Chapter 6 Discrete Functions

6.1 Sequences as Discrete Functions

KEY CONCEPTS

- A sequence of numbers can be represented by a discrete function. The graph of a discrete function is a distinct set of points, not a smooth curve.
- The domain of a function representing a sequence is the set or a subset of the natural numbers, $\mathbb{N}$.
- Given the explicit formula for the *n*th term, t_n or $f(n)$, of a sequence, the terms can be written by substituting the term numbers for n. Examples of explicit formulas are $t_n = 3n + 2$ and $f(n) = 5n + 3$.
- An explicit formula for the *n*th term of a sequence can sometimes be determined by finding a pattern among the terms.

Example

Write the first four terms of a sequence that satisfies each of the following. Write an explicit formula for the *n*th term of each of your sequences using function notation.

a) The terms of the sequence are determined by adding a constant value.

b) The terms of the sequence are determined by dividing by a constant value.

Solution

a) Select a number for the first term, say 3. Select a constant value to be added to each term, say 7.

The first four terms of the sequence are 3, 10, 17, and 24.

The explicit formula is $f(n) = -4 + 7n$.

b) Select a number for the first term, say −20. Select a constant value that each term is divided by, say 4.

The first four terms of the sequence are $-20, -5, -\frac{5}{4}$, and $-\frac{5}{16}$.

The explicit formula is $f(n) = -\frac{20}{4^{n-1}}$, or $f(n) = -20(4^{-n+1})$.

A Practise

1. Write the first three terms of each sequence, given the explicit formula for the nth term of the sequence.

   a) $t_n = 2n + 1$

   b) $t_n = 4 - 3n$

   c) $f(n) = 1 + 2(n - 2)$

   d) $t_n = 7 + 2n$

   e) $t_n = 2^{n-1}$

   f) $f(n) = -2(3)^{n+1}$

2. Write the 16th term, given the explicit formula for the nth term of the sequence.

   a) $f(n) = 5 - 2n$

   b) $t_n = 3n + 2$

   c) $t_n = \sqrt{n} + 2$

   d) $f(n) = (-2)^{10-n}$

3. Describe the pattern in each sequence. Write the next three terms of each sequence.

   a) 5, 25, 125, 625, …

   b) 9, 7, 5, 3, …

   c) −4, −8, −16, −32, …

   d) 300, 30, 3, 0.3, …

   e) 3, 9, 27, 81, 243, …

   f) ar^3, ar^2, ar, a, …

   g) 0.11, −0.33, 0.99, −2.97, …

4. For each sequence, make a table of values using the term number and term, and calculate the finite differences. Then, determine an explicit formula in function notation and specify the domain.

   a) 6, 12, 18, 24, …

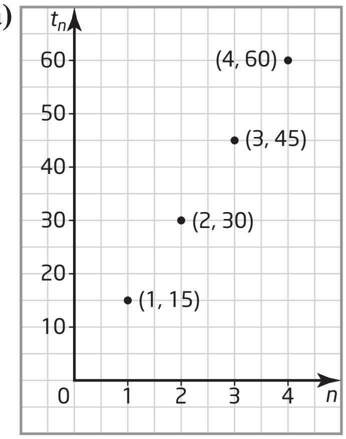
   b) 7, 4, 1, −2, …

   c) 2, 5, 10, 17, …

   d) 4, 13, 26, 43, …

5. The graphs show the terms in a sequence. Write each sequence in function notation and specify the domain.

   a)



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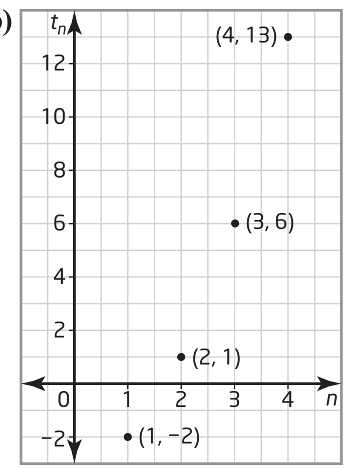

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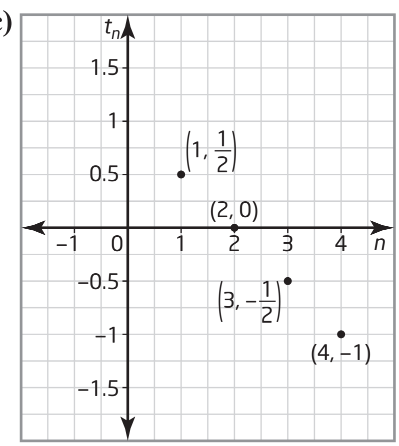
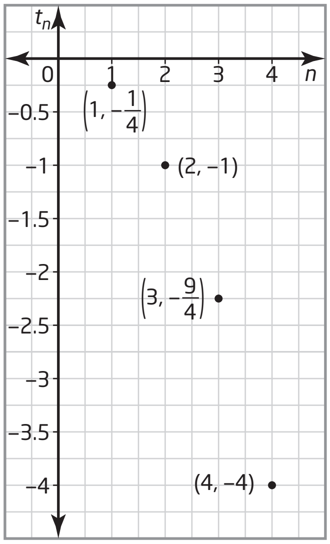


6. The graphs show the terms in a sequence. Write each sequence in function notation and specify the domain.

a)



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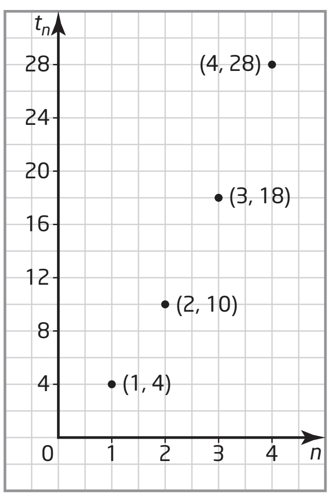

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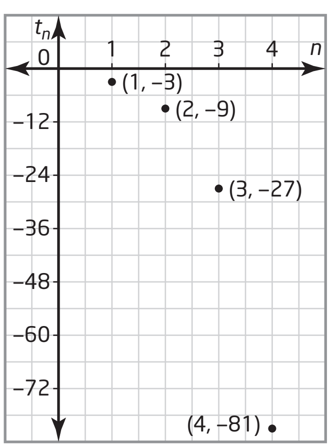
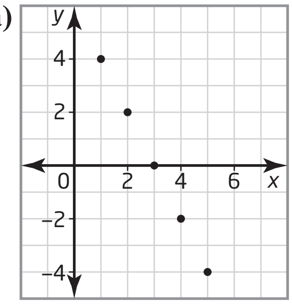


7. For each graph, specify whether the function is discrete or continuous and explain your choice.

a)



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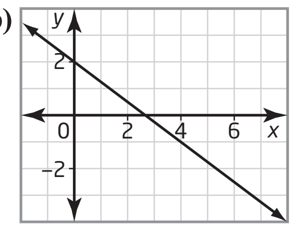

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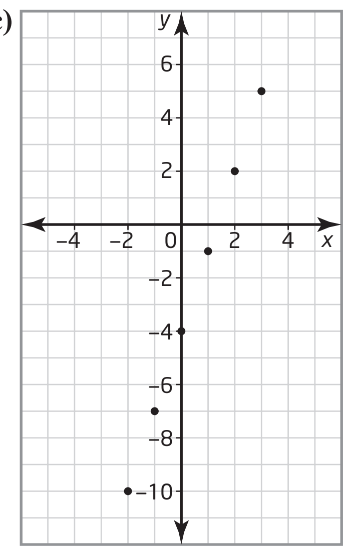

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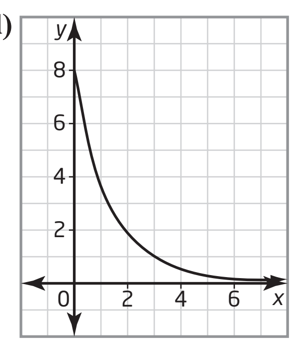


B Connect and Apply

☆**8.** Describe the pattern in each sequence and write the next four terms.

a) 2, −5, 4, −10, 6, −15, 8, −20, 10, …

b) −7, 9, −1, 2, 5, −5, 11, …

c) $4, \frac{1}{27}, 9, \frac{1}{9}, 14, \frac{1}{3}, 19, \ldots$

d) 8, −1, 0.8, −1, 0.08, −1, …

9. Consider the sequence 9, 18, 27, 36, …. Determine whether each of the following numbers is part of this sequence. Explain your thinking.

a) 135

b) 179

c) 653

d) 423

10. Determine an explicit formula for the *n*th term of each sequence. Use the formula to write the 15th term.

a) −16, 8, −4, 2, …

b) $1, \frac{2}{3}, \frac{3}{5}, \frac{4}{7}, \ldots$

c) $2, \frac{2}{\sqrt{2}}, \frac{2}{\sqrt{3}}, 1, \ldots$

d) 1, 3, 9, 27, …

e) $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}, \ldots$

11. Write the first four terms of a sequence that satisfies each of the following. Write an explicit formula for the *n*th term of each of your sequences in function notation.

a) The terms of the sequence are determined by subtracting a constant value.

b) The terms of the sequence are determined by multiplying a constant value.

