made by considering the effect of changing a parameter on the graph of the original equation.
- The graph or equation can be used to determine values.

Example

The table shows the daily high temperature, as recorded every 30 days, for the town of Mathville, for two years beginning January 1, 2007.

Day Number	High Temperature (°C)
0	−1.9
30	0.1
60	5.5
90	13.1
120	20.5
150	25.9
180	27.9
210	26.0
240	20.3
270	13.2
300	5.6
330	0.2
360	−1.9
390	0.1
420	5.5
450	13.1
480	20.5
510	25.9
540	27.9
570	26.0
600	20.3
630	13.2
660	5.6
690	0.2
720	−1.9

a) Draw a graph to represent the data.

b) Does the graph model a periodic function? Explain.

c) Determine the amplitude of the function.

d) Determine the period of the function.

e) Explain how your graph can be used to predict the daily high temperature for another cycle of 360 days.

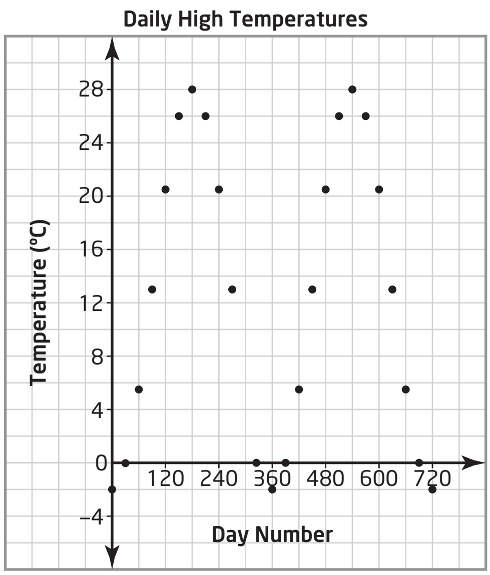
f) Extend your graph. Predict the daily high for day 840. Will this be the actual daily high temperature for day 840? Explain.

g) Construct a model for the daily high temperature by writing an equation using a sine function.

h) Construct a model for the daily high temperature by writing an equation using a cosine function.

Solution

a)



b) The graph models a periodic function because the values along the vertical axis are repeated. So, the graph contains a repeating pattern.

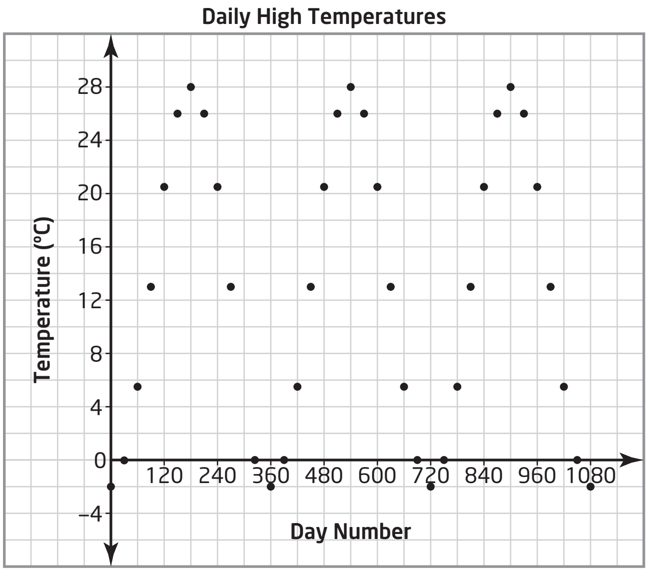
c) From the data, the minimum value is −1.9 and the maximum value is 27.9.

The amplitude is $\frac{27.9 - (-1.9)}{2}$, or 14.9.

d) The first cycle begins at day 0 and ends at day 360. So, the period is 360 days.

e) The graph can be extended by drawing another cycle for the next 360 days. The values on the graph can then be used to predict the daily high temperature for 30-day periods.

f) From the graph, the predicted daily high temperature for day 840 is approximately 20 °C. There is no guarantee that this will be the actual daily high temperature for day 840, since weather patterns may change.



g) Let n represent the day number and T represent the temperature, in degrees Celsius.

