Mathematical Process Expectations

The table shows questions that provide good opportunities for students to use the mathematical processes.

Process Expectation	Selected Questions
Problem Solving	1–19
Reasoning and Proving	1–19
Reflecting	4–10, 12–19
Selecting Tools and Computational Strategies	2–13, 15–17
Connecting	1–19
Representing	4, 8, 10, 11
Communicating	4, 6, 9, 13, 18, 19

4.6 Trigonometric Identities

Student Text Pages
270–275

Suggested Timing
60–70 min

Tools
- graphing calculator

Related Resources
- BLM 4–2 Trigonometric Ratios of Special Angles
- BLM 4–12 Section 4.6 Practice

Teaching Suggestions

- Some teachers prefer to teach identities after covering the graphs of trigonometric functions. Therefore, you may prefer to move this section to the end of Chapter 5.
- The formula in the introductory text for this section is only true for the given speed. The general form of the trajectory equation is beyond the scope of this chapter.
- Before beginning the **Investigate**, establish the difference between equations that are not identities and those that are. An equation such as $\sin\theta = \cos\theta$ may be true for certain values of θ, such as $\theta = 45°$, but not true for other values, such as $\theta = 60°$. An identity, such as $\tan\theta = \frac{\sin\theta}{\cos\theta}$, is true for any value of θ. Have students try different values of θ for each of these.
- After completing the **Investigate**, emphasize what does and what does not constitute a proof.
- Before presenting **Example 3**, it may be helpful to review the concept of difference of squares.
- After completing the **Examples**, encourage students to look at both sides of an equation before beginning a proof.
- Consider having students add the Pythagorean, quotient, and reciprocal identities to their summary sheet of exact trigonometric ratios (**BLM 4–2 Trigonometric Ratios of Special Angles**), for use on homework assignments, quizzes, and tests.
- Students will likely have little experience with formal proofs. Take time to establish why the L.S./R.S. method must be used.
- Be sure students understand that only one counterexample is needed to show that an equation is not an identity.

Differentiated Instruction

- Add the term *identity* and the equations for the Pythagorean identity, the quotient identity, and the reciprocal identities to the **word wall** constructed in previous sections.
- Use **graffiti** to practise proving various identities. Put students in groups of two to four. Assign a different problem to each group. Within their group, students pass around a large sheet of paper and take turns adding lines to the solution. Post the completed solutions around the room. Use a **gallery walk** and have groups circulate to review the work of their peers. Each group could complete a formal check of at least one other solution in the gallery walk.

Common Errors

- Students often find proofs confusing, and may not know how to begin.

R_x Reassure students that most mathematicians often use a trial and error method when first approaching a proof that is not obvious. There is no harm in going up a blind alley, then trying another tack when it does not work. In fact, a student can learn just as much from something that does not work as from something that does.

- It is often tempting to assume that all proofs involving squares such as $\sin^2\theta$ should be pursued using the Pythagorean identity, but this is not always the case.

R_x Encourage students to keep an open mind when attempting proofs, and to consider more than one approach before starting to write down a sequence of steps.

- Students will often implicitly assume that an identity is true as part of a proof and fail to use the L.S./R.S. method. For example, in question 4a) it is tempting to just divide both sides by csc A and simplify.

R_x Take the time to explain to students why this approach is not a legitimate form of proof, and why the L.S./R.S. method must be used.

Investigate Answers (pages 270–271)

1. The primary trigonometric ratios are $\sin\theta = \frac{y}{r}$, $\cos\theta = \frac{x}{r}$, and $\tan\theta = \frac{y}{x}$.

2. a)
$$\frac{\sin\theta}{\cos\theta} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{y}{x}$$

b) This ratio is equivalent to the tangent ratio.

c) $\frac{\sin\theta}{\cos\theta} = \tan\theta$

3. Answers may vary.

4. a)
$$\begin{aligned}\sin^2\theta + \cos^2\theta &= \left(\frac{y}{r}\right)^2 + \left(\frac{x}{r}\right)^2\\ &= \frac{y^2}{r^2} + \frac{x^2}{r^2}\\ &= \frac{x^2+y^2}{r^2}\\ &= \frac{r^2}{r^2}\\ &= 1\end{aligned}$$

b) $\sin^2\theta + \cos^2\theta = 1$

5. Answers may vary. Sample answer: If an equation is shown to hold for several different angles, that does not constitute a proof that the equation is an identity. It could still be the case that there is an angle for which the equation does not hold.

6. Answers may vary. Sample answer: It is called the Pythagorean identity because it involves using the identity $x^2 + y^2 = r^2$, which is an instance of the Pythagorean theorem involving a point (x, y) on a circle of radius r.

