4.1 Special Angles

Student Text Pages
222–231

Suggested Timing
60–70 min

Tools
- geometry set
- grid paper
- computer with *The Geometer's Sketchpad®*

Related Resources
- G–1 Grid Paper
- T–2 *The Geometer's Sketchpad®* 4
- BLM 4–2 Trigonometric Ratios of Special Angles
- BLM 4–3 Section 4.1 Practice

Differentiated Instruction
- Prior to starting the chapter, use an **anticipation guide** to form conjectures about the questions "Are there trigonometric ratios for angles greater than 90°?" and "If so, how are these ratios calculated?" Discuss the questions as a large group to review prior trigonometry skills. Post the conjectures in the classroom to compare to new learning as the chapter unfolds.
- Construct a **word wall** of terms relevant to this chapter. Start with the terms *unit circle*, *initial arm*, *terminal arm*, *reference angle*, and *quadrant(s)*. Include a picture with each definition for quick reference throughout the chapter.
- For Investigate C, use **Think-Pair-Share** to discuss why sin 230° and cos 230° are negative, but tan 230° is positive.
- Encourage students to create a method for determining the sign of a trigonometric ratio and to write their responses in a math **journal**.

Teaching Suggestions

- Each **Investigate** has a distinct focus:
  - **Investigate A** considers how to find exact trigonometric ratios for special angles using the unit circle.
  - **Investigate B** considers how to find trigonometric ratios for angles greater than 90°.
  - **Investigate C** considers how to find approximate trigonometric ratios for any angle.
- Consider pausing the class briefly after each **Investigate** to ensure that all students have grasped the concept. If an interactive whiteboard (such as a SMART Board™) is available, consider having students take turns adding to the solution.
- You may wish to have students work in pairs or small groups.
- If students are using *The Geometer's Sketchpad®* for **Investigate A**, these steps produce the required construction:
  - Create the unit circle, then select the point tool and place a point at the intersection of the positive x-axis and the circle. Select this new point and the origin, and then choose **Segment** from the **Construct** menu. The resulting line segment will be the initial arm.
  - Select the initial arm, and then choose **Rotate** from the **Transform** menu. Enter 45° to product the terminal arm.
  - Select point A at the intersection of the terminal arm and the unit circle. Choose **Perpendicular Line** from the **Construct** menu. Point B is at the intersection of this perpendicular line and the x-axis.
  - Construct a line segment between point B and the origin.
  - Students can select points A, O (the origin), and B, and then choose **Angle** from the **Measure** menu to verify their construction. This angle should be 45°.

  If needed, use **T–2 *The Geometer's Sketchpad®* 4** to support this activity.
- If students have used *The Geometer's Sketchpad®* for **Investigate A**, they can select line segment OA and then choose **Reflect** from the **Transform** menu to complete the construction required for **Investigate B**. An alternative method is to select line segment OA, then choose **Rotate** from the **Transform** menu and enter 90°.
- After finishing **Example 1**, demonstrate how a computer algebra system (CAS) displays either exact or approximate values for calculations, including trigonometric ratios.
- Point out that $\tan \theta$ equals the slope of the terminal arm. This will provide another context for undefined tangents such as tan 90° and tan 270°.
- Use a calculator to evaluate sin 80°, sin 85°, sin 89°, sin 89.9°, sin 89.99°, and so on, to show how the value of sine approaches 1 as the angle approaches 90°. Do a similar activity for sin 0°, cos 90°, cos 0°, tan 90°, and tan 0°.
- As you work through succeeding examples, have students begin to complete the summary sheet of the exact trigonometric ratios for special angles given in **BLM 4–2 Trigonometric Ratios of Special Angles**. Space has been left on the sheet for the reciprocal ratios that students will study later. As students work through the homework exercise, they can fill in more of the table. Advise students to look for patterns as they become apparent.
- **Example 3** demonstrates to students how they can solve a problem involving the cosine law and an "unfriendly angle" without resorting to "black box" calculator operations.