Since the amplitude, a, is 14.9 and the maximum is 27. 9, the vertical shift, c, is 13. The sine wave starts at day 90, so the phase shift, d, is 90. From part d), the period is 360 days. So, $k = 1$. A sine equation that models this function is $T = 14.9 \sin (n - 90) + 13$.

h) The amplitude, the period, and the vertical shift remain the same as in part g). The start of the first cosine wave is at day 180, so the phase shift, d, is 180 to the right.
A cosine equation that models this function is $T = 14.9 \cos (n - 180) + 13$.

A Practise

1. The height, h, in metres, of the tide in a given location on a given day at t hours after midnight is modelled using the sinusoidal function
$h(t) = 4.5 \sin [30(t - 4)] + 6$.

a) Find the maximum and minimum values for the depth, h, of the water.

b) What time is high tide? What time is low tide?

c) What is the depth of the water at 11:00 a.m.?

d) Find all the times during a 24-h period when the depth of the water is 6 m.

2. The number of tourists, V, visiting a popular city is modelled using the function $V = 3500 \sin [30(t - 8)] + 5600$, where t is the number of months after New Year's Day.

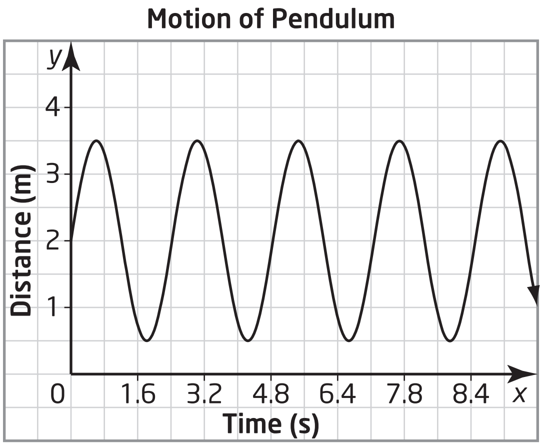
a) Find the maximum and minimum number of tourists visiting the city over the period of a year.

b) When is the number of tourists a maximum? When is it a minimum?

c) How many tourists visit the city on July 30?

d) When is the number of tourists visiting the city about 3000?

3. A motion sensor is used to gather data on the motion of a pendulum. The table of values is exported to a computer, and graphing software is used to draw the graph shown.



a) Use the graph to estimate the maximum and minimum values. Then, use these values to find the amplitude, a.

b) Sketch a horizontal reference line. Estimate the vertical shift, c.

c) Use the horizontal reference line to estimate the phase shift, d.

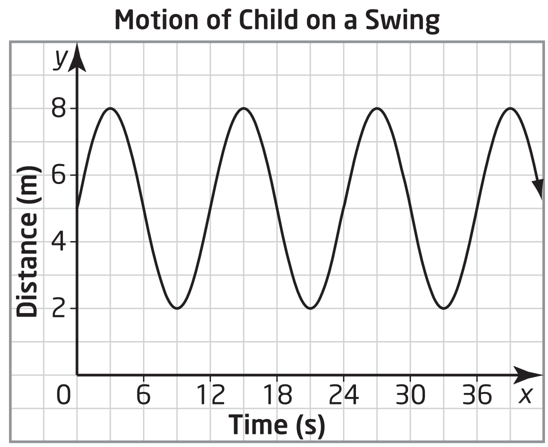
d) Use the horizontal reference line to estimate the period. Use the period to find the value of k.

e) Construct a model for the motion by writing an equation using a sine function.

f) Construct a model for the motion by writing an equation using a cosine function.

g) **Use Technology** Use technology to graph your models from parts e) and f). Compare your models to the graph shown. If you see any significant difference, check and adjust your model.