☆**12.** A new small business plans to triple its sales every day for the first 2 weeks. Sales on the first day are $20.

a) Write the sequence that represents the sales for the first 6 days according to the plan.

b) Write an explicit formula to determine the sales on any of the first 14 days.

c) Use your formula to determine the sales on the 14th day. Is this reasonable? Why or why not?

13. The number of tourists visiting a small seaside town each summer is decreasing. This year the number of visitors was 3400, and it has been predicted that every year there will be 260 fewer tourists.

a) Write an explicit formula to determine the number of tourists in any given year.

b) How many tourists are expected to visit in 6 years?

c) How long will it be before the number of tourists drops below 1500?

C Extend

14. Consider the sequence $\sqrt{5}, \sqrt{\sqrt{5}}, \sqrt{\sqrt{\sqrt{5}}}, \ldots$.

a) Write the next two terms of the sequence.

b) Express each of the five terms as a power.

c) Write an explicit formula, in function notation, to represent the terms in the sequence.

d) Express your formula in part c) in a different form.

e) Write a power to represent the 50th term in this sequence.

15. Determine a formula for the *n*th term of each sequence.

a) $1 \times 1, 3 \times 4, 5 \times 7, 7 \times 10, \ldots$

b) $\frac{3}{8}, \frac{15}{24}, \frac{35}{48}, \frac{63}{80}, \ldots$

6.2 Recursive Procedures

KEY CONCEPTS

- A recursive procedure is one where a process is performed on an initial object or number and then the result is put through the steps of the process again. This is repeated many times over.
- A sequence can be defined recursively if each term can be calculated from the previous term or terms.
- A recursion formula shows the relationship between the terms of a sequence.
- A sequence can be represented by a pattern, an explicit formula, or a recursion formula. Formulas can also be written using function notation.

  For example:

  Pattern: 1, 3, 5, 7, …

  Explicit formula: $t_n = 2n - 1$ or $f(n) = 2n - 1$

  Recursion formula: $t_1 = 1$, $t_n = t_{n-1} + 2$, or $f(1) = 1$, $f(n) = f(n-1) + 2$
- In an explicit or a recursion formula for a sequence, n is a natural number because it is a term number. To find the terms of a sequence using a recursion formula, begin with the next natural number that is not used in the formula.

Example

Aniia and Marco paid \$350 000 for their first home. The real estate agent told them that the house will appreciate in value by 4% per year.

a) Copy and complete the table to show the value of the house for the next 8 years.

Year	House Value (\$)
0	350 000
1	$350\,000 + 0.04 \times 350\,000 = 364\,000$

b) Write the value of the house for the first 8 years as a sequence.

c) Write a recursion formula to represent the value of the house.

d) Predict the value after 25 years.

Solution

a)

Year	House Value ($)
0	350 000
1	350 000 + 0.04 × 350 000 = 364 000
2	364 000 + 0.04 × 364 000 = 378 560
3	378 560 + 0.04 × 378 560 = 393 702.40
4	393 702.40 + 0.04 × 393 702.40 = 409 450.50
5	409 450.50 + 0.04 × 409 450.50 = 425 828.52
6	425 828.52 + 0.04 × 425 828.52 = 442 861.66
7	442 861.66 + 0.04 × 442 861.66 = 460 576.13
8	460 576.13 + 0.04 × 460 576.13 = 478 999.18

b) The terms of the sequence are the final amounts in the second column of the table:
350 000, 364 000, 378 560, 393 702.40, 409 450.50, 425 828.52, 442 861.66, 460 576.13, 478 999.18.

c) The first term will be the house value in year 0, $350 000.

To find a recursion formula, note the pattern of calculations in each row and the common factor on the left side of the equation.

For example, in row 2 there is a common factor of 350 000 on the left side of the equation.

$$\begin{aligned} 350\,000 + 0.04 \times 350\,000 &= 350\,000(1 + 0.04) \\ &= 350\,000(1.04) \\ &= 364\,000 \end{aligned}$$

Similarly, the calculation in row 3 may also be written as 364 000(1.04) = 378 560, and so on for subsequent rows.

Therefore, a recursive formula to represent the value of the house is
$t_1 = 350\,000$, $t_n = 1.04t_{n-1}$.

d) The value of the house in 25 years can be obtained by multiplying the original amount by 1.04, 25 times, which is equivalent to multiplying by $(1.04)^{25}$.

$350\,000(1.04)^{25} = 933\,042.72$

In 25 years, the value of the house will be $933 042.72.

A Practise

1. Write the first four terms of each sequence, where $n \in \mathbb{N}$.

   a) $t_1 = 2,$
   $t_n = t_{n-1} + 6$

   b) $t_1 = 4,$
   $t_n = 3t_{n-1} - 2$

   c) $t_1 = -1,$
   $t_n = 0.5t_{n-1} + 0.5$

   d) $t_1 = 500,$
   $t_n = \dfrac{t_{n-1}}{5}$

   e) $t_1 = 3,$
   $t_n = 3n - 2t_{n-1}$

   f) $t_1 = 90,$
   $t_n = \dfrac{4t_{n-1}}{3}$

2. Write the first four terms of each sequence, where $n \in \mathbb{N}$.

   a) $f(1) = -3,$
   $f(n) = f(n-1) + 4$

   b) $f(1) = 0.25,$
   $f(n) = -2f(n-1)$

   c) $f(1) = 4,$
   $f(n) = \dfrac{f(n-1)}{n+1}$

   d) $f(1) = -7,$
   $f(n) = -f(n-1) + 3$

   e) $f(1) = -1.5,$
   $f(n) = f(n-1) + n$

3. Determine a recursion formula for each sequence.

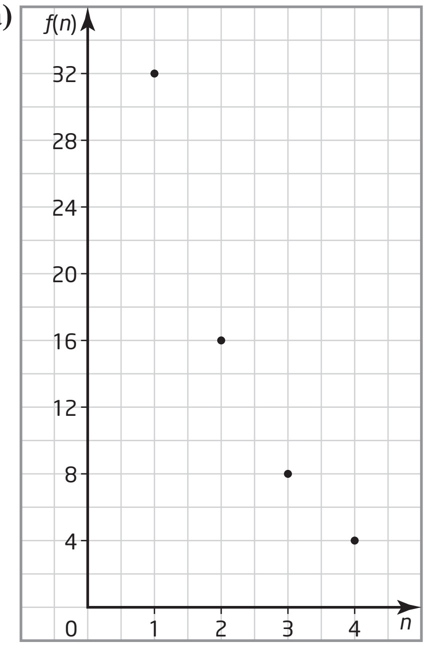
   a) 2, 5, 11, 23, …

   b) 3, −1, −7, −15, …

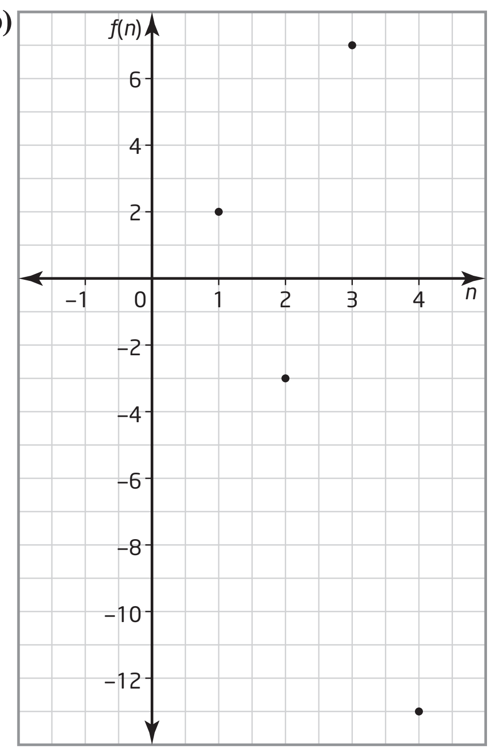
   c) −2, −3, −5, −8, …

4. For each graph, write the sequence of terms and determine a recursion formula using function notation.

   a)



   b)



5. The first term of a constant sequence is $f(1) = 3.14$.

a) State $f(n)$ for this sequence.

b) Write the first five terms of this sequence.

