6.2

Recursive Procedures

Student Text Pages
365 to 372

Suggested Timing
50–75 min

Tools
- grid paper
- isometric dot paper
- computer with *The Geometer's Sketchpad®* (optional)
- computer with spreadsheet software (optional)

Related Resources
- G–1 Grid Paper
- BLM 6–5 Section 6.2 Practice

Differentiated Instruction
- Use **cooperative task groups** to complete the Investigate. Have one student read the instructions, another draw the diagrams, and a third complete the table. Repeat for Example 3. Have students hide the textbook solution until they are ready to check their work.
- Use **Think-Pair-Share** to discuss questions C1 to C4.
- Complete question 17 using **cooperative task groups**. Have one group present their solution using **think-aloud**.

Teaching Suggestions

- Have students work in pairs to complete the **Investigate**. If students are having difficulty visualizing the diagrams, provide square tiles in two colours and encourage them to build representative models.
- After students complete the diagrams and chart, lead a discussion to help them come up with strategies they can use to determine a formula to represent the sequence.
- Reinforce the meaning of *recursion* with students before proceeding to the examples.
- In **Example 2** part **b**), have students write the terms of the sequence from the graph.
- Assist students to see the similarities and differences between term notation and function notation.
- Completing the **Check Your Understanding** questions will provide students with opportunities to work with the **Read/Write** connection by describing in words some of the concepts they have just been learning.
- For an alternative method for **Example 3**, students could use Microsoft® *Excel*.

Investigate Answers (pages 365–366)

1.

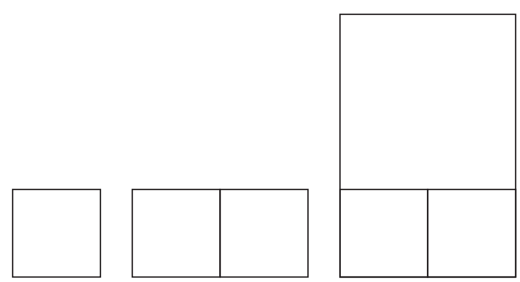

Diagram 1 Diagram 2 Diagram 3

2.

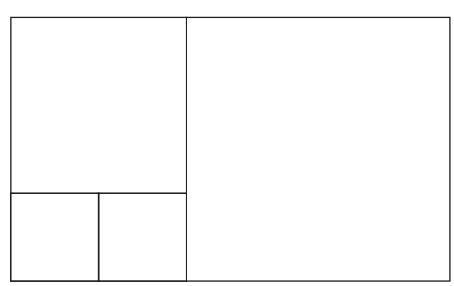

Diagram 4

3.

Diagram Number	Side Length of a Square (units)
1	1
2	1
3	2
4	3
5	5
6	8

4. a) 1, 1, 2, 3, 5, 8, 13, …

b) Answers may vary. Sample answer: The first term in the sequence is 1, and the second term is also 1. The subsequent terms are found by adding the two previous terms together. Formula for the *n*th term: $t_n = t_{n-2} + t_{n-1}$.

Communicate Your Understanding Responses (page 369)

C1 Answers may vary. Sample answer: You have to know that each term in the sequence can be calculated from the previous term or terms.

C2 Answers may vary. Sample answers:

a) The first part gives the value of the first term in the sequence, 5. The second part gives an equation to calculate the second term and each successive term in the sequence using the calculated value of the previous term.

b) A recursion formula has at least two parts because it is used to calculate each term in a sequence using the value of the previous term. To calculate the second term in a sequence, you need the value of the first term and an equation to calculate the second term. The second term is then used to calculate the third term, the third term is used to calculate the fourth term, and so on.

c) The terms in the sequence are calculated by applying a process to the initial number in the sequence to get the second number. The process is then applied to the second number in the sequence to get the third number, and so on.

C3 Answers may vary. Sample answer: The first term in the sequence is 4. The second term is calculated by multiplying the first term by −2 and then adding 5 to get −3. The third term is calculated by multiplying the second term by −2 and then adding 5 to get 11.

C4 Answers may vary. Sample answer: It might be more convenient to use the explicit formula for the *n*th term of the sequence if you wanted to create a discrete graph of the terms in the sequence using a graphing calculator.