☆ 4. A motion sensor is used to gather data on the motion of a child on a swing. The table of values is exported to a computer, and graphing software is used to draw the graph shown.



a) Use the graph to estimate the maximum and minimum values. Then, use these values to find the amplitude, a.

b) Sketch a horizontal reference line. Estimate the vertical shift, c.

c) Use the horizontal reference line to estimate the phase shift, d.

d) Use the horizontal reference line to estimate the period. Use the period to find the value of k.

e) Construct a model for the motion by writing an equation using a sine function.

f) Construct a model for the motion by writing an equation using a cosine function.

g) **Use Technology** Use technology to graph your models from parts e) and f). Compare your models to the graph shown. If you see any significant difference, check and adjust your model.

B Connect and Apply

☆**5.** The height above the ground of a rider on a Ferris wheel can be modelled by the sine function $h(x) = 25 \sin (x - 90°) + 27$, where $h(x)$ is the height, in metres, and x is the angle, in degrees, that the radius to the rider makes with the horizontal.

a) Complete the table for one revolution of the Ferris wheel.

x	$h(x) = 25 \sin (x - 90°) + 27$
0°	
30°	
60°	
90°	
120°	
150°	
180°	
210°	
240°	
270°	
300°	
330°	
360°	

b) Predict the values for a second revolution of the Ferris wheel. Explain your reasoning.

x	$h(x) = 25 \sin (x - 90°) + 27$
390°	
420°	
450°	
480°	
510°	
540°	
570°	
600°	
630°	
660°	
690°	
720°	

c) Graph the function for two revolutions.

d) Determine the maximum and minimum heights of the rider.

e) Use the graph to predict the measures of the angle when the height of the rider is 40 m.

f) Use the cosine function to write an equation to model the height of the rider on the Ferris wheel.

6. The period of a pendulum, T, in seconds, is related to the length, l, in metres, according to the relation $T = 2\pi\sqrt{\frac{l}{g}}$, where g is the acceleration due to gravity, about 9.8 m/s^2, near the surface of Earth.

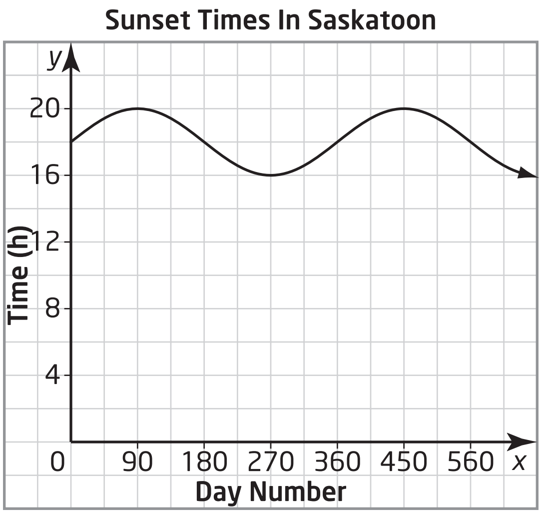
a) If the length is tripled, by what factor does the period increase?

b) If you want a pendulum with double the period of the given pendulum, what must you do to the length?

☆**7.** **a)** The graph shows the time of sunset in Saskatoon over a 720-day period. Determine a sine equation that models the time of sunset. Justify your answer.

b) Determine a cosine equation that models the time of sunset.

c) State the range of the function.



8. The Ferris wheel at a carnival has a diameter of 18 m and descends to 3 m above ground level at its lowest point. Assume that a rider enters a car from a platform that is located 40° around the rim before the car reaches its lowest point.

a) Model the rider's height above ground versus angle using a transformed sine function.

b) Model the rider's height above ground versus angle using a transformed cosine function.

c) Suppose that the platform is moved to 50° around the rim from the lowest position of the car. How will the equations in parts a) and b) change? Write the new equations.

9. Suppose that the axle of the Ferris wheel in question 8 is moved upward 2.5 m, but the platform is left in place at a point 40° before the car reached its lowest point. How do the equations in parts a) and b) of question 8 change? Write the new equations.

10. The movement of a piston in the engine of a certain vehicle can be modelled by the function $y = 45 \sin (9000t) + 18$, where y is the distance, in centimetres, from the crankshaft and t is the time, in seconds.

a) What is the period of the motion?

b) Determine the maximum, the minimum, and the amplitude.

c) When do the maximum and minimum values occur?

d) What is the vertical position of the piston at $t = \frac{1}{20}$ s?