- Consider having students add the diagrams shown in the **Key Concepts**, including the "friendly triangles," to their summary sheet of exact trigonometric ratios (**BLM 4–2 Trigonometric Ratios of Special Angles**).
- You can extend **question C1** by asking students what would happen if the terminal arm were moved in a clockwise direction.
- Although the entire section can be done using primarily pencil and paper, with a scientific calculator to check answers or evaluate trigonometric ratios, take advantage of any opportunity to at least demonstrate alternate technology, especially the use of the computer algebra system on the TI-Nspire™ CAS calculator. If you have provided students with home copies of GSP, you can assign technology use as part of the homework. If an interactive whiteboard is available, consider having students take turns demonstrating alternative solutions using technology.

Investigate Answers (pages 222–224)

Investigate A

3. $\triangle AOB$ is a right isosceles triangle. $\angle B$ is a right angle and both $\angle A$ and $\angle O$ measure 45°.

4. The length of OB = AB = $\frac{1}{\sqrt{2}}$ units, and the length of OA = 1 unit.

5. a) Using the side lengths gives $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, and $\tan 45° = 1$.

b) Evaluating these expressions to 4 decimal places gives $\sin 45° \doteq 0.7071$, $\cos 45° \doteq 0.7071$, and $\tan 45° = 1$.

6. Using a calculator gives $\sin 45° \doteq 0.7071$, $\cos 45° \doteq 0.7071$, and $\tan 45° = 1$.

7. a) The values found using the side lengths are the same as those found using a calculator.

b) The length of side OB is equal to the value of $\cos 45°$.

c) The length of side AB is equal to the value of $\sin 45°$.

d) Answers may vary. Sample answer: The coordinates of point A are (0.7071, 0.7071). Using the results of parts b) and c), this can be represented as ($\cos 45°$, $\sin 45°$).

Investigate B

1. The measure of the angle between OC and the negative x-axis is 45°.

2. The coordinates of point C are $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

3. Using the coordinates of point C, $\cos 135° = -\frac{1}{\sqrt{2}}$ and $\sin 135° = \frac{1}{\sqrt{2}}$.

4. Answers may vary. Sample answer: Since the tangent ratio of an angle is the ratio of the opposite side to the adjacent side, $\tan 135° = \frac{\frac{1}{\sqrt{2}}}{-\frac{1}{\sqrt{2}}}$, which simplifies to -1.

5. a) $\tan 135° = -1$

b) The slope of OC and the tangent of 135° are both equal to -1.

6. Answers may vary. Sample answer: The values of $\sin 135°$, $\cos 135°$, and $\tan 135°$ found in steps 3 and 5 are the same as the calculator values for those ratios.

Investigate C

3. Answers may vary. Sample answer: The length of OE is approximately 0.75 units, and the length of ED is approximately 0.65 units. So, the coordinates of point D are (-0.65, -0.75).

4. Using a calculator, $\sin 230° \doteq -0.7760$, $\cos 230° \doteq -0.6428$, $\tan 230° \doteq 1.1918$.

5. Answers may vary. Sample answer: In $\triangle EOD$, sides OE and ED are both negative, but the hypotenuse of $\triangle EOD$ is positive. So, the sine and cosine ratios are both negative. The tangent of an angle is the ratio of the opposite side to the adjacent side. Since the opposite side and the adjacent side are both negative, the tangent of 230° will be positive.

Communicate Your Understanding Responses (page 228)

C1 Answers may vary. Sample answer: From 0° to 90° the cosine ratio is positive, and changes to negative from 90° to 270°, and once again becomes positive from 270° to 360°. The x-value for a point on the unit circle changes sign in the same way. From 0° to 180°, the sine ratio is positive, and changes to negative from 180° to 360°. From 0° to 90°, the tangent ratio is positive, and changes to negative from 90° to 180°. It changes back to positive from 180° to 270°, and then changes back to negative from 270° to 360°.

