Chapter 1 Polynomial Functions

1.1 Power Functions

KEY CONCEPTS

- A polynomial function has the form
  $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \ldots + a_3x^3 + a_2x^2 + a_1x + a_0$, or where n is a whole number

Examples

$f(x) = 7x^3 + 5x^2 + 10x + 2$

$g(x) = 9x^5 - 8x^2 + x - 15$

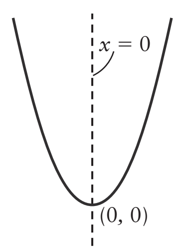
- A power function is a polynomial of the form $y = ax^n$, where n is a whole number.

Examples

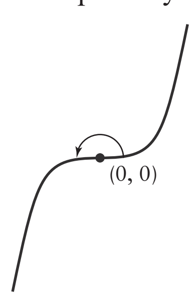
$f(x) = -6x^3$

$A(r) = \pi r^2$

- Power functions have similar characteristics depending on whether their degree is even or odd.
- Even-degree power functions have line symmetry in the y-axis, $x = 0$.



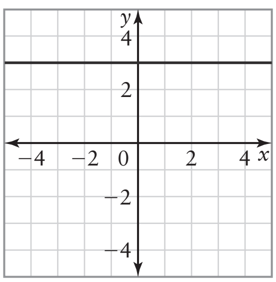
- Odd-degree power functions have point symmetry about the origin, (0, 0).



Power Functions Summary

Function	$y = a$	$y = ax$	$y = ax^2$	$y = ax^3$	$y = ax^4$
Degree	0	1	2	3	4
Name	Constant	Linear	Quadratic	Cubic	Quartic
Type		Odd-degreed	Even-degreed	Odd-degreed	Even-degreed
Domain	$x \in \mathbb{R}$	$x \in \mathbb{R}$	$x \in \mathbb{R}$	$x \in \mathbb{R}$	$x \in \mathbb{R}$
Range	$y = a$	$y \in \mathbb{R}$	$y \geq 0, y \in \mathbb{R}$	$y \in \mathbb{R}$	$y \geq 0, y \in \mathbb{R}$
End Behaviour (for $a > 0$)	Extends from quadrant 2 to quadrant 1	Extends from quadrant 3 to quadrant 1	Extends from quadrant 2 to quadrant 1	Extends from quadrant 3 to quadrant 1	Extends from quadrant 2 to quadrant 1
Symmetry	Line	Point	Line	Point	Line
Graph	a	b	c	d	e

(a)



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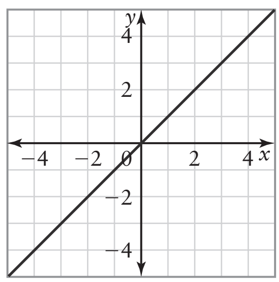

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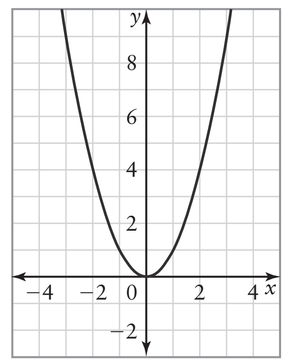

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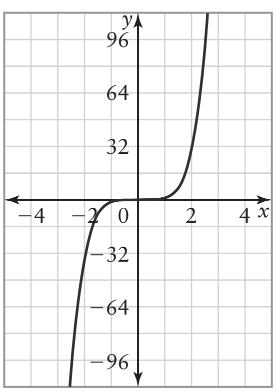

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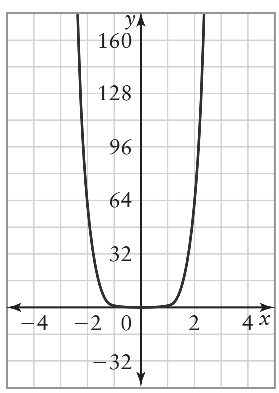