C Extend

11. The diameter of a motorcycle tire is 60 cm. While the motorcycle is being driven, a small sharp stone lodges itself in one of the grooves of the tire. How high above the ground is the stone 1 km after the motorcycle hit the stone?

12. Aubrey is riding on a Ferris wheel at a constant rate of 10 km/h. He boards the Ferris wheel from a platform that is 1.5 m high. The diameter of the wheel is 16 m. Determine an equation, in terms of the cosine function, that models Aubrey's height above the ground given that Aubrey begins at the highest point on the Ferris wheel.

13. **a)** High-voltage electricity with a maximum of 1100 V is transmitted at a frequency of 260 Hz, or 260 cycles per second. Determine a sinusoidal equation that represents the voltage as a function of time.

b) A transformer station reduces the voltage by half and the frequency by one quarter. Determine a sinusoidal equation for the new voltage.

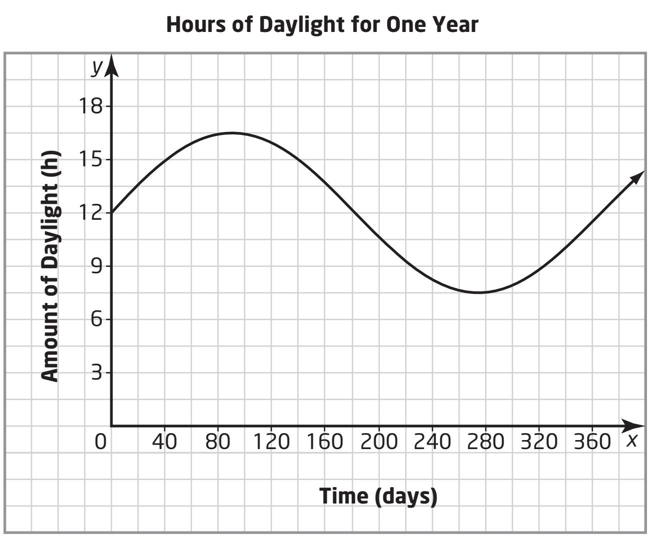
5.6 Use Sinusoidal Functions to Model Periodic Phenomena Not Involving Angles

KEY CONCEPTS

- Sinusoidal functions can be used to model periodic phenomena that do not involve angles as the independent variable.
- The amplitude, phase shift, period, and vertical shift of the basic sine or cosine function can be adjusted to fit the characteristics of the phenomenon being modelled.
- Technology can be used to quickly draw and analyse the graph modelled by the equation.
- The graph can be used to solve problems related to the phenomenon.

Example

The number of hours of daylight varies throughout the year. The graph shows the amount of daylight in southern Ontario over one year, starting on March 21, the first day of spring, which has 12 h of daylight. December 21 is the "shortest day," and June 21 is the "longest day."



a) What is the number of daylight hours on December 21?
b) What is the number of daylight hours on June 21?
c) Determine the amplitude, the phase shift, the vertical shift, and the period of the graph.
d) Determine a sinusoidal equation that models the graph. Justify your answer.

Solution

a) December 21 is the shortest day, so it corresponds to the minimum number of daylight hours, which is approximately 7.5 h.

b) June 21 is the longest day, so it corresponds to the maximum number of daylight hours, which is approximately 16.5 h.

c) The amplitude is $\frac{16.5 - 7.5}{2}$, or 4.5. Therefore $a = 4.5$. Alternatively, determine the distance between the horizontal reference line and the maximum or minimum value.

Here the horizontal reference line is $y = 12$. Then, the distance from 12 to 16.5 is 4.5. There is no phase shift since the graph begins at the y-axis. So, $d = 0$.
One way to determine the vertical displacement is to subtract the amplitude from the maximum value, that is, $16.5 - 4.5$, or 12. Another way to find the vertical displacement is to add the amplitude to the minimum value, that is, $7.5 + 4.5$, or 12. Therefore, $c = 12$.
The period of the graph is 365 days, since the pattern of number of daylight hours repeats on a yearly basis.

d) The equation will be of the form $y = a \sin [k(x - d)] + c$. Use the period to determine the value of k.

$$\frac{360}{k} = 365$$
$$k = \frac{360}{365}$$
$$k = \frac{72}{73}$$

Substitute $a = 4.5$, $k = \frac{72}{73}$, $d = 0$, and $c = 12$.