B Connect and Apply

6. Annette and Gordon paid \$275 000 for a new home. They were told that they can expect the house to appreciate in value by 3.5% per year.

a) Express 3.5% as a decimal.

b) Copy and complete the table to show the value of the house for the next 8 years.

Year	House Value ($)
0	275 000
1	
2	
3	
4	
5	
6	
7	
8	

c) Write the value of the house for the first 8 years as a sequence.

d) Write a recursion formula to represent the value of the house.

e) Use your formula in part d) to predict the value after 20 years.

7. The auditorium in a new school has 42 seats in the first row, 51 in the second row, 65 in the third row, 84 in the next row, and so on.

a) Represent the number of seats in the rows as a sequence.

b) Describe the pattern in the number of seats per row.

c) Write a recursion formula to represent the number of seats in any row.

☆8. A bacteria culture begins with 12 bacteria and doubles every hour.

a) Write the sequence for the first 7 h.

b) Write an explicit formula for the nth term of the sequence.

c) Write a recursion formula to represent the number of bacteria.

d) Which of the above formulas was easier to determine? Explain.

e) Determine the number of bacteria in the culture after 12 h. Which formula did you use? Why?

f) After how many hours will there be 1 572 864 bacteria?

9. Given the explicit formula of a sequence, write the first five terms and then determine a recursion formula for the sequence.

a) $f(n) = 6n - 5$

b) $t_n = 3n - 1$

c) $t_n = (-2)^{-n}$

d) $t_n = n(n - 2)$

10. Use the given recursion formula to determine the first five terms of each sequence.

a) $f(1) = 1,$
$f(2) = 1,$
$f(n) = f(n-1) + f(n-2)$

☆**b)** $f(1) = 4,$
$f(2) = -1,$
$f(n) = f(n-1) - 2f(n-2)$

c) $t_1 = 2,$
$t_2 = 3,$
$t_n = 3t_{n-2} - t_{n-1}$

d) $t_1 = -1,$
$t_2 = 4,$
$t_n = -2t_{n-2} + t_{n-1}$

e) $t_1 = 3,$
$t_2 = -2,$
$t_n = t_{n-2} \times t_{n-1} + n$

f) $t_1 = 5,$
$t_2 = 1,$
$t_3 = -1,$
$t_n = t_{n-3} - 2t_{n-2} + t_{n-1}$

11. Write the first four terms of each sequence.

a) $t_1 = 1,$
$t_n = 1 - (t_{n-1})^2$

b) $f(1) = 64,$
$f(n) = \dfrac{f(n-1)}{-4}$

c) $t_1 = -3,$
$t_n = -5t_{n-1}$

d) $t_1 = -2,$
$t_n = 11 + 3t_{n-1}$

e) $t_1 = \frac{1}{8},$
$t_n = 4t_{n-1} - 1$

f) $f(1) = a - 2b,$
$f(n) = f(n-1) + 3b$

g) $f(1) = 2c + 3d,$
$f(n) = f(n-1) - c$

h) $f(1) = m - 5n,$
$f(n) = f(n-1) + 2m + n$

12. Given the recursion formula, write the first four terms of the sequence and then determine the explicit formula for the sequence.

a) $t_1 = 4,$
$t_n = t_{n-1} - 7$

b) $t_1 = 81,$
$t_n = -\frac{1}{3}t_{n-1}$

c) $t_1 = 0,$
$t_n = t_{n-1} + 2n - 1$

d) $t_1 = -5,$
$t_n = t_{n-1} + 3$

C Extend

13. Write the first five terms of each sequence, starting at $f(1)$.

a) $f(2) = -5,$
$f(n) = -3f(n-1) + 1$

b) $f(3) = 7,$
$f(n) = f(n-1) - 2n$

14. Is it possible for a sequence to have two different recursion formulas? Justify your answer with an example.

6.3 Pascal's Triangle and Expanding Binomial Powers

KEY CONCEPTS

- Pascal's triangle is a triangular array of natural numbers in which the entries can be obtained by adding the two entries immediately above.
  $t_{n,\,r} = t_{n-1,\,r-1} + t_{n-1,\,r}$, where n is the horizontal row number and r is the diagonal row number.

- Many number patterns can be found in Pascal's triangle. For example, the sums of the terms of the rows form a sequence of the powers of 2 and the terms in diagonal row 2 are triangular numbers.

- The coefficients of the terms in the expansion of $(a + b)^n$ correspond to the terms in row n of Pascal's triangle.

Example

Write $t_{6,\,4}$ as the difference of two terms, each in the form $t_{n,\,r}$.

Solution

Use the formula $t_{n,\,r} = t_{n-1,\,r-1} + t_{n-1,\,r}$.
Since this formula involves a sum, rearrange the terms to create a formula that involves the difference of two terms. There are two possibilities.

i) $t_{n,\,r} - t_{n-1,\,r-1} = t_{n-1,\,r}$
ii) $t_{n,\,r} - t_{n-1,\,r} = t_{n-1,\,r-1}$

Formula i):

Let $t_{6,\,4}$ be represented by $t_{n-1,\,r-1}$ in the formula $t_{n,\,r} - t_{n-1,\,r} = t_{n-1,\,r-1}$.
Then, $n - 1 = 6$ and $r - 1 = 4$.
So, $n = 7$ and $r = 5$.
Therefore, $t_{6,\,4}$ can be expressed as the difference of two terms as $t_{6,\,4} = t_{7,\,5} - t_{6,\,5}$.

Formula ii):

Let $t_{6,\,4}$ be represented by $t_{n-1,\,r}$ in the formula $t_{n,\,r} - t_{n-1,\,r-1} = t_{n-1,\,r}$.
Then, $n - 1 = 6$ and $r = 4$.
So, $n = 7$ and $r - 1 = 3$.
Therefore, $t_{6,\,4}$ can be expressed as the difference of two terms as $t_{6,\,4} = t_{7,\,4} - t_{6,\,3}$.

A Practise

1. The following hockey stick pattern is one of many found in Pascal's triangle.

```
                1
              1   1
            1   2   1
          1   3   3   1
        1   4   6   4   1
      1   5  10  10   5   1
    1   6  15  20  15   6   1
  1   7  21  35  35  21   7   1
1   8  28  56  70  56  28   8   1
```

a) Is this the only hockey stick pattern that has these numbers? Explain.

b) Copy Pascal's triangle shown above and extend it to row 10. Use your triangle to list the numbers in a hockey stick pattern with each of the following sums.

i) 84
ii) 120
iii) 56
iv) 252

2. Determine the sum of the terms in each of the following rows of Pascal's triangle.

a) row 11

b) row 19

c) row 22

d) row 30

e) row $n + 1$

3. Express as a single term from Pascal's triangle in the form $t_{n,\,r}$.

a) $t_{5,\,7} + t_{5,\,8}$

b) $t_{9,\,2} + t_{9,\,3}$

c) $t_{3,\,4} + t_{3,\,5}$

d) $t_{10,\,7} + t_{10,\,8}$

e) $t_{15,\,9} + t_{15,\,10}$

f) $t_{a+1,\,b} + t_{a+1,\,b+1}$

4. Write each as the sum of two terms, each in the form $t_{n,\,r}$.

a) $t_{19,\,11}$

b) $t_{25,\,16}$

c) $t_{14,\,7}$

d) $t_{a,\,2}$

e) $t_{x+2,\,x-3}$

5. Use Pascal's triangle to expand each power of a binomial.

a) $(x + 1)^8$

b) $(y - 2)^7$

c) $(2 + t)^6$

d) $(1 - m^2)^4$

e) $(a + 2b)^3$

6. How many terms will there be in each expansion?

a) $(6a - 7b)^7$

b) $(8x + 11)^{20}$

c) $(9t - 4)^1$

d) $(-7x + 2)^{38}$

e) $\left(\frac{1}{x} + 3\right)^{54}$

f) $(5b + 2a)^{n+1}$

g) $(2t^2 + 3t - 5)^0$

7. Use patterns in the terms of the expansion to determine the value of k in each term of $(x + y)^{10}$.

a) ky^{10}

b) $252x^ky^5$

c) $128x^7y^k$

d) kx^6y^4

e) kx^2y^8

f) $210x^ky^6$

B Connect and Apply

8. What row number of Pascal's triangle has each indicated row sum?

a) 32 768

b) 128

c) 131 072

d) 1024

e) 1 048 576

f) 8192

g) 512

h) 4096

9. Write each as the difference of two terms, each in the form $t_{n,\,r}$.

a) $t_{7,\,3}$

b) $t_{9,\,5}$

c) $t_{13,\,2}$

d) $t_{26,\,17}$

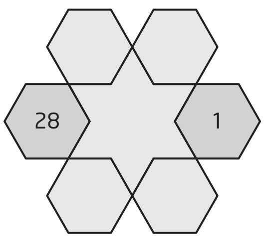
e) $t_{24,\,3}$

f) $t_{10,\,10}$

g) $t_{n,\,r}$

10. Look for patterns in Pascal's triangle. What are the missing numbers in each diagram?

a)