Bracket Interval	Inequality	Number Line	In Words
			The set of all real numbers x such that
(a, b)	$a < x < b$	a b $\mathbb{R}$	x is greater than a and less than b
$(a, b]$	$a < x \leq b$	a b $\mathbb{R}$	x is greater than a and less than or equal to b
$[a, b)$	$a \leq x < b$	a b $\mathbb{R}$	x is greater than or equal to a and less than b
$[a, b]$	$a \leq x \leq b$	a b $\mathbb{R}$	x is greater than or equal to a and less than or equal to b
$[a, \infty)$	$x \geq a$	a $\mathbb{R}$	x is greater than or equal to a
$(-\infty, a]$	$x \leq a$	a $\mathbb{R}$	x is less than or equal to a
(a, ∞)	$x > a$	a $\mathbb{R}$	x is greater than a
$(-\infty, a)$	$x < a$	a $\mathbb{R}$	x is less than a
$(-\infty, \infty)$	$-\infty < x < \infty$	$\mathbb{R}$	x is an element of the real numbers

A

1. Identify the degree of each of the following power functions.

a) $y = 8x^2$

b) $y = -3x^3$

c) $y = 0.25x^{10}$

d) $y = -2x$

e) $y = 3x^3$

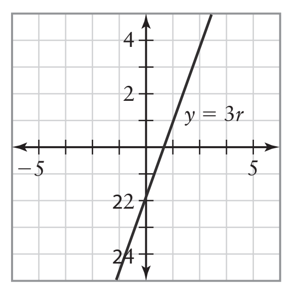
f) $y = x^4$

g) $y = -\frac{1}{3}x^6$

h) $y = 5x$

2. Which of the functions in question 1 have line symmetry? Which have point symmetry? Justify your choices.

3. Describe the end behaviour of each of the functions in question 1. For example, the graph of $y = 3x$ is an odd-degree function with a positive leading coefficient. The end behaviour is from quadrant 3 to quadrant 1.



4. Identify whether each of the following is a polynomial function. Justify each answer.

a) $y = 0.6x^4 - 3x^2 + 2$

b) $y = 3 \cos x$

c) $y = \frac{1}{x^3 - 4}$

d) $y = 3^{x-1} - 11$

e) $y = 3x - 2$

f) $y = -7$

g) $y = \sqrt{3x^2 - 5x}$

B

5. Determine a possible equation for a power function for each of the following:

a) the function has point symmetry at (0, 0)

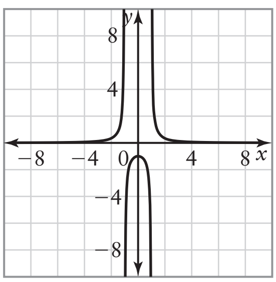
b) a positive function

c) an odd negative function

d) a function that begins in quadrant 3 and ends in quadrant 4

6. Determine which graphs below represent polynomial functions. Justify your choices.

a)



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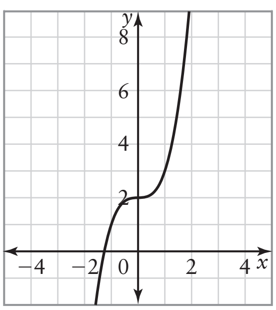

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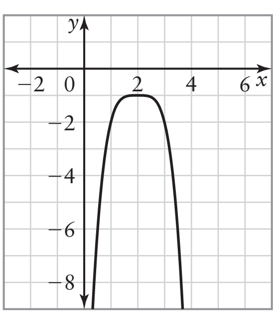

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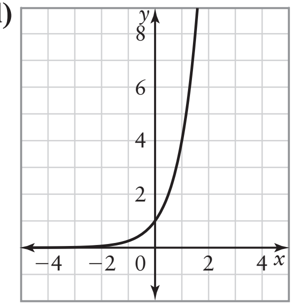


☆**7.** A stone is dropped into a pond and its ripple increases in a circular pattern. The area of the circle, in square centimetres, and the ripple's distance from the centre, as represented by the radius of the circle, r, can be modelled by the function $A(r) = \pi r^2$.

a) Graph the function for $0 \le r \le 30$.

b) Determine the domain and range of the function.

c) Describe the similarities and differences between the graph of $A(r)$ and the graph of a quadratic function.