An equation that models the number of daylight hours in southern Ontario is
$y = 4.5 \sin \frac{72}{73}x + 12$.

A Practise

1. The sinusoidal function $h(t) = 7 \sin [30(t - 2.5)]$ models the height, h, of tides in a particular location on a particular day at t hours after midnight.

a) Determine the maximum height and the minimum height of the tides.

b) At what times do high tide and low tide occur?

c) Use a cosine function to write an equivalent equation.

2. Refer to question 1. On a different day, the maximum height is 4.5 m, the minimum is −4.5 m, and low tide occurs at 5:00 a.m.

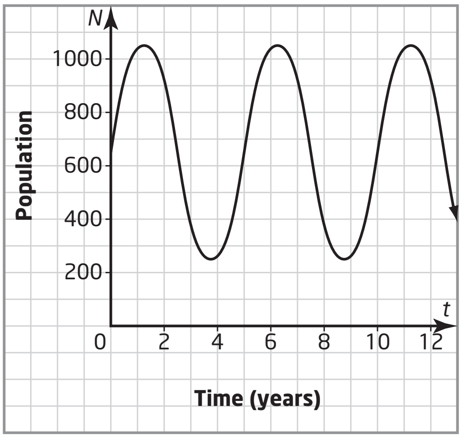
a) Modify the sine function so that it matches the new data.

b) Predict the times for the next low and high tides.

c) Modify your equation from part a) so that low tide occurs at 5:30 a.m.

d) Write an equivalent equation using a cosine function for your answer to part c).

3. The population of prey in a predator-prey relation is shown. Time is in years since 1985.



a) Determine the maximum and minimum values of the population. Use these to find the amplitude.

b) Determine the vertical shift, *c*.

c) Determine the phase shift, *d*.

d) Determine the period. Use the period to determine the value of *k*.

e) Model the population versus time with a sinusoidal function.

f) Graph your function. Compare it to the graph shown.

4. Refer to question 3. Suppose the period was 8 years. How would your equation in part e) change? Write the new equation.

B Connect and Apply

☆**5.** Refer to the Example at the beginning of this section. Use the equation you found in part d) to determine the number of hours of daylight on

a) April 1

b) September 1

☆**6.** The depth of the water at the end of a pier is 2 m at low tide and 12 m at high tide. There are 6 h between low tide and high tide. The first high tide occurs 6 h after midnight. One complete cycle takes 12 h.

a) Write a sinusoidal equation that represents the depth of the water.

b) Determine the depth of the water at 4:15 a.m. and 3:30 p.m.

c) Graph your equation from part a) and plot the points represented by your answers in part b).

7. The depth of water, *d*, in metres, of a seaside inlet on a given day can be modelled by the function $d = 7 \sin [30(t - 4)] + 11$, where *t* is the time past midnight, in hours.

a) Determine the maximum and minimum depths of water in the inlet.

b) What is the period between maximum values?

c) Graph the water level over a period of 24 h.

d) When is the water 5 m deep?

8. Angelina constructs a model alternating current (AC) generator in physics class and cranks it by hand at 5 revolutions per second. She is able to light up a flashlight bulb that is rated for 8 V.

a) What is the period of the AC produced by the generator?

b) Determine the value of *k*.

c) What is the amplitude of the voltage function?

d) Model the voltage with a suitably transformed sine function.