C2 Answers may vary. Sample answer: The tangent ratio is undefined at angles of 90° and 270°, since the tangent ratio is defined as the value of the sine ratio divided by the value of the cosine ratio, and at these two angles the cosine ratio is zero. Since division by zero is undefined, the tangent ratio will be undefined at these angles.

C3 Answers may vary. Sample answer: In the fourth quadrant, the cosine ratio is positive, but both the sine and tangent ratios are negative. A point in the fourth quadrant has a positive x-value (which corresponds to a positive adjacent side) and a negative y-value (which corresponds to negative opposite side). Since the hypotenuse is always positive, the sign of a trigonometric ratio depends only on the signs of the other sides in the ratio.

Common Errors

- Some students may become confused about which side lengths should be used for a desired trigonometric ratio, especially when a diagram does not sit in the same orientation as triangles in the unit circle.

R_x Have students label the desired angle on the diagram. Then, identify and label the opposite side, adjacent side, and hypotenuse, with reference to the angle.

- Some students will forget to use the scale factor when using the unit circle to determine trigonometric ratios of angles other than the "friendly" angles.

R_x Continually reinforce the use of the scale factor at every opportunity, until it becomes second nature.

Practise, Connect and Apply, Extend

- Students can use the results of **questions 5** and **6** to fill in more of their summary sheet of exact trigonometric ratios (**BLM 4–2 Trigonometric Ratios of Special Angles**). When finished, students should see patterns emerging. The patterns can be used to add approximate values for each ratio with only a few calculator operations required.
- **Question 10** shows students how to develop and apply the CAST rule. The diagram can be added to the summary sheet of exact trigonometric ratios (**BLM 4–2 Trigonometric Ratios of Special Angles**).
- **Question 13** requires students to reflect on and reason through the details to produce a diagram representing the situation. Students will also have to reflect and reason to find the length of the second wire without finding the angle that the second wire makes with the ground. Then, students communicate their findings. To solve the algebraic problems in this question, students will select appropriate tools and connect this situation to their mathematical knowledge.
- **Question 14** reinforces the patterns formed by the values of trigonometric ratios as one proceeds around the unit circle. These patterns will be important when studying the periodic functions in the next chapter, and when solving trigonometric equations.
- **Question 15** gives each student the opportunity to develop his or her own real-world problem and to solve it using reflecting and reasoning skills. To solve their own problem and a classmate's problem, students will have to select adequate tools and use connecting skills from past mathematical knowledge. Students will then communicate a judgment of the mathematical correctness of their classmate's solution.
- **Questions 19** and **20** form a trigonometric proof of one of Euclid's geometrical propositions. You can ask students to do a brief investigation of who Euclid was and what he contributed to mathematics.
- Use **BLM 4–3 Section 4.1 Practice** for remediation or extra practice.

Ongoing Assessment

Achievement Check, question 18, on student text page 230.

Achievement Check, question 18, student text page 230

This performance task is designed to assess the specific expectations covered in Section 4.1. The following mathematical process expectations can be assessed.

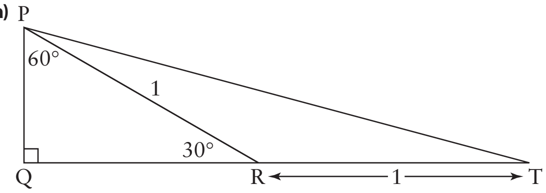
- Problem Solving
- Reasoning and Proving
- Reflecting
- Selecting Tools and Computational Strategies
- Connecting
- Representing
- Communicating

Ongoing Assessment

Achievement Check, question 18, on student text page 230.

Sample Solution

a)



b) $\angle PRT = 180° - 30°$ (straight angle or supplementary angle)
$= 150°$

Since PR = RT, $\triangle PRT$ is isosceles and $\angle RPT = \angle T$.

$\angle T = (180° - 150°) \div 2$ (Angle Sum of a Triangle theorem)
$= 30° \div 2$
$= 15°$

Therefore, $\angle T = 15°$.