8. The circumference, in centimetres, of the ripple in question 7 can be modelled by the function $C(r) = 2\pi r$.

a) Graph the function for $0 \le r \le 30$.

b) Determine the domain and range of the function.

c) Describe the similarities and differences between the graph of $C(r)$ and the graph of a linear function.

9. a) Graph the functions $f(x) = x^5$, $g(x) = (x + 2)^5$, and $h(x) = (x - 2)^5$ on the same axes. Compare the graphs. Describe how they are related.

b) Repeat part a) for the functions $f(x) = x^4$, $g(x) = (x + 2)^4$, and $h(x) = (x - 2)^4$.

c) Make a conjecture for the relationship between $y = x^n$ and $y = (x + b)^n$, where $b \in \mathbb{R}$ and n is a whole number.

d) Test the accuracy of your conjecture for different values of n and b.

☆**10. a)** Sketch graphs of the functions $f(x) = 3x^3$, $g(x) = 3x^3 - x$, and $h(x) = 3x^3 + x$ on the same axes.

b) Compare and describe the key features of the graphs of these functions.

11. a) Use Technology Graph the following functions on the same set of axes.

i) $y = 3x^4$

ii) $y = 3(x - 2)^4$

iii) $y = 3(x - 2)^4 - 1$

iv) $y = -3(x - 2)^4 - 1$

b) Compare and describe the key features of the graphs of these functions. Discuss domain and range, symmetry, and end behaviour.

12. The surface area of a spherical snowball is given by the function $S(r) = 4\pi r^2$, where r is the radius of the snowball, in centimetres, and $r \in [0, 12]$.

a) Graph $S(r)$.

b) State the domain and the range.

c) Describe the similarities and differences between the graphs of $S(r)$ and $y = x^2$.

13. a) Describe the relationship between the graph of $y = x^3$ and the graph of $y = -\frac{1}{2}(x + 1)^3 - 4$.

b) Predict the relationship between the graph of $y = x^5$ and the graph of $y = -\frac{1}{2}(x + 1)^5 - 4$.

c) Verify the accuracy of your prediction by graphing the functions in part b).

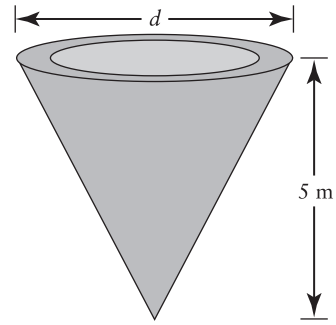
14. a) Graph $y = x^5$, $y = (x + 3)^5$, and $y = (x - 3)^5$ on the same set of axes.

b) Describe the similarities and differences among the graphs in part a).

c) Make a conjecture about the relationship between the graphs of $y = x^n$ and $y = (x - h)^n$, where $h \in \mathbb{R}$ and n is an odd whole number.

C

☆**15.** A conical reservoir is filling with water at a constant rate. The reservoir is 5 m deep and has a maximum diameter of 10 m.



a) Determine a function $V(r)$ that describes the volume of the reservoir.

b) Graph $V(r)$.

c) What are the domain and range of this function?

16. Use Technology Graph each pair of functions below, using graphing technology. What is the degree of each function? Describe any similarities and differences in each pair.

a) $f(x) = x^2$, $g(x) = x^2 - 2x$

b) $f(x) = x^3$, $g(x) = x^3 - 5x^2 - x - 10$

c) $f(x) = x^4$, $g(x) = x^4 - 8x^2 + 15$

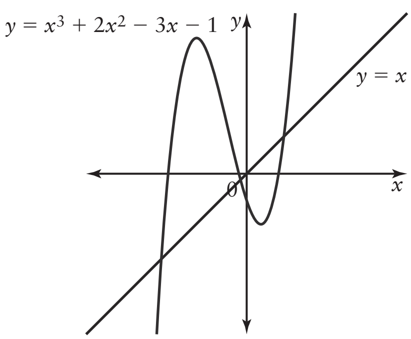
d) $f(x) = x^5$, $g(x) = (x - 3)^2 (x + 2)^3$