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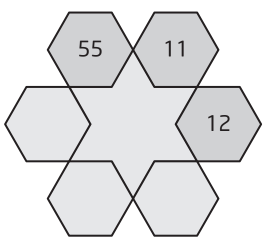

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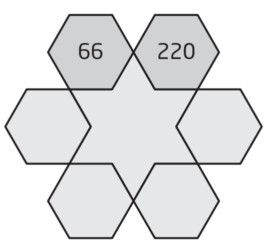

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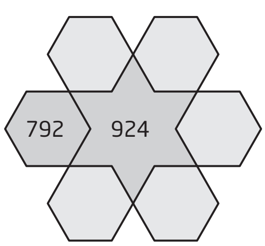

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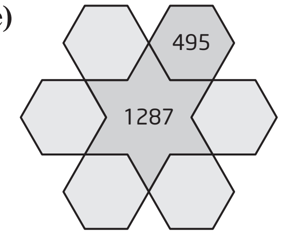

11. Use Pascal's triangle to expand and simplify.

a) $(x + y)^4 + (x - y)^4$

b) $(x + y)^4 - (x - y)^4$

12. Use Pascal's triangle to expand each power of a binomial.

a) $\left(2x + \frac{3}{x}\right)^6$

b) $\left(\sqrt{x} - \frac{2}{\sqrt{x}}\right)^6$

C Extend

13. a) Describe how Pascal's triangle can be used to expand and simplify $(a + b + c)^3$.

b) Apply your method from part a) to expand and simplify the trinomial $(a + b + c)^3$.

14. The first three terms in the expansion of $(1 + ax + bx^2)^4$ are 1, $8x$, and $32x^2$. Determine the value(s) of a and b.

15. Determine the coefficient of x^{14} in the expansion of $(x^4 + 3)^3(2 - x^2)^5$.

16. The coefficients of $(a + b)^n$ in Pascal's triangle may be written in the form $\binom{n}{r}$, where $\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$, $n! = n \times (n - 1) \times (n - 2) \times \ldots \times 3 \times 2 \times 1$, $0! = 1$, and $0 \le r \le n$.

a) Show that the above is true for $(a + b)^5$.

b) Write the coefficients of $(a + b)^{10}$ in the form $\binom{n}{r}$.

6.4 Arithmetic Sequences

KEY CONCEPTS

- An arithmetic sequence is a sequence in which the difference between consecutive terms is a constant.
- The difference between consecutive terms of an arithmetic sequence is called the common difference.
- The formula for the general term of an arithmetic sequence is $t_n = a + (n - 1)d$, where a is the first term, d is the common difference, and n is the term number.

Example

Determine the number of terms in the arithmetic sequence −28, −22, −16, …, 260.

Solution

First, determine the general term of the sequence.
The first term is $a = -28$ and the common difference is $d = 6$.
Substitute the values into the formula for the general term.

$$\begin{aligned} t_n &= a + (n - 1)d \\ &= -28 + (n - 1)(6) \\ &= -28 + 6n - 6 \\ &= -34 + 6n \end{aligned}$$

A formula for the general term is $t_n = -34 + 6n$.

To find the number of terms in the sequence, solve for n when $t_n = 260$.

$$\begin{aligned} 260 &= -34 + 6n \\ 294 &= 6n \\ n &= 49 \end{aligned}$$

There are 49 terms in the sequence.

A Practise

1. For each arithmetic sequence, determine the values of a and d. Then, write the next four terms.

 a) 5, 8, 11, …

 b) −3, 2, 7, …

 c) 1.5, 0.7, −0.1, …

 d) 33, 31.2, 29.4, …

 e) $\frac{1}{4}, \frac{1}{2}, \frac{3}{4}, \ldots$

 f) 0.25, 0.26, 0.27, …

2. State whether or not each sequence is arithmetic. Justify your answer.

 a) 1, 4, 7, 10, …

 b) 3, 2, 3, 4, …

 c) −12, −5, 2, 9, …

 d) 0.41, 0.32, 0.23, …

 e) $-\frac{5}{4}, -\frac{4}{3}, -\frac{3}{2}, \ldots$

 f) $\frac{19}{12}, \frac{5}{4}, \frac{11}{12}, \ldots$

3. Given the values of a and d, write the first three terms of the arithmetic sequence. Then, write the formula for the general term.

 a) $a = 12, d = -5$

 b) $a = -9, d = 2$

 c) $a = 11, d = -\frac{7}{8}$

 d) $a = -\frac{2}{3}, d = \frac{1}{2}$

 e) $a = x^2, d = 1.3x^2$

4. Given the formula for the general term of an arithmetic sequence, determine t_{16}.

 a) $t_n = 2n + 7$

 b) $f(n) = 3 - 5n$

 c) $t_n = -\frac{3}{4}n + 3$

 d) $f(n) = 16 - 2.8n$

 e) $t_n = 4n + 11$

 f) $f(n) = 19 - 12n$

5. Given the formula for the general term of an arithmetic sequence, write the first five terms. Graph the discrete function that represents each sequence.

 a) $t_n = 4n - 1$

 b) $f(n) = -n + 3$

 c) $f(n) = 2(1 - 5n)$

 d) $t_n = -6n + 17$

 e) $f(n) = \frac{3n + 2}{2}$

 f) $t_n = 0.9n + 0.3$

 g) $f(n) = 0.1(3 + 2n)$

6. Which term in the arithmetic sequence −15, −8, −1, … has the value 125?

7. Determine the number of terms in each arithmetic sequence.

 a) 5, 11, 17, …, 179

 b) 8, 16, 24, …, 424

 c) −13, −9, −5, …, 327

 d) 8, 3, −2, …, −152

 e) $\frac{11}{5}, -\frac{4}{5}, -\frac{19}{5}, \ldots, -\frac{949}{5}$

 f) $2x^2 + x, 3x^2 + 6x, 4x^2 + 11x, \ldots, 39x^2 + 186x$

B Connect and Apply

☆8. **a)** Verify that the sequence determined by the recursion formula $t_1 = -5$, $t_n = t_{n-1} + 3$ is arithmetic.

 b) Determine the formula for the general term of the sequence.

9. For each sequence, determine the values of a and d and write the next three terms.

 a) $\frac{1}{5}, \frac{7}{15}, \frac{11}{15}, \ldots$

 b) $-1, -\frac{5}{3}, -\frac{7}{3}, \ldots$

 c) $2, \frac{5}{4}, \frac{1}{2}, \ldots$

 d) $-\frac{3}{8}, \frac{1}{2}, \frac{11}{8}, \ldots$

 e) $-4x + y, -3x + 5y, -2x + 9y, \ldots$

 f) $3m - \frac{5}{6}n, -3m - \frac{2}{3}n, -9m - \frac{1}{2}n, \ldots$

10. Determine a and d, and then write the formula for the nth term of each arithmetic sequence with the given terms.

a) $t_3 = 11$ and $t_8 = 46$

b) $t_{12} = 52$ and $t_{22} = 102$

c) $t_{50} = 140$ and $t_{70} = 180$

d) $t_{34} = 96$ and $t_{46} = 132$

e) $t_{21} = 34.5$ and $t_{38} = 60$

f) $t_{19} = -91.8$ and $t_{41} = -223.8$

g) $t_9 = 29 + 41x$ and $t_{16} = 29 + 76x$

h) $t_5 = -x^3 - 6$ and $t_{18} = -14x^3 - 19$

11. Write a recursion formula for each sequence in question 10.

12. Eight fence posts are to be equally spaced between two corner posts that are 117 m apart.

a) What should be the distance between two posts?

b) Write a sequence to represent this situation.

c) Write the formula for the general term of the sequence in part b).

d) How is the value you found in part a) related to the general term of the formula you found in part c)?