Alternative Method

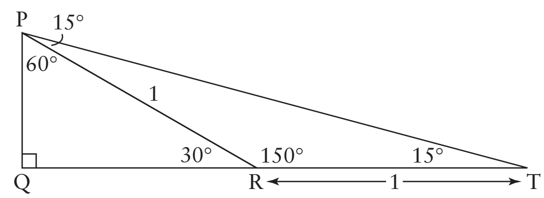
Since PR = RT, then $\triangle PRT$ is isosceles and $\angle RPT = \angle T$.

Let $\angle RPT = \angle T = x$

Then in $\triangle PQT$ the sum of the angles is 180°.

$$90° + 60° + x + x = 180°$$
$$150° + 2x = 180°$$
$$2x = 180° - 150°$$
$$2x = 30°$$
$$x = 30° \div 2$$
$$= 15°$$

Therefore, $\angle T = 15°$.



c) Since $\tan T = \dfrac{\text{opposite}}{\text{adjacent}}$

$= \dfrac{PQ}{QT}$

In $\triangle PQT$, QT = QR + RT where RT = 1 (since RT = PR).

The two lengths that we need to know are PQ and QR.

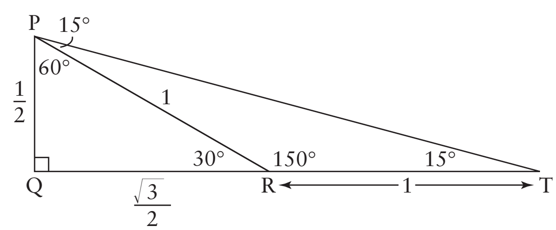
d) Since the angle measures of $\triangle PQR$ are 30°, 60°, and 90°, and the length of the hypotenuse is 1, $\triangle PQR$ is a special triangle with $\sin 30° = \dfrac{1}{2}$ and $\cos 30° = \dfrac{\sqrt{3}}{2}$.

Therefore, $PQ = \dfrac{1}{2}$ and $QR = \dfrac{\sqrt{3}}{2}$.

$QT = QR + RT$

$= 1 + \dfrac{\sqrt{3}}{2}$

$= \dfrac{2 + \sqrt{3}}{2}$



e) $\tan T = \dfrac{PQ}{QT}$ Substitute the values for PQ and QT found in part d).

$$= \frac{\frac{1}{2}}{\frac{2 + \sqrt{3}}{2}}$$

$$= \frac{1}{2} \times \frac{2}{2 + \sqrt{3}}$$

$$= \frac{1}{2 + \sqrt{3}}$$

Level 3 Notes

Look for the following:

- Diagram(s) are mostly labelled and mostly correct
- Angle measures are mostly correct
- Understanding of which lengths are needed to find the tangent ratio is mostly evident
- Understanding of how to use the special triangles to find unknown lengths is mostly evident
- Exact answers are mostly correct and in mostly simplified form
- Justifications and explanations of solutions, where required, are mostly valid

What Distinguishes Level 2

- Diagram(s) are partially labelled and somewhat correct
- Angle measures are somewhat correct
- Understanding of which lengths are needed to find the tangent ratio is somewhat evident
- Understanding of how to use the special triangles to find unknown lengths is somewhat evident
- Exact answers are somewhat correct and in somewhat simplified form
- Justifications and explanations of solutions, where required, are somewhat valid

What Distinguishes Level 4

- Diagram(s) are clearly labelled and accurate
- Angle measures are accurate
- Understanding of which lengths are needed to find the tangent ratio is clearly evident
- Understanding of how to use the special triangles to find unknown lengths is clearly evident
- Exact answers are accurate and in fully simplified form
- Justifications and explanations of solutions, where required, are accurate

Mathematical Process Expectations

The table shows questions that provide good opportunities for students to use the mathematical processes.

Process Expectation	Selected Questions
Problem Solving	13, 15–17, 20, 21
Reasoning and Proving	10–21
Reflecting	10, 13, 15–17, 19–21
Selecting Tools and Computational Strategies	12–14, 16–20
Connecting	3, 5–8, 10–21
Representing	3–9, 11, 13, 14, 16, 18
Communicating	13