17. Draw the graph of a function that is not a polynomial function, and explain why it is not.

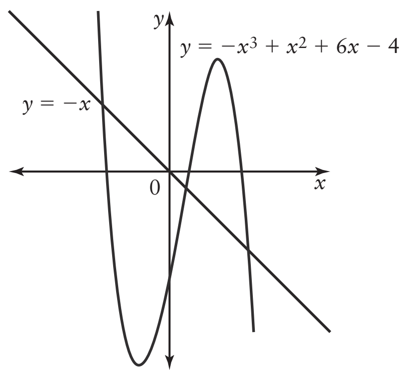
KEY CONCEPTS

Odd-Degree Polynomial Functions

Positive Leading Coefficient
- The graph extends from quadrant 3 to quadrant 1 (similar to the graph of $y = x$).



Negative Leading Coefficient
- The graph extends from quadrant 2 to quadrant 4 (similar to the graph of $y = -x$).

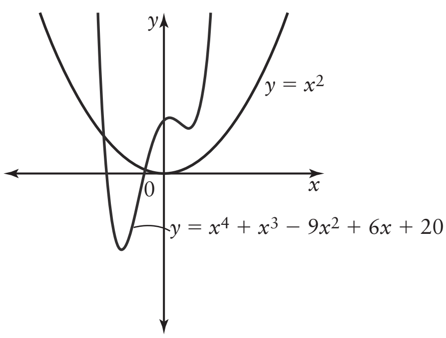


- Odd-degree polynomials have at least one x-intercept, up to a maximum of n x-intercepts, where n is the degree of the function.
- The domain of all odd-degree polynomials is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}\}$. Odd-degree functions have no maximum point and no minimum point.
- Odd-degree polynomials may have point symmetry.

Even-Degree Polynomial Functions

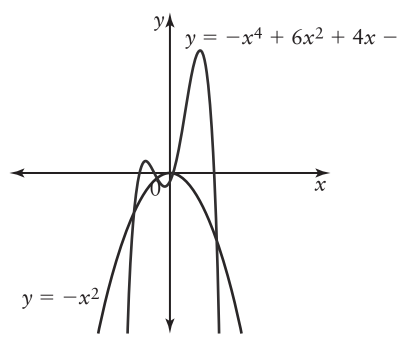
Positive Leading Coefficient

- The graph extends from quadrant 2 to quadrant 1 (similar to the graph of $y = x^2$).
- The range is $\{y \in \mathbb{R}, y \geq a\}$, where a is the minimum value of the function.
- An even-degree polynomial with a positive leading coefficient will have at least one minimum point.



Negative Leading Coefficient

- The graph extends from quadrant 3 to quadrant 4 (similar to the graph of $y = -x^2$).
- The range is $\{y \in \mathbb{R}, y \leq a\}$, where a is the maximum value of the function.
- An even-degree polynomial with a negative leading coefficient will have at least one maximum point.



- Even-degree polynomials may have from zero to a maximum of n x-intercepts, where n is the degree of the function.
- The domain of all even-degree polynomials is $\{x \in \mathbb{R}\}$.
- Even-degree polynomials may have line symmetry.

Key Features of Graphs of Polynomial Functions

- A polynomial function of degree n, where n is a whole number greater than 1, may have at most $n - 1$ local minimum and local maximum points.
- For any polynomial function of degree n, the nth differences
  - are equal (or constant)
  - have the same sign as the leading coefficient
  - are equal to $a[n \times (n - 1) \ldots \times 2 \times 1]$, where a is the leading coefficient $= a(n!)$

A

1. State the degree of the polynomial function that corresponds to each constant finite difference. Determine the value of the leading coefficient for each polynomial function.

a) first difference $= -3$

b) second difference $= 2$

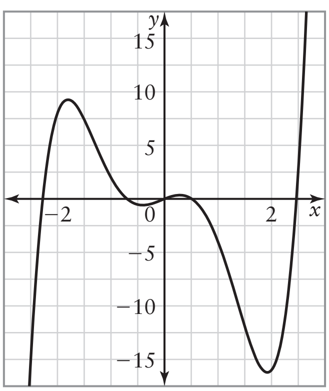
c) fifth difference $= -240$

d) fourth difference $= 144$

e) third difference $= 31.5$

2. Each graph represents a polynomial function of degree 3, 4, 5, or 6. Determine the least possible degree of the function corresponding to the graph. Give reasons for your choice.