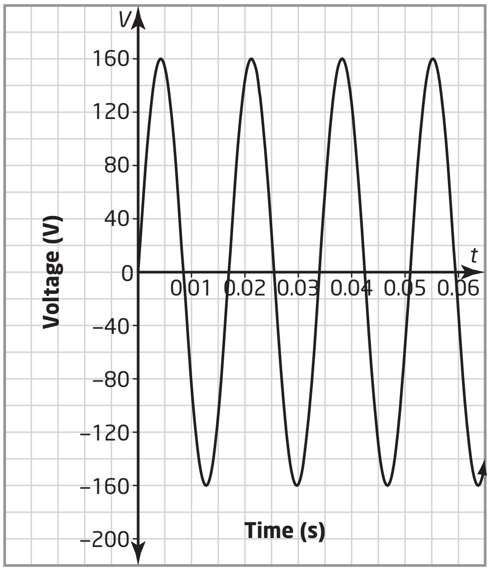
9. The number of sunlight hours on each day of the year for a particular location can be modelled by a periodic function. In London, Ontario, the maximum number of sunlight hours, 15.3 h, occurs on June 21. The minimum number of sunlight hours, 9.3 h, occurs on December 21.

a) Use a cosine function to model the number of sunlight hours in London, Ontario. Note that June 21 is day 172 and December 21 is day 355.

b) Determine the number of sunlight hours on April 1.

c) Determine a day when there are 14 h of sunlight.

10. The graph shows how the voltage, V, in volts, varies with time, t, in seconds, for the electricity provided to an electrical appliance.



a) Identify the period and amplitude of the function.

b) What is the maximum voltage?

c) Write an equation to model this situation.

11. The table shows the temperature readings every 2 h over a 24-h period on an early summer day.

Time	Temperature (°C)
13:00	27.3
15:00	28.9
17:00	27.8
19:00	26.0
21:00	22.0
23:00	18.1
01:00	16.0
03:00	15.1
05:00	16.1
07:00	18.2
09:00	22.1
11:00	25.5
13:00	27.4

a) Use the data to write an equation that models the temperature over the 24-h period.

b) Explain how you can check the accuracy of your equation.

c) Use your equation to predict the temperature at the following times:

i) 04:00
ii) 16:00
iii) 20:30

12. The blades on a windmill turn at a frequency of 20 revolutions per minute. The length of each blade is 3.5 m, and the tips of the blades are 8 m off the ground when they are at their lowest point.

a) Use a sinusoidal function to model the height above the ground of one of the blade tips as a function of time.

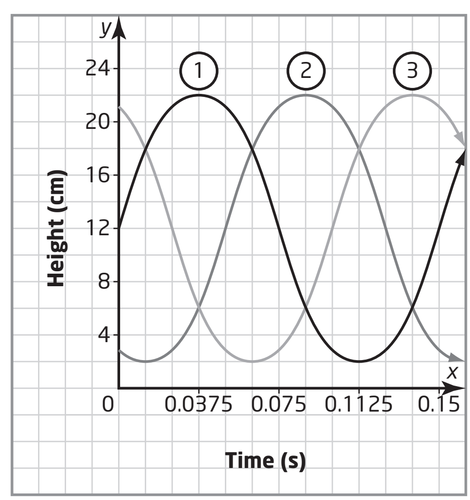
b) Graph the function.

c) Determine the times in the first cycle when the tip of a blade is 10 m off the ground.

C Extend

13. Pistons in a vehicle engine move up and down many times each second. The graph shows how the height varies with time for the pistons in a 6-cylinder engine.

Pistons in 6-Cylinder Engine



a) Determine an equation for each graph.

b) Determine the frequency of the motion.

Chapter 5 Review

5.1 Modelling Periodic Behaviour

1. Which of the following values do you expect to follow a periodic pattern? Justify your answer for each case.
   a) the distance from the centre as a grandfather clock pendulum swings back and forth
   b) the cost of sending a package by courier, which varies depending on the weight of the package
   c) the value of a stock over a 1-year period

2. **a)** Sketch a periodic function, $f(x)$, with a maximum value of 8, a minimum of −5, and a period of 3.
   b) Select a value a for x, and determine $f(a)$.
   c) Determine two other values, b and c, such that $f(a) = f(b) = f(c)$.

5.2 The Sine Function and the Cosine Function

3. **a)** Copy and complete the table of values for $y = \sin 2x$.