☆**13.** An architect's starting salary is $73 000. The company has guaranteed a raise of $2275 every 6 months with satisfactory performance.

a) Write a sequence to represent this situation. Is this sequence arithmetic? Explain.

b) State the general term of the sequence in part a).

c) Write a recursion formula for the sequence in part a).

d) What will the architect's salary be after 8 years?

e) When will the architect's salary be $127 600?

14. A number, m, is called an arithmetic mean between a and b if a, m, and b form an arithmetic sequence.

a) Determine the arithmetic mean between 3 and 27.

b) Determine five arithmetic means between 5 and 29.

15. Determine the general term of an arithmetic sequence such that the 11th term is 53 and the sum of the 5th and 7th terms is 56.

C Extend

16. An original painting is purchased for $230, and each year it increases in value by 22% of its original value.

a) What is the painting's value after 12 years?

b) When is the painting worth $1242?

17. Determine all possible arithmetic sequences formed by the numbers p, q, and r such that $q = 3$ and $p^2 + r^2 = 68$.

18. The sum of the first three terms of an arithmetic sequence is 15. The sum of their squares is 147. Determine the sequence.

6.5 Geometric Sequences

KEY CONCEPTS

- A geometric sequence is a sequence in which the ratio of consecutive terms is a constant.
- The ratio between consecutive terms of a geometric sequence is called the common ratio.
- The formula for the general term of a geometric sequence is $t_n = a(r)^{n-1}$, where a is the first term, r is the common ratio, and n is the term number.

Example

In a geometric sequence, the fifth term is 1875 and the seventh term is 46 875.

a) Determine the formula for the general term of the sequence.

b) Write the first four terms of the sequence.

Solution

a) The formula for the general term of a geometric sequence is $t_n = ar^{n-1}$.

The fifth term occurs when $n = 5$.

Therefore, $t_5 = ar^4$ and $ar^4 = 1875$. ①

The seventh term occurs when $n = 7$.

Therefore, $t_7 = ar^6$ and $ar^6 = 46\,875$. ②

To determine the common ratio, divide ② by ①.

$\frac{ar^6}{ar^2} = \frac{46\,875}{1875}$ Simplify each side.

$r^2 = 25$

$r = \pm 5$

There are two possible geometric sequences, one with $r = 5$ and the other with $r = -5$.

Determine a for each value of r using $ar^4 = 1875$.

Since $5^4 = (-5)^4$, you only need to check one value of r.

$a(5)^4 = 1875$

$625a = 1875$

$a = 3$

The formula for the general term is $t_n = 3(5)^{n-1}$ or $t_n = 3(-5)^{n-1}$.

b) The first four terms of the sequence $t_n = 3(5)^{n-1}$ are 3, 15, 75, and 375.

The first four terms of the sequence $t_n = 3(-5)^{n-1}$ are 3, −15, 75, and −375.

A Practise

1. Determine whether the sequence is arithmetic, geometric, or neither. Give a reason for your answer.

   a) 7, 5, 3, 1, …

   b) 4, −16, 64, −256, …

   c) 3, 0.3, 0.03, 0.003, …

   d) 8, 8.8, 8.88, 8.888, …

   e) 1, 3, 9, 27, …

   f) $ab, ab^2, ab^3, ab^4, \ldots$

2. State the common ratio for each geometric sequence and write the next three terms.

   a) 3, 6, 12, 24, …

   b) −5, 20, −80, 320, …

   c) $\frac{1}{2}, -1, 2, -4, \ldots$

   d) 8000, 800, 80, 8, …

   e) $-\frac{1}{6}, -\frac{1}{2}, -\frac{3}{2}, -\frac{9}{2}, \ldots$

   f) 2.5, 0.5, 0.1, 0.02, …

   g) $\frac{(x+3)^2}{3}, \frac{(x+3)^5}{12}, \frac{(x+3)^8}{48}, \frac{(x+3)^{11}}{192}, \ldots$

3. For each geometric sequence, determine the formula for the general term and use it to determine the indicated term.

   a) 4096, 1024, 256, …, t_8

   b) 12, 6, 3, …, t_{12}

   c) $6, -2, \frac{2}{3}, \ldots, t_8$

   d) 13.45, 2.69, 0.538, …, t_{10}

   e) $\frac{1}{32}, \frac{1}{8}, \frac{1}{2}, \ldots, t_{13}$

   f) $\frac{a^2}{b}, \frac{a^3}{2b}, \frac{a^4}{4b}, \ldots, t_{16}$

4. Write the first four terms of each geometric sequence.

   a) $t_n = 3(-1)^{n-1}$

   b) $a = 22, r = -2$

   c) $f(n) = \frac{1}{3}(2)^{n-1}$

   d) $f(n) = 4(\sqrt{5})^{n-1}$

   e) $a = -2, r = \frac{2}{3}$

   f) $t_n = -1111(-0.3)^{n-1}$

5. Determine the number of terms in each geometric sequence.

   a) 2, −10, 50, …, −156 250

   b) $64, 32, 16, \ldots, \frac{1}{256}$

   c) $12, 4, \frac{4}{3}, \ldots, \frac{4}{729}$

   d) 5, 35, 245, …, 588 245

   e) $a, ab, ab^2, \ldots, ab^{12}$

B Connect and Apply

6. Determine if each sequence is arithmetic, geometric, or neither. If it is arithmetic, state the values of a and d. If it is geometric, state the values of a and r.

   a) $3m, 7m, 11m, \ldots$

   b) $-1, -\frac{2}{x}, -\frac{4}{x^2}, \ldots$

   c) $3x - 4y, 5x - 6y, 7x - 8y, \ldots$

   d) 5.440, 54.40, 544.0, …

   e) $\frac{8}{7}, \frac{6}{5}, \frac{4}{3}, \ldots$

   f) $7, 4 + s, 1 + 2s, \ldots$

7. Find the unknown terms, m and n, in each geometric sequence.

   a) $7, m, 63, n$

   b) $-2, -10, m, n$

   c) $\frac{2}{9}, m, \frac{1}{2}, n$

   d) $m, 6, n, 216$

   e) $m, n, 2, \frac{1}{2}$

   f) $4, m, n, 500$

☆8. Which term of the geometric sequence $\frac{2}{81}, \frac{4}{27}, \frac{8}{9}, \ldots$ has a value of 6912?

9. Which term of the geometric sequence $5, 1, \frac{1}{5}, \ldots$ has a value of $\frac{1}{78\,125}$?

☆**10.** Without writing the terms of the sequence, determine the general term of the geometric sequence that corresponds to each recursion formula.

a) $t_1 = 4$, $t_n = -3xt_{n-1}$

b) $t_1 = -28m^3$, $t_n = \frac{1}{2}t_{n-1}$

c) $t_1 = \frac{5}{3}$, $t_n = \left(\frac{3}{4} + c\right)t_{n-1}$

11. Chad, a champion show dog, had 2 parents one generation ago, 4 grandparents two generations ago, 8 great-grandparents three generations ago, and so on.

a) Write the general term of the geometric sequence that represents this situation.

b) Determine how many ancestors Chad had each number of generations ago.
i) 6
ii) 10
iii) 14

c) How many generations ago did Chad have 8192 ancestors?

12. In a bacteria strain, the number of bacteria doubles every 20 min. There were 8 bacteria to start with.

a) How many bacteria will there be after 3 h?

b) Write an expression to represent the term that corresponds to the number of bacteria after 1 day.

13. The geometric mean of a set of n numbers is the nth root of the product of the numbers. For example, given two non-consecutive terms of a geometric sequence, 6 and 24, their product is 144 and the geometric mean is $\sqrt{144}$, or 12. The numbers 6, 12, 24 form a geometric sequence. Determine the geometric mean of

a) 2 and 200

b) 8 and 128

14. A super bouncy ball is thrown from a balcony that is 180 m above the ground. When it hits the ground the ball bounces back to 0.75 of its original height.

a) How high does the ball bounce after the 8th bounce?

b) On which bounce is the ball's height 5.7 m?

C Extend

15. Determine the value of y if $y - 2$, $5y + 10$, and $y - 50$ are consecutive terms in

a) a geometric sequence

b) a arithmetic sequence

16. Determine the values of p, q, r, and s such that $\frac{1}{18}$, p, q, r, s, 432 forms a geometric sequence.

17. The product of the first two terms of a geometric sequence is 27, and the product of the first three terms is also 27.

a) State the general term of the sequence.

b) Which term in the sequence is $\frac{1}{729}$?