a)



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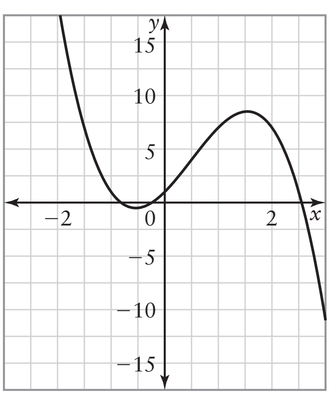

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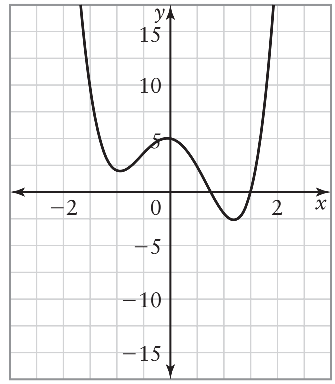

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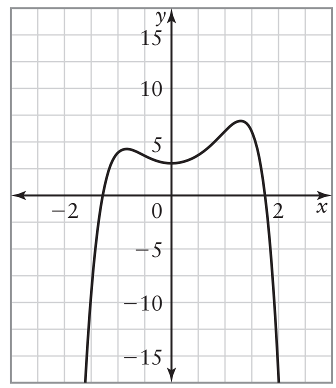


3. Refer to question 2. For each graph, do the following.

a) State the sign of the leading coefficient. Justify your answer.

b) Describe the end behaviour.

c) Identify any symmetry.

d) State the number of minimum and maximum points and local minimum and local maximum points. How are these related to the degree of the function?

4. Use the degree and the sign of the leading coefficient in each of the following functions to

i) describe the end behaviour of each polynomial function

ii) state which finite differences will be constant

iii) determine the value of the constant finite differences

a) $y = x^4 - 3x^2 + 2$

b) $f(x) = 9 - 7x$

c) $y = -3x^7 + 5x^2$

d) $g(x) = 8x - 6x^2$

e) $y = -0.5x^4 - 8x^3 + 4x^2 - 3x + 1$

f) $h(x) = 27x^3 - 81x$

B

5. Match each polynomial function below to the appropriate graph. Justify your choice.

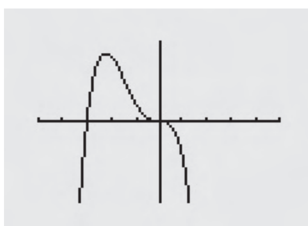
a) $y = x^4 + 2x^3 - x^2 - 6$

b) $y = x^5 - 3x^3 + 5x + 1$

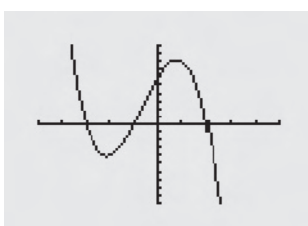
c) $y = -x^3 + 2x^2 + 5x + 6$

d) $y = -x^4 - 3x^3$

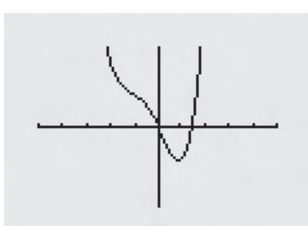
i)



ii)



iii)



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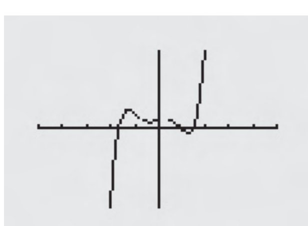


☆**6.** Use finite differences to determine

i) the degree of each polynomial function

ii) the sign of the leading coefficient

iii) the value of the leading coefficient

a)

x	y
−3	140
−2	37
−1	8
0	5
1	4
2	5
3	21

b)

x	y
−3	0
−2	−4
−1	0
0	6
1	8
2	0
3	−24

c)

x	y
−3	36
−2	16
−1	4
0	0
1	4
2	16
3	36

☆**7.** Flipus Discus manufactures flying discs for a recreational sports league. The company determines that its profit, P, in thousands of dollars, can be modelled by the function, $P(x) = 0.65x^4 - 3.5x^2 - 12$, where x represents the number, in hundreds, of flying discs sold.

