# 3.1 The Nature of Exponential Growth

**Student Text Pages**
150 to 157

**Suggested Timing**
75 min

**Tools**
- coloured tiles or linking cubes
- grid paper
- graphing calculator
- computer with *The Geometer's Sketchpad®*

**Related Resources**
- G–1 Grid Paper
- BLM 3–2 Section 3.1 Practice

**Differentiated Instruction**
- Use **Think-Pair-Share** to complete the Investigate and answer the Reflect questions.
- Construct a **Frayer model** on exponential growth. Use the Examples and Key Concepts from this section for definitions, characteristics, examples, and non-examples of exponential growth.

## Teaching Suggestions

- The **Investigate** provides a vivid illustration of the dramatic nature of exponential growth. After building just two more stages of each pattern, students will see that the pattern exhibiting exponential growth grows far more rapidly than the other patterns.
- Encourage students to use a graphing calculator or graphing software in step 6a) of the **Investigate**.
- An alternative way to explore exponential growth is to challenge students to fold a piece of paper neatly in half ten times. Students will expect this task to be straightforward, but it will quickly become apparent that it is impossible to get ten folds. Students can be asked to produce the exponential function that gives the number of layers at each stage.
- Note that both the paper folding activity described above and the **Investigate** involve a doubling pattern. **Example 1** can be used to further emphasize the more general patterns of finite differences for exponential functions.
- In **Example 1**, students explore the nature of exponential growth within a contextual situation. As they work through the problem there are opportunities for them to see exponential growth patterns represented
  - as a table of values
  - in the form of an equation
  - as a graph

  The analysis of first and second differences illustrates that an exponential growth function increases at an increasing rate over its domain. Students should discover that the ratio of successive terms is constant within each set of finite differences. Ask students, "What will the third differences look like? Is this true for all exponential functions?"
- Graphing technology can be a powerful tool for expediting the numerical calculations and for viewing the various representations in **Examples 1** and **2**. In **Example 1** part **e**), students can use a TI-83 Plus or TI-84 Plus to calculate first and second differences. In **Example 2**, students can use the TI-83 Plus or TI-84 Plus to determine the $y$-intercept of an exponential graph.
- In **Example 2**, a patterning approach is used to reinforce student understanding of zero exponents. By tracing an exponential function to its $y$-intercept, some graphing calculators (or graphing software) can be used to verify this result. Note that not all calculators or software recognize $0^0$ as undefined (see **question 6**). Spreadsheet software or a graphing calculator can be used to perform the repeated calculations used in the pattern or to replicate the same result using
  a different base.

**Investigate Answers (pages 150–151)**

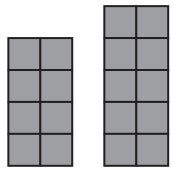
**1.** Answers may vary. Sample answer: Pattern 1 increases by adding 2 to the previous term, pattern 2 increases by adding consecutive odd numbers to the previous term, and pattern 3 increases by doubling the number of tiles in the previous term.

**2.** Answers may vary. Sample answer:

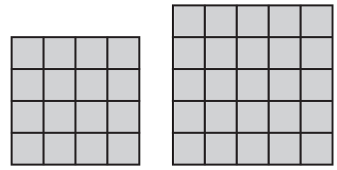
**a)** Pattern 2 is growing fastest, since it has the most squares in the third term.

**b)** Pattern 1 is growing slowest, since it has the fewest squares in the third term.

**3. a)** Pattern 1



Pattern 2



Pattern 3

**b)** Answers may vary. Sample answer: Yes.

**4. a)** Pattern 1

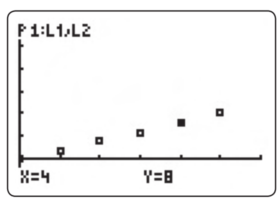
| Term Number, $n$ | Number of Squares, $t$ | First Differences | Second Differences |
|---|---|---|---|
| 1 | 2 | 2 | |
| 2 | 4 | 2 | 0 |
| 3 | 6 | 2 | 0 |
| 4 | 8 | 2 | 0 |
| 5 | 10 | | |

**b)** Answers may vary. Sample answer: The first differences are constant and the second differences are zero.

**c)** Answers may vary. Sample answer: Linear, since the number of squares increases by a constant value of 2 as the pattern is extended.