18. Determine the first three terms of a geometric sequence such that the sum of the second and third terms is 24 and the sum of the seventh and eighth terms is 5832.

19. Determine the first five terms of two different geometric sequences that satisfy the given conditions.

a) The sum of the first two terms of the sequence is 3 and the sum of the next two terms is $\frac{4}{3}$.

b) The sum of the first three terms of the sequence is 3 and the sum of the third, fourth, and fifth terms is 12.

6.6 Arithmetic Series

KEY CONCEPTS

- An arithmetic series is the indicated sum of the terms of an arithmetic sequence. For example, 4, 9, 14, 19, … is an arithmetic sequence, while $4 + 9 + 14 + 19 + \cdots$ is an arithmetic series.
- Given the first term, the last term, and the number of terms of an arithmetic series, the sum of the series can be found using the formula $S_n = \frac{n}{2}(a + t_n)$ or the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$.
- Given the first terms of an arithmetic series, the sum of the first n terms can be found using the formula $S_n = \frac{n}{2}[2a + (n - 1)d]$.

Example

Determine an expression for the sum, S_n, of the terms of an arithmetic series where the terms are represented by $t_n = 6n - 14$.

Solution

To determine t_1, or a, substitute $n = 1$ into the general term $t_n = 6n - 14$.

$$\begin{aligned} t_1 &= 6(1) - 14 \\ &= 6 - 14 \\ &= -8 \end{aligned}$$

The last term is $t_n = 6n - 14$, and there are n terms in the series.

Since there is no specified number of terms for the series, the expression for the sum of the series is found by substituting $a = -8$ and $t_n = 6n - 14$ into the formula $S_n = \frac{n}{2}(a + t_n)$.

$$\begin{aligned} S_n &= \frac{n}{2}(a + t_n) \\ &= \frac{n}{2}(-8 + 6n - 14) \\ &= \frac{n}{2}(-22 + 6n) \\ &= -11n + 3n^2 \end{aligned}$$

An expression for the sum of the series is $S_n = 3n^2 - 11n$.

A Practise

1. Determine the sum of each arithmetic series.

   a) $a = 3, t_n = 15, n = 8$

   b) $a = 14, d = -3, n = 16$

   c) $a = 2, t_n = -34, n = 15$

   d) $a = -1, t_n = 11, n = 21$

   e) $a = \frac{4}{3}, d = \frac{1}{4}, n = 8$

   f) $a = 7x, t_n = 32x, n = 17$

2. For each arithmetic series, state the values of a and of d. Then, determine the sum of the first 10 terms.

   a) $3 + 7 + 11 + \cdots$

   b) $5 + 12 + 19 + \cdots$

   c) $2 + 8 + 14 + \cdots$

   d) $6 + 18 + 30 + \cdots$

   e) $\frac{3}{2} + \frac{1}{2} - \frac{1}{2} - \cdots$

   f) $5.6 + 5.9 + 6.2 + \cdots$

3. The first and last terms in each arithmetic series are given. Determine the sum of the series.

   a) $a = \frac{1}{3}, t_8 = 729$

   b) $a = 6, t_{15} = 62$

   c) $a = -3, t_{36} = -73$

   d) $a = 6.6, t_{23} = -19.8$

   e) $a = -\sqrt{5}, t_{24} = 15\sqrt{5}$

   f) $a = 9, t_9 = -\frac{1}{729}$

4. Determine the sum of each arithmetic series.

   a) $2 + 4 + 6 + \cdots + 2000$

   b) $4 + 8 + 12 + \cdots + 400$

   c) $-2 - 8 - 14 - \cdots - 128$

   d) $-17 - 10 - 3 + \cdots + 74$

   e) $2 + 7 + 12 + \cdots + 62$

   f) $4 + 2.5 + 1 + \cdots - 33.5$

5. Determine the sum of each arithmetic series.

   a) $5 + 10 + 15 + \cdots + 265$

   b) $-20 - 18 - 16 - \cdots - 2$

   c) $1 + 0.9 + 0.8 + \cdots - 5.3$

   d) $\frac{1}{6} + \frac{1}{3} + \frac{1}{2} + \cdots + \frac{5}{3}$

   e) $\sqrt{3} + 2\sqrt{3} + 3\sqrt{3} + \cdots + 20\sqrt{3}$

   f) $1 + 5 + 9 + \cdots + 77$

B Connect and Apply

6. Find the specified sum for each arithmetic series.

   a) S_{16} for $3 + 7 + 11 + \cdots$

   b) S_{20} for $100 + 85 + 70 + \cdots$

   c) S_{11} for $2 + 10 + 18 + \cdots$

   d) S_{10} for $1 + 6 + 11 + \cdots$

   e) S_{20} for $-21 - 19 - 17 - \cdots$

   f) S_{15} for $2 - 1 - 4 - \cdots$

   g) S_{18} for $\frac{2}{3} + 1 + \frac{4}{3} + \cdots$

   h) S_{21} for $a + (2a + b) + (3a + 2b) + \cdots$

   i) S_{15} for $5 + 9 + 13 + \cdots$

7. Determine the number of terms, n, for each arithmetic series with the given sum.

   a) $1 + 2 + 3 + \cdots + t_n = 78$

   b) $3 + 7 + 11 + \cdots + t_n = 1830$

   c) $15 + 20 + 25 + \cdots + t_n = 1250$

   d) $10 + 8 + 6 + \cdots - t_n = -350$

   e) $-30 - 26 - 22 - \cdots - t_n = -120$

   f) $5 + 1 - 3 - \cdots - t_n = -345$

   g) $28 + 26 + 24 + \cdots + t_n = 154$

   h) $8 + 2 - 4 - \cdots - t_n = -300$

   i) $19 + 15 + 11 + \cdots - t_n = -441$

8. Find the specified sum for each arithmetic series with the given general term.

   a) $t_n = 2n - 1; S_{100}$

   b) $t_n = 2n; S_{2000}$

   c) $t_n = 3n - 2; S_{80}$

9. The general term and the sum of the first n terms of each arithmetic series is given. Determine the value of n.

a) $t_n = 3n - 1$; $S_n = 950$

b) $t_n = 2n - 1$; $S_n = 1\ 234\ 321$

10. The sum of the first 5 terms of an arithmetic sequence is 625 and the sum of the first 10 terms is 100. Determine the sum of the first 15 terms.

11. The third term of an arithmetic sequence is 18 and the seventh term is 30. Find the sum of the first 23 terms.

12. Determine the sum of each arithmetic series.

a) $-8\sqrt{6} - 6\sqrt{6} - 4\sqrt{6} - \cdots + 34\sqrt{6}$

b) $53x + 47x + 41x - \cdots - 55x$

c) $(9a - 5b) + (12a - 4b) + (15a - 3b) + \cdots + (93a + 23b)$

d) $\frac{1}{x^2} + \frac{4}{x^2} + \frac{7}{x^2} + \cdots + \frac{94}{x^2}$

13. Which are arithmetic series? Justify your answers.

a) $-5 - 10 - 14 - 17 - \cdots$

b) $6x^2 + 2x^2 - 2x^2 + \cdots$

c) $3m + (5m - b) + (7m - 2b) + \cdots$

d) $-\frac{8}{b} + \frac{4}{b} - \frac{2}{b} + \cdots$

14. In a grocery store, soup cans are displayed in a triangular formation. There are 55 cans in the bottom row and 1 can in the top row. Each row has 1 can less than the previous row. How many cans are in this display?

☆15. Cool Juices makes a profit of \$350 in the first week of a 16-week summer season. After the first week, the profit increases by \$75 per week.

a) Explain why the total profit for the season represents an arithmetic series.

b) Determine the total profit for the season.

☆16. Bashira has a choice between two summer jobs, each for a period of 16 weeks.
Job A: He would be paid \$450 every two weeks.
Job B: He would be paid \$100 the first week and then an additional \$25 per week for each successive week.
Which job should Bashira accept to earn the most money? Justify your answer.

17. Determine an expression for the sum, S_n, of the terms of an arithmetic series where the terms are represented by

a) $t_n = 5 + 2(n - 1)$

b) $t_n = 4n + 1$

c) $t_n = 5n - 2$

d) $t_n = 7n + 4$

e) $t_n = 12 - 3n$

18. Determine an expression for the sum of

a) the first n even natural numbers

b) the first n odd natural numbers

19. Determine an arithmetic series such that the fifth term of the series is 16 and the sum of the first 10 terms is 145.