Communicate Your Understanding Responses (page 273)

Answers may vary. Sample answer:

C1 While the angle she has chosen solves the equation, it is only an example. Particular cases cannot be used to prove identities.

C2 An equation may be true for some values and not true for others, but an equation that is an identity will be true for all values of the variable.

C3 A counterexample for the equation in C1 is $\theta = 90°$.

Practise, Connect and Apply, Extend

- In principle, all required identities can be proved from basic definitions of trigonometric ratios, as in **question 1**. However, it is usually faster and easier to use known identities as a starting point.
- In **question 10**, graphing calculators are used to demonstrate whether an equation may or may not be an identity. Emphasize that this approach only provides guidance, and is not a method of proof (see **question 11**). You can further illustrate this point by asking students to consider an equation like $\cos\theta = 1.01\cos\theta$.
- **Question 12** requires students to reflect and reason in order to select the tools needed to prove the given identity. Students will connect newly acquired concepts to their prior mathematical knowledge in order to prove the identity.
- For **questions 12**, **13**, and **17**, it may be helpful to remind students to find suitable window settings for their graphs, if they choose to use a graphing calculator. In each case, the window settings from **question 10** could be used.
- **Question 15** gives students the opportunity to select the tools needed to draw the diagram. Students use their reasoning and reflecting skills to interpret the diagram and complete the required proof.

- When discussing **question 18**, especially part **b**), reinforce the understanding that showing that an equation is true for selected values does not constitute a proof that the equation is an identity.
- Use **BLM 4–12 Section 4.6 Practice** for remediation or extra practice.

Ongoing Assessment

Achievement Check, question 17, on student text page 275.

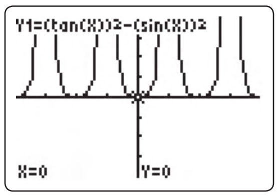
Achievement Check, question 17, student text page 275

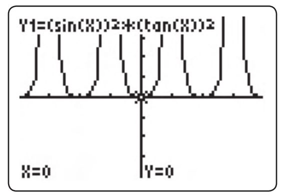
This performance task is designed to assess the specific expectations covered in Section 4.6. The following mathematical process expectations can be assessed.

- Reasoning and Proving
- Reflecting
- Selecting Tools and Computational Strategies
- Connecting
- Representing
- Communicating

Sample Solution

a) Yes, it appears to be an identity because the graphs of $y = \tan^2\theta - \sin^2\theta$ and $y = \sin^2\theta\tan^2\theta$ appear to be the same.





b) Answers may vary. Sample answer: Use the quotient identity $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$, so $\tan^2\theta = \dfrac{\sin^2\theta}{\cos^2\theta}$.

c)

$$
\begin{aligned}
\text{L.S.} &= \tan^2\theta - \sin^2\theta \\
&= \frac{\sin^2\theta}{\cos^2\theta} - \sin^2\theta && \text{Use the quotient identity.} \\
&= \frac{\sin^2\theta}{\cos^2\theta} - \frac{\sin^2\theta\cos^2\theta}{\cos^2\theta} \\
&= \frac{\sin^2\theta(1 - \cos^2\theta)}{\cos^2\theta} && \text{Find a common factor.} \\
&= \sin^2\theta \times \frac{\sin^2\theta}{\cos^2\theta} && \text{Use the Pythagorean identity.} \\
&= \sin^2\theta\tan^2\theta && \text{Use the quotient identity.} \\
&= \text{R.S.}
\end{aligned}
$$

d) Answers may vary. Yes and no are both possible answers, depending on the approach students use in their solutions.

Level 3 Notes

Look for the following:

- Understanding of how to verify the identity graphically is mostly evident
- Selects identities that result in a mostly correct proof of the identity
- L.S./R.S. reasoning is mostly used to prove the identity
- Mostly valid reasoning/justification is provided to support answers, where required

What Distinguishes Level 2

- Understanding of how to verify the identity graphically is somewhat evident
- Selects identities that result in a somewhat correct proof of the identity
- L.S./R.S. reasoning is somewhat used to prove the identity
- Somewhat valid reasoning/justification is provided to support answers, where required

What Distinguishes Level 4

- Understanding of how to verify the identity graphically is clearly evident
- Selects identities that result in a correct proof of the identity
- L.S./R.S. reasoning is properly used to prove the identity
- Clearly valid reasoning/justification is provided to support answers, where required

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Process Expectation	Selected Questions
Problem Solving	1, 3–9, 12–15, 18–20
Reasoning and Proving	1, 3–9, 11–20
Reflecting	7–9, 11–15, 18–20
Selecting Tools and Computational Strategies	3–9, 12–15, 17–20
Connecting	1–20
Representing	2, 15, 17–20
Communicating	2, 10, 11, 17