**d)** $t = 2n$

**e)**




**5.** Pattern 2

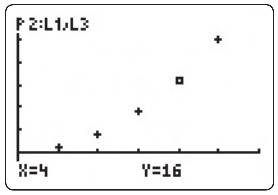
**a)**

| Term Number, $n$ | Number of Squares, $t$ | First Differences | Second Differences |
|---|---|---|---|
| 1 | 1 | 3 | |
| 2 | 4 | 5 | 2 |
| 3 | 9 | 7 | 2 |
| 4 | 16 | 9 | 2 |
| 5 | 25 | | |

**b)** Answers may vary. Sample answer: The first differences are increasing by 2 and the second differences are constant.

**c)** Answers may vary. Sample answer: Non-linear, since the number of squares does not increase by a constant value in the first differences column.

**d)** $t = n^2$

e) 

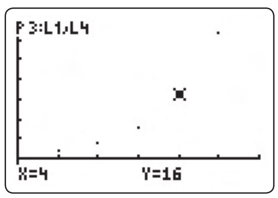

Pattern 3

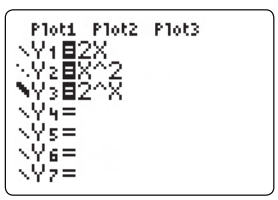
| Term Number, *n* | Number of Squares, *t* | First Differences | Second Differences |
|---|---|---|---|
| 1 | 2 | | |
| | | 2 | |
| 2 | 4 | | 2 |
| | | 4 | |
| 3 | 8 | | 4 |
| | | 8 | |
| 4 | 16 | | 8 |
| | | 16 | |
| 5 | 32 | | |

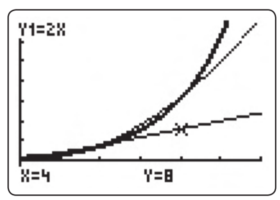
**b)** Answers may vary. Sample answer: The first differences and the second differences are not constant. Each term is twice the value of the previous term.

**c)** Answers may vary. Sample answer: Non-linear, since the number of squares does not increase by a constant value in the first differences column.

**d)** $t = 2^n$

**e)** 


**6. a)** 




**b)** Answers may vary. Sample answer: They are alike in that they all increase, but they differ by the rate of this increase.

**c)** Answers may vary. Sample answer: Blue tiles, since the number of squares in pattern 3 increases faster as the term number increases than in the other patterns.

**Communicate Your Understanding Responses (page 154)**

**C1 a)** C

**b)** Answers may vary. Sample answer: The coefficient 50 corresponds to the initial population, and 3 represents the tripling factor.

**C2** Answers may vary. Sample answers:

**a)** The functions differ in that the first is quadratic, the second is linear, and the third is exponential. The graph of the exponential function increases much more dramatically than the others, and the quadratic function is the only one for which the $y$-values increase as the $x$-values decrease.

**b)** They all have a range that is $\{x \in \mathbb{R}\}$, but their domains differ. The quadratic function has range $\{y \in \mathbb{R}, y \geq 0\}$, the linear function has range $\{y \in \mathbb{R}\}$, and the exponential function has range $\{y \in \mathbb{R}, y > 0\}$.

**C3** $y = 3x$ is linear, $y = 3^x$ is exponential

**C4** Answers may vary. Sample answers:

**a)** A linear function, like $y = 2x + 3$, has constant first differences.

**b)** A quadratic function, like $y = 2x^2 + 3$, has constant second differences.

**c)** An exponential function, like $y = 2^x + 3$, has a repeating pattern of finite differences: the ratio of successive finite differences is constant.

**C5** Answers may vary. Sample answer: Yes. Both simplify to $1 - 1 = 0$.

**Common Errors**

- Some students may intuitively think that a base raised to a zero exponent equals zero.

$R_x$ Have students explore zero exponents using the methods presented in Example 2 (patterning and graph tracing) and question 3 (algebraic reasoning). Emphasize that raising a base to the exponent zero is not the same as multiplying a value by zero.