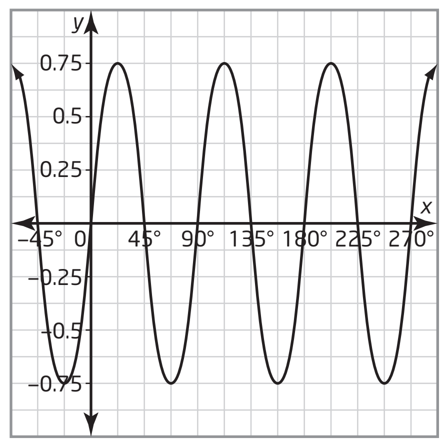
x	$2x$	$y = \sin 2x$
0°		
45°		
90°		
135°		
180°		
225°		
270°		
315°		
360°		

   b) Use the table values to sketch a graph of $y = \sin 2x$. On the same set of axes, sketch a graph of $y = \sin x$
   c) Compare the graphs in part b). Describe the similarities and differences.

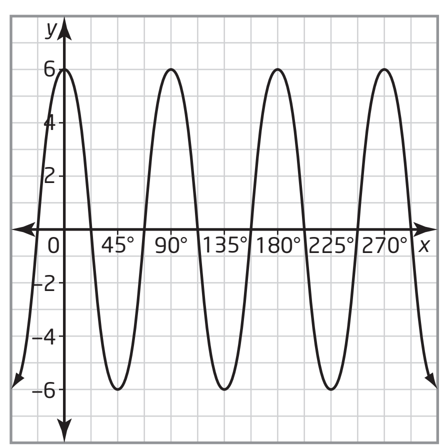
4. **a)** Predict what the graph of $y = -\csc x$ looks like. Use a table of values or technology to sketch the graph. Is your prediction correct?
   b) Sketch the graph of $y = \csc x$ on the same set of axes.
   c) Describe the similarities and differences between the graphs.

5.3 Investigate Transformations of Sine and Cosine Functions

5. Write two equations, one in the form $y = a \sin kx$ and one in the form $y = a \cos [k(x - d)]$, to match the graph.



6. Write two equations, one in the form $y = a \cos kx$ and one in the form $y = a \sin [k(x - d)]$, to match the graph.



7. Determine the amplitude, the period, the phase shift, and the vertical shift of each function.

a) $y = 5 \sin [3(x - 40°)] + 6$

b) $y = \frac{1}{4} \cos [4x + 400°] - 2$

c) $y = -7 \sin \left[\frac{2}{3}(x + 75°)\right] - 1$

d) $y = 0.4 \cos [3.5(x - 60°)] + 5.6$

5.4 Graphing and Modelling With $y = a \sin [k(x - d)] + c$ and $y = a \cos [k(x - d)] + c$

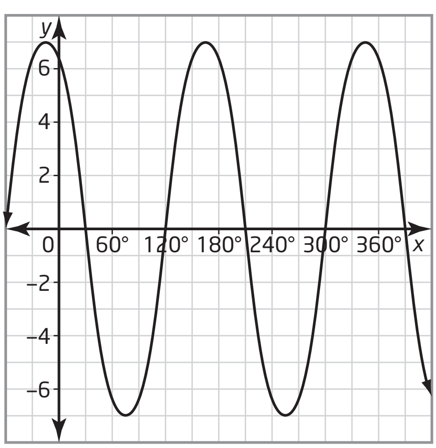
8. **a)** Transform the graph of $f(x) = \sin x$ to $g(x) = -7 \sin \left[\frac{1}{2}(x - 30°)\right] + 1$. Show each step in the transformation.

b) State the domain and range of $f(x)$ and $g(x)$.

9. **a)** Transform the graph of $f(x) = \cos x$ to $g(x) = \frac{2}{3} \cos [5(x + 28°)] - 4$. Show each step in the transformation.

b) State the domain and range of $f(x)$ and $g(x)$.

10. **a)** Determine the equation of a sine function that represents the graph shown.



b) Determine the equation of a cosine function that represents the graph.

5.5 Data Collecting and Modelling

11. The table shows the mid-season high temperatures for winter, spring, summer, and fall over a 3-year period for a city in Ontario.