Practise, Connect and Apply, Extend

- Use an example from **question 3** or a supplementary example to provide students with a model of how to describe a formula in words.
- Point out that when using an explicit or a recursion formula to determine a sequence, *n* is a natural number because it is a term number. With a recursion formula students need to start with the next *n*-value after those defined recursively.
- Students could use a spreadsheet or a graphing calculator to complete **question 7**.
- For **question 11**, provide some Cube-a-Links and allow students to construct a model of the pyramid.
- **Question 13** gives students the opportunity to use reasoning and reflecting skills to connect material learned in this chapter with previously learned mathematical concepts. They need to select tools to represent the given explicit formulas by recursion formulas.
- **Question 19** allows students to solve the problem of creating three different sequences that start with 2, 3, and 4 by using reasoning and reflecting skills. They must also reason out and reflect on how to solve the problem of representing those sequences by recursion formulas. Students are required to select tools and to use connecting skills to determine the next two terms in each sequence. Each student, being challenged by a classmate, is required to communicate the recursive rule that the classmate has used in developing the formulas.
- In **question 20**, students explore a convergent sequence. Discuss the meaning of the word *converge* as it relates to sequences. Provide a supplementary example for students to study prior to attempting the question. For example, the sequence 2.1, 2.01, 2.001, 2.0001, … converges to 2.
- Use **BLM 6–5 Section 6.2 Practice** for remediation or extra practice.

Common Errors

- Students confuse the explicit and recursion formulas.

R_x Stress that the recursion formula is based on an operation or process done to the previous term.

- Students have difficulty writing recursive formulas for sequences.

R_x Provide examples in which some steps are completed as students try to find a recursive formula for a sequence. It may help them to write each term as a function of the previous term and then change to use the notation t_n.

Ongoing Assessment

Achievement Check, question 17, on student text page 372.

Achievement Check, question 17, student text page 372

This performance task is designed to assess the specific expectations covered in Sections 6.1 and 6.2. The following mathematical process expectations can be assessed.

- Reflecting
- Selecting Tools and Computational Strategies
- Connecting
- Representing
- Communicating

Sample Solution

a)

Square	Number of Pennies
1	1
2	2
3	4
4	8
5	16
6	32
7	64
8	128
9	256
10	512
11	1024
12	2048

b) The number of pennies on each square is twice the number of pennies on the previous square. The recursion formula is $f(1) = 1$, $f(n) = 2f(n - 1)$, which can also be expressed as $t_1 = 1$, $t_n = 2t_{n-1}$.

The number of pennies on square 20 corresponds to $n = 20$. Since this formula is recursive, the previous terms need to be calculated, as shown below.

$t_{20} = 2t_{19}$

$t_{19} = 2t_{18}$

$t_{18} = 2t_{17}$

$t_{17} = 2t_{16}$

$t_{16} = 2t_{15}$

$t_{15} = 2t_{14}$

$t_{14} = 2t_{13}$

$t_{13} = 2t_{12}$

$= 2(2048)$ Substitute the value for t_{12} from the table in part a).

$= 4096$

Now back-substitute to determine the other required values all the way up to t_{20}.

$t_{14} = 2(4096)$
$= 8192$

$t_{15} = 2(8192)$
$= 16\ 384$

$t_{16} = 2(16\ 384)$
$= 32\ 768$

$t_{17} = 2(32\ 768)$
$= 65\ 536$

$t_{18} = 2(65\ 536)$
$= 131\ 072$

$t_{19} = 2(131\ 072)$
$= 262\ 144$

$t_{20} = 2(262\ 144)$
$= 524\ 288$

c) The terms are found by repeated multiplication of 2, so each term in a power of 2. That is, $t_1 = 2^0$, $t_2 = 2^1$, $t_3 = 2^2$, $t_4 = 2^3$, ….
The explicit formula is $t_n = 2^{n-1}$. When $n = 20$,
$$t_{20} = 2^{19} = 524\,288$$
This verifies the result in part b).

d) The number of pennies on each square is represented only by a positive integer, so this function is discrete. The graph of this function would be defined by a set of distinct points, not a smooth curve.

Level 3 Notes

Look for the following:

- Calculates the values in parts a) and b) with considerable accuracy
- Understanding of how to determine a recursive formula is mostly evident
- Understanding of how to determine an explicit formula is mostly evident
- Determines the recursive formula and the explicit formula with considerable accuracy
- Justification and explanations in parts c) and d) are mostly valid

What Distinguishes Level 2

- Calculates the values in parts a) and b) with some accuracy
- Understanding of how to determine a recursive formula is somewhat evident
- Understanding of how to determine an explicit formula is somewhat evident
- Determines the recursive formula and the explicit formula with some accuracy
- Justification and explanations in parts c) and d) are somewhat valid

What Distinguishes Level 4

- Calculates the values in parts a) and b) with a high degree of accuracy
- Understanding of how to determine a recursive formula is clearly evident
- Understanding of how to determine an explicit formula is clearly evident
- Determines the recursive formula and the explicit formula with a high degree of accuracy
- Justification and explanations in parts c) and d) are clearly valid

Mathematical Process Expectations

The table shows questions that provide good opportunities for students to use the mathematical processes.

Process Expectation	Selected Questions
Problem Solving	15, 17–20
Reasoning and Proving	1–21
Reflecting	3, 12–21
Selecting Tools and Computational Strategies	10–17, 19–21
Connecting	1–21
Representing	3–7, 10–17, 19, 20
Communicating	6, 9, 15, 17, 19, 20