## Practise, Connect and Apply, Extend

- **Question 1** gives students an opportunity to confirm the observations of the **Investigate** and **Example 1** regarding the nature of exponential growth.
- When answering **question 2** students are likely to apply the patterning approach similar to that used in **Example 2**. **Question 3** provides an alternative method to develop the same result for a zero exponent.
- The approach taken in **question 3** requires students to apply prior knowledge of exponent rules and algebraic reasoning to confirm the meaning of a zero exponent.
- For **question 5**, consider having students work in pairs or small groups. Suggest that they start with a table and work out the value of the expression when $n = 1, 2, 3$, and 4.
- **Question 5** requires students to use their reasoning and reflecting skills to select tools to design a growing pattern. Students have to draw diagrams to represent the first four terms of $t(n) = 3n$, and they will use their connecting and communicating skills to answer questions related to this function.
- For **question 6**, note that the TI-84 Plus gives the correct result in part **c**), while *The Geometer's Sketchpad*® does not.
- **Question 7** should prompt an interesting discussion about the application of exponential functions in contexts that are not obviously related.
- When discussing answers to **question 8**, prompt students to see if they can find a connection between this scenario and the three growing patterns presented in the **Investigate**.
- **Question 9** requires students to reason and reflect on the given information in order to represent it as an equation. They will select tools and use their connecting skills to sketch a graph of the relationship during its first month. They will use their communicating skills to answer questions related to the graph.
- The financial applications introduced in **questions 10** and **12** are revisited in greater depth in Chapter 7.
- **Question 13** provides an opportunity to begin a stockpile of resources that can be re-used from year to year.
- Graphing technology may be a useful tool for students when approaching **question 14**.
- Use **BLM 3–2 Section 3.1 Practice** for remediation or extra practice.

### Mathematical Process Expectations

The table shows questions that provide good opportunities for students to use the mathematical processes.

| Process Expectation | Selected Questions |
|---|---|
| Problem Solving | 8, 14, 15 |
| Reasoning and Proving | 1, 2, 5–10, 12–15 |
| Reflecting | 1, 5, 6, 8, 9, 12–15 |
| Selecting Tools and Computational Strategies | 4–15 |
| Connecting | 1–15 |
| Representing | 1, 5, 6, 9, 10 |
| Communicating | 1–3, 5, 6, 8–10, 12, 13, 15 |

# Use Technology

**Student Text Pages**
158 to 159

**Suggested Timing**
10–15 min

**Tools**
- TI-Nspire™ CAS graphing calculator

## Use the Lists and Trace Features on a TI-Nspire™ CAS Graphing Calculator

### Teaching Suggestions

- In **A**, the **Lists & Spreadsheet** application is used to perform repeated finite difference calculations that occur in **Example 1** of Section 3.1. In **B** and **C**, the **Lists & Spreadsheet** and **Graphs & Geometry** applications are used to represent and explore the exponential relation presented in **Example 2** of Section 3.1. These two representations can be viewed simultaneously by setting the viewing window to split-screen view. Refer to step 7 of the Use Technology activity on page 87 for instructions.
- If you know the name of the operation that you want—for example, δlist(pop) in step 3—you can type it directly into the formula cell without having to access the catalogue.
- Note that using the **Trace** function, as required in **C**, can cause the calculator to reset your window settings.

# 3.2 Exponential Decay: Connecting to Negative Exponents

**Student Text Pages**
160 to 169

**Suggested Timing**
75–150 min

**Tools**
- graphing calculator
- computer with spreadsheet software

**Related Resources**
- BLM 3–3 Section 3.2 Practice

**Differentiated Instruction**
- Complete the Investigate and answer the Reflect questions using **Think-Pair-Share**.
- Construct a **Frayer model** for exponential decay. Use the Examples and Key Concepts for definitions, characteristics, examples, and non-examples of exponential decay.
- Use **timed retell** to answer Communicate Your Understanding questions C1, C3, and C4.

### Teaching Suggestions

- This lesson may be split over two periods. Monitor student progress through the **Investigate** and **Example 1**, leaving **Examples 2** to **4** and the related exercises for the next period, if necessary.
- The purpose of the **Investigate** is to illustrate the similarities and differences between exponential growth and exponential decay. Nuclear decay is one type of exponential decay encountered in the physical sciences. The method used in the **Investigate** is to perform an exponential regression analysis on data generated in a table. Students should examine the equation produced by this regression and reflect on why it makes sense.
- If you have used the paper folding activity described in the Teaching Suggestions for Section 3.1, you can use the same activity again to illustrate exponential decay. Instead of focusing on the number of layers, focus on the change in the visible area as the sheet of paper is repeatedly folded. After one fold, $\frac{1}{2}$ of the original area is showing; after two folds, $\frac{1}{4}$ of the original area is showing; and so on.
- **Example 1** requires students to develop an equation to model exponential decay. The equation is used to find an unknown appearing in the exponent. Three methods of solution are presented: using a table or spreadsheet, carrying out a systematic trial, and using graphing technology. Many students will benefit from seeing this variety of approaches. Students will learn an algebraic method to solve this type of problem when they study logarithms in grade 12.
- **Examples 2** to **4** highlight the relationship between powers with negative exponents and corresponding powers with positive exponents.