C Extend

20. The sum of the first 10 terms of an arithmetic sequence is 250. All the terms are positive. Determine the general term for a sequence that satisfies this condition.

21. A formula for the sum of the first n terms of an arithmetic series is $S_n = n^2 + 4n$. Determine the first four terms of the series.

22. Determine an arithmetic series such that $t_{12} = 35$ and $S_{20} = 610$.

23. The sum of an arithmetic series is $S_n = \frac{1}{2}(3n^2 - n)$. Determine the general term for the series.

6.7 Geometric Series

KEY CONCEPTS

- A geometric series is the sum of the terms in a geometric sequence. For example, $-3 + 6 - 12 + 24 - \cdots$ is a geometric series.
- The formula for the sum of the first n terms of a geometric series with first term a and common ratio r is $S_n = \dfrac{a(r^n - 1)}{r - 1}$, $r \neq 1$.

Example

Find the third term of a geometric series given that the first term is 7, the last term is 448, and the sum of the series is 889.

Solution

The formula for the general term of a geometric series is $t_n = ar^{n-1}$, and $t_1 = 7$, $t_n = 448$, and $S_n = 889$.
Since $a = t_1$, then $a = 7$.
Since $t_n = ar^{n-1}$, then $ar^{n-1} = 448$. ①
Since $S_n = 889$, then $\dfrac{a(r^n - 1)}{r - 1} = 889$. ②
Substitute $a = 7$ in ①.

$$7r^{n-1} = 448$$
$$r^{n-1} = 64$$

Now, 64 may be expressed as **i)** 2^6 or **ii)** $(-2)^6$.

Case i):
$r^{n-1} = 2^6$, $r = 2$, and $n - 1 = 6$, or $n = 7$.
Substitute $a = 7$, $r = 2$, and $n = 7$ in ② to check if these values satisfy the sum of the series.

$$\begin{aligned} \text{L.S.} &= \frac{a(r^n - 1)}{r - 1} & \text{R.S.} = 889 \\ &= \frac{7(2^7 - 1)}{2 - 1} \\ &= 7(127) \\ &= 889 \end{aligned}$$

L.S. = R.S.
The values $a = 7$, $r = 2$, and $n = 7$ satisfy the sum of the series.

Case ii):

$r^{n-1} = (-2)^6$, $r = -2$, and $n - 1 = 6$, or $n = 7$.

Substitute $a = 7$, $r = -2$, and $n = 7$ in ② to check if these values satisfy the sum of the series.

$$\text{L.S.} = \frac{a(r^n - 1)}{r - 1} \qquad \text{R.S.} = 889$$

$$= \frac{7[(-2)^7 - 1]}{-2 - 1}$$

$$= \frac{7[-128 - 1]}{-2 - 1}$$

$$= \frac{-903}{-3}$$

$$= 301$$

L.S. $\neq$ R.S.

The sum of the series is not satisfied by $r = -2$.

The general term is $t_n = 7(2)^{n-1}$ and $t_3 = 7(2)^2$, or 28.

A Practise

1. Determine whether each series is geometric, arithmetic, or neither. Justify your answer.

a) $1 + 10 + 100 + \cdots$

b) $6 + 12 + 18 + 24 + \cdots$

c) $-4 + 8 - 16 + 32 - \cdots$

d) $7 - 8 - 10 - 13 - \cdots$

e) $6571 - 2187 + 729 - 243 + \cdots$

f) $\frac{1}{8} + \frac{1}{7} + \frac{1}{6} + \frac{1}{5} + \cdots$

2. For each geometric series, determine the values of a and r. Then, determine the specified sum.

a) S_9 for $3 + 6 + 12 + \cdots$

b) S_{12} for $5 - 10 + 20 - \cdots$

c) S_8 for $\frac{1}{8} + \frac{1}{4} + \frac{1}{2} + \cdots$

d) S_{13} for $-0.2 + 0.6 - 1.8 + \cdots$

e) S_{50} for $9 - 9 + 9 - \cdots$

f) S_{17} for $2 + 0.2 + 0.02 + \cdots$

3. Determine S_n for each geometric series.

a) $a = \frac{1}{81}$, $r = 3$, $n = 7$

b) $f(1) = 6$, $r = -\frac{1}{2}$, $n = 11$

c) $f(1) = 4$, $r = -2$, $n = 10$

d) $f(1) = -6$, $r = 4$, $n = 9$

e) $a = 16$, $r = 5$, $n = 8$

f) $a = -2$, $r = 2$, $n = 13$

g) $f(1) = 0.4$, $r = 1.5$, $n = 6$

h) $a = -4$, $r = -1$, $n = 20$

i) $a = 20$, $r = 6$, $n = 5$

4. Determine the sum of each geometric series.

a) $1 + 5 + 25 + \cdots + 3125$

b) $3 + 9 + 27 + \cdots + 59\,049$

c) $\frac{1}{3} + 1 + 3 + \cdots + 6561$

d) $5 + 20 + 80 + \cdots + 20\,480$

e) $-4 + 12 - 36 + \cdots + 8748$

f) $2700 + 270 + 27 + \cdots + 0.0027$

g) $243 + 81 + 27 + \cdots + \frac{1}{27}$

5. Determine the sum of each geometric series.

a) $2 + 1 + \frac{1}{2} + \cdots + \frac{1}{128}$

b) $3 + 9 + 27 + \cdots + 6561$

c) $100 + 25 + 6.25 + \cdots + 0.390\,625$

d) $5 + \frac{5}{2} + \frac{5}{4} + \cdots + \frac{5}{512}$

e) $5 - \frac{5}{2} + \frac{5}{4} - \cdots - \frac{5}{512}$

B Connect and Apply

6. Determine the specified sum for each geometric series.

a) S_8 for $\sqrt{5} - 5 + 5\sqrt{5} - \cdots$

☆ b) S_{13} for $x + x\sqrt{7} + 7x + 7\sqrt{7}x + \cdots$

c) S_{14} for $4 + 4x + 4x^2 + \cdots$

d) S_{11} for $2 + 2x^2 + 2x^4 + \cdots$

7. Determine the sum of each geometric series.

a) $1 + \frac{5}{2} + \frac{25}{2} + \cdots + \frac{15\,625}{64}$

b) $3\sqrt{6} - 18 + 18\sqrt{6} - \cdots + 839\,808\sqrt{6}$

c) $500 + 500(1.2) + 500(1.2)^2 + \cdots + 500(1.2)^{11}$

d) $8 + 16x^3 + 32x^6 + \cdots + 32\,768x^{36}$

8. Find an expression for the sum of the first n terms of the series with the given general term. Then, use your expression to find the sum when $n = 9$.

☆ a) $t_n = -2(3)^{n-1}$

b) $t_n = 18\left(\frac{2}{3}\right)^{n-1}$

c) $t_n = x^2(x^2)^{n-1}$

9. Determine the sum of the first 10 terms of a geometric series with common ratio 2 and whose tenth term is 16.

10. The sum of the first seven terms of a geometric series is 70 993 and the common ratio is 4. Determine the first term.

11. Determine the fourth term of a geometric series given that the sum of the first seven terms of the series is 1093 and the common ratio is 3^{-1}.

12. Tatiana is training for a marathon that will take place in four months. This week she ran 45 km and intends to increase her distance by 10% each week. Determine the total distance that Tatiana will have run after 10 weeks.

13. In a lottery, the first ticket drawn wins a prize of $25, the second ticket drawn receives a prize of $75, the third ticket drawn receives a prize of $225, and so on. How many prizes can be given out if the total amount of prize money is $1 million?

14. Kayla tried to convince her dad that during the month of April he should pay her allowance in the following manner: 1 penny on the first day of the month, 2 pennies on the second day of the month, 4 pennies on the third day, 8 pennies on the fourth day, and so on until the last day of the month. What is the total allowance Kayla would receive if her dad agrees to her idea? Do you think he will agree? Explain.

15. If the second term of a geometric series is 15 and the sum of the first three terms is 93, determine the general term of the series.

C Extend

16. Show that the sum of $4 + 2 + 1 + 0.5 + 0.25 + \cdots + t_n$ is always less than 8 for any natural number n.

17. Determine the sum of the factors of 3^8.

18. a) Determine an expression for the sum of the first n terms of a series with general term $t_n = 2^n - 3^{n-1}$.

b) Use your expression from part a) to find the sum of the first six terms of the series with the given general term.