Month	Temperature (°C)
February	–9
May	16
August	25
November	3
February	–10
May	17
August	27
November	3
February	–10
May	16
August	26
November	3

a) Use the table to determine a sinusoidal model for the mid-season high temperature.

b) Graph the points in the table on the same axes as your model to verify the fit.

c) Is the fit as you expect? Explain any discrepancies.

5.6 Use Sinusoidal Functions to Model Periodic Phenomena Not Involving Angles

12. During a 12-h period, the tides in one area of the Bay of Fundy cause the water level to rise to 6 m above average sea level and to fall 6 m below average sea level. The depth of the water at low tide is 2 m.

a) Suppose the water is at average sea level at midnight and the tide is coming in. Draw a graph that shows the height of the tide over a 24-h period. Explain how you obtained the graph.

b) Determine an equation that represents the tide for part a) using
i) a sine function
ii) a cosine function

c) Suppose the water is at average sea level at 3 a.m. and the tide is coming in. Write an equation that represents this new situation. Explain your reasoning.

Chapter 5 Math Contest

1. Determine the value of x such that $\frac{8^{29} - 8^{28}}{7} = 2^x$.

A 29 **B** 84 **C** 58 **D** 56

2. The sum of $300 - 299 + 298 - 297 + 296 - 295 + \cdots - 3 + 2 - 1$ is

A 149 **B** 151 **C** 300 **D** 150

3. Determine the values of three positive integers x, y, and z given that $x + \cfrac{1}{y + \cfrac{1}{z}} = \frac{38}{11}$.

4. If the fraction $\frac{5}{13}$ is written in expanded decimal form, what is the 153rd digit after the decimal point?

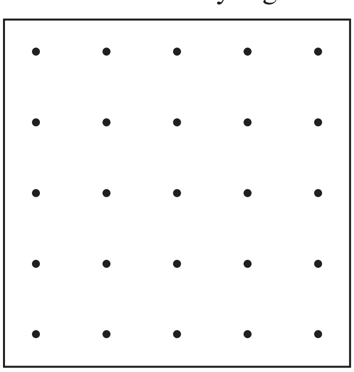
A 4 **B** 6 **C** 8 **D** 1

5. Three consecutive integers less than 20 have a sum of 27 and a product of 720. What are the numbers?

6. If $f(x) = \frac{1}{x^2 - 4}$ and $g(x) = \frac{2x}{x - 2}$, determine the value of x such that $(f + g)(x) = 2$.

A -2 **B** $-\frac{5}{2}$ **C** $-\frac{9}{4}$ **D** $\frac{7}{4}$

7. How many different-sized squares can be made on a 5-by-5 geoboard?



A 43 **B** 10 **C** 6 **D** 8

8. A sequence of numbers has the terms 0.3, 0.33, 0.333, 0.3333, …. What value does the sequence approach?

A 0.34 **B** 0.4 **C** $\frac{1}{3}$ **D** 1

9. A sequence of numbers has the terms $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \sqrt{2\sqrt{2\sqrt{2\sqrt{2}}}}, \ldots$. What is the limit of the value of the terms in the sequence?

A 2 **B** 1.5 **C** 3 **D** 1.4

10. When a number is divided by 18, the remainder is 13. What is the remainder when the number is divided by 6?

A 2 **B** 1 **C** 3 **D** 5

11. The period of $y = |5 \sin (4x - 120°)|$ is

A 45°

B 90°

C 120°

D 360°

12. If the number 6912 can be written in the form $2^x3^y4^z$, then the value of $x^2 + y^2 + z^2$ is

A 9 **B** 14 **C** 33 **D** 29

13. If $y = \sqrt[3]{x}$, what is the value of $x^{21} + 3\sqrt{2} - y^{63}$?

14. If $\binom{n}{r} = \frac{n!}{r!(n - r)!}$, where $n! = n \times (n - 1) \times (n - 2) \times \ldots 3 \times 2 \times 1$, determine the value of $\frac{\binom{100}{98}}{2}$.

A 4950

B 2475

C 9900

D 10 540