Chapter 6 Review

6.1 Sequences as Discrete Functions

1. Write the first three terms of each sequence, given the explicit formula for the nth term of the sequence.

a) $f(n) = 3^{-n}$

b) $t_n = \dfrac{n + 2}{n + 1} + 1$

2. Write the 16th term, given the explicit formula for the nth term of the sequence.

a) $f(n) = n^2 - 6$

b) $t_n = \dfrac{n - 2}{n}$

3. Describe the pattern in each sequence. Write the next three terms of each sequence.

a) 7, −14, 21, −28, …

b) $-\frac{1}{2}, \frac{1}{4}, -\frac{1}{8}, \frac{1}{16}, \ldots$

c) $3x, 6x^2, 12x^3, 24x^4, \ldots$

4. For each sequence, create a table of values using the term number and term, and calculate the finite differences. Then, determine an explicit formula, in function notation, and specify the domain.

a) −5, −10, −15, −20, …

b) 22, 23, 18, 7, …

6.2 Recursive Procedures

5. Write the first four terms of each sequence, where $n \in \mathbb{N}$.

a) $f(1) = 2, f(n) = 5n - f(n - 1)$

b) $f(1) = 5, f(n) = 0.4f(n - 1)$

c) $f(1) = 10, f(n) = \dfrac{f(n - 1)}{1 - n}$

6. Determine a recursion formula for each sequence.

a) 1, 4, 13, 40, …

b) 3, 5, 7, 9, …

c) −2, 2, −10, 26, …

7. Write the first four terms of each sequence.

a) $t_1 = \frac{1}{8}, t_n = 4t_{n-1} - 1$

b) $f(1) = a - 2b, f(n) = f(n - 1) + 3b$

8. Anita convinces her dad to increase her allowance by 12% per week. In the first week she receives \$5.

a) Write a sequence to represent Anita's allowance for 4 weeks.

b) Write a recursion formula to represent her weekly allowance.

c) Write an explicit formula for the nth term of the sequence.

d) What will her allowance be after 10 weeks?

e) After how many weeks will her allowance be \$75.89?

6.3 Pascal's Triangle and Expanding Binomial Powers

9. Use Pascal's triangle to list the numbers in a hockey stick pattern with each of the following sums.

a) 210

b) 70

c) 126

10. Determine the sum of the terms in each of these rows of Pascal's triangle.

a) row 21

b) row 18

11. Express as a single term from Pascal's triangle in the form $t_{n,\,r}$.

a) $t_{8,\,6} + t_{8,\,7}$

b) $t_{12,\,2} + t_{12,\,3}$

c) $t_{23,\,21} + t_{23,\,22}$

12. Use Pascal's triangle to expand each power of a binomial.

 a) $(a + 1)^5$

 b) $(4x^2 - 3y^3)^4$

 c) $\left(1 - \frac{1}{x}\right)^5$

6.4 Arithmetic Sequences

13. For each arithmetic sequence, determine the values of a and d and the formula for the general term. Then, write the next four terms.

 a) $-19, -25, -31, \ldots$

 b) $\frac{8}{3}, \frac{34}{15}, \frac{28}{15}, \ldots$

14. Given the formula for the general term of an arithmetic sequence, determine t_{16}.

 a) $f(n) = 1 + \frac{1}{2}n$

 b) $t_n = 3.2n + 0.8$

15. A jogger running along a path that goes up a hill runs a distance of 423 m in the first minute. As the hill becomes steeper, the jogger runs 14 m less in each subsequent minute.

 a) Determine the general term for the sequence that represents this situation.

 b) How far does the jogger run in the 12th minute?

 c) In which minute does the jogger run 157 m?

6.5 Geometric Sequences

16. Determine whether the sequence is arithmetic, geometric, or neither. Give a reason for your answer.

 a) $\frac{3}{5}, \frac{5}{7}, \frac{7}{9}, \frac{9}{11}, \ldots$

 b) $2, 2\sqrt{3}, 6, 6\sqrt{3}, \ldots$

 c) $x + 7y, 2x + 10y, 3x + 13y, \ldots$

17. For each geometric sequence, determine the formula for the general term and use it to determine the specified term.

 a) $-3, 15, -75, \ldots, t_9$

 b) $-\frac{2}{625}, \frac{2}{125}, -\frac{2}{25}, \ldots, t_{11}$

18. A chain e-mail starts with three people each sending out five e-mail messages. Each of the recipients sends out five messages, and so on. How many e-mail messages will be sent in the ninth round of e-mailing?

6.6 Arithmetic Series

19. For each arithmetic series, state the values of a and d. Then, determine the sum of the first 10 terms.

 a) $21 + 15 + 9 + \cdots$

 b) $-4 - 9 - 14 - \cdots$

20. Determine the sum of each arithmetic series.

 a) $52 + 47 + 42 + \cdots - 48$

 b) $3 + 5.5 + 8 + \cdots + 133$

21. A ball picks up speed as it rolls down a long steep hill. The ball travels 0.75 m in the first second, 1.25 m in the second, 1.75 m in the next second, and so on. Determine the total distance travelled by the ball in 40 s.

22. Determine an arithmetic series such that the sum of the first 9 terms of the series is 162 and the sum of the first 12 terms is 288.

6.7 Geometric Series

23. For the geometric series $100 - 200 + 400 - \cdots$, determine the values of a and r. Then, determine S_8.

24. Determine the sum of each geometric series.

 a) $2 + 6 + 18 + \cdots + 1458$

 b) $1 - \frac{1}{3} + \frac{1}{9} - \cdots - \frac{1}{19\,683}$

25. The sum of the first two terms of a geometric series is 12 and the sum of the first three terms is 62. Determine the series.

Chapter 6 Math Contest

1. Given $f(x) = 5^x$ and $f(x + 3) - f(x + 2) = mf(x)$, determine the value of m.
   A 125 **B** 100 **C** 25 **D** 5

2. Solve the equation $\sqrt{x + 30} = 12 - x$.
   A −5 **B** 5 **C** −6 **D** 6

3. When $f(x)$ is divided by $4x + 1$, a quotient of $2x - 3$ and a remainder of 8 are obtained. Determine $f(x)$.

4. Determine the smallest positive integers x and y such that $\frac{1}{x} + \frac{x}{y} + \frac{1}{xy} = \frac{11}{10}$.

5. A quadratic equation $f(x)$ satisfies $f(1) = 6$, $f(-2) = -9$, and $f(0) = -3$. Determine the value of $f(-3)$.
   A 9 **B** −8 **C** −6 **D** 4

6. Given that $m = x + y$ and $n = xy$, determine $x^2 + y^2$ in terms of m and n.
   A $m^2 - 2n$
   B $m^2 + 2n$
   C $n^2 - 2m$
   D $n^2 + 2m$

7. Without using a calculator, determine which of the following is equivalent to 2^{200}.
   A $2^{100} + 2^{100}$
   B $2^{40} \times 2^{50}$
   C 400
   D 16^{50}

8. Determine the sum of the series $300 - 299 + 298 - 297 + 296 - \cdots + 2 - 1$.
   A 1 **B** 150 **C** 151 **D** 300

9. The sum of the first n terms of a sequence is $S_n = (2n + 1)(n - 1)$. Determine the 15th term of the sequence.
   A 57 **B** 434 **C** 377 **D** 61

10. State an explicit formula for the general terms of two different geometric sequences, such that the sum of the first two terms is 2 and the sum of the first three terms is 3.

11. Consider the sequence 4, −6, x, y. The first three terms of this sequence form an arithmetic sequence. The last three terms form a geometric sequence. Determine the values x and y.

12. Find the sum of the 52 terms in the series $\sin 90° + \sin 180° + \sin 270° + \sin 360° + \sin 450° + \cdots + \sin 4590° + \sin 4680°$.
   A 0 **B** −1 **C** 26 **D** 1

13. Determine two different geometric sequences such that $t_1 + t_2 + t_3 = 3$ and $t_3 + t_4 + t_5 = 12$.

14. Find the sum of the factors of 2^{10}.
   A 2047
   B 20
   C 2104
   D 1024

15. How many four-digit numbers greater than 5000 can be formed from 1, 3, 5, and 7 if the numbers can be repeated?
   A 98 **B** 12 **C** 128 **D** 102

16. A school has 8 doors that lead outside. In how many ways can a student enter the school through one door and leave through a different door?
   A 65 **B** 56 **C** 16 **D** 7