a) What type of function is $P(x)$?

b) Without calculating, determine which finite differences are constant for this polynomial function. What is the value of the constant finite difference? Explain how you know.

c) Describe the end behaviour of this function, assuming that there are no restrictions on the domain.

d) State the restrictions on the domain in this situation.

e) What do the x-intercepts of the graph represent for this situation?

f) If the Disc Golf League wants to buy 500 flying discs to distribute to its members, what is Flipus Discus's profit from the sale?

8. Consider the function $f(x) = -x^4 + 3x^3 + 2x^2 - x + 2$.

a) How do the degree and the sign of the leading coefficient correspond to the end behaviour of the polynomial function?

b) Sketch a graph of the polynomial function.

c) What can you tell about the value of the fourth differences for this function?

9. Explain why odd-degree polynomials have unrestricted domain and range. What does this tell you about the number of maximums or minimums? What does this tell you about the number of x-intercepts?

☆**10. Use Technology** The data below show the stopping distance required for various speeds. The stopping distance is a function of speed, reaction time, and braking time.

Speed (km/h)	Total Stopping Distance (m)
30	18
50	35
70	57
100	98
110	114

SOURCE: http://www.mpi.mb.ca/PDFs/WatchYourSpeed.pdf

a) Graph the data in the table using a graphing calculator.

b) Use the regression feature of the graphing calculator to determine an equation for the function that models this situation. What type of polynomial function appears to be the best fit? Justify your choice.

C

11. Sketch a graph of a quintic (degree 5) polynomial function

a) with point symmetry

b) without point symmetry

c) with the maximum number of zeros

d) with the minimum number of zeros

12. Describe how to determine the number of x-intercepts as well as the number of maximum and minimum points from the degree of a polynomial function.

☆**13.** A farmer wants to construct a cylindrical grain silo where the ratio of radius to height is 1:6.

a) Write a polynomial function to represent the surface area of the silo (not including the top and bottom) in terms of the radius, r.

b) Write a polynomial function to represent the volume of the silo, in terms of the radius, r.

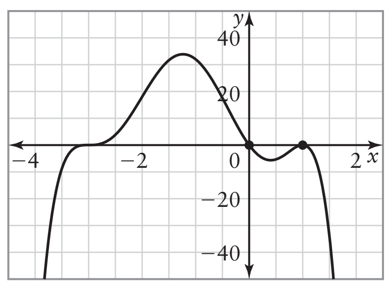
c) Describe the key features of the graph that correspond to each of the above functions.

KEY CONCEPTS

- The graph of a polynomial function can be sketched using the x-intercepts, the degree of the function, and the sign of the leading coefficient.
- When a polynomial function is in factored form, the zeros can be easily determined from the factors. When a factor is repeated n times, the corresponding zero has order n.
- The graph of a polynomial function changes sign only at x-intercepts that correspond to zeros of odd order. At x-intercepts that correspond to zeros of even order, the graph touches but does not cross the x-axis.

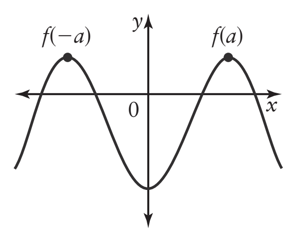
Analysis of Graphs of Polynomial Functions

$f(x) = -x(x - 1)^2(x + 3)^3$

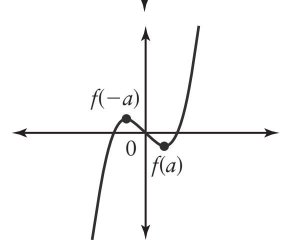


Degree	The product of all factors will give a multiple of x^6. The function is degree 6.
Leading Coefficient	The product of all the x-coefficients is $-1 \times 1 \times 1 = -1$.
End Behaviour	An even-degree polynomial with a negative leading coefficient extends from quadrant 3 to quadrant 4.
Zeros and x-intercepts	The zeros are 3 (order 3), 0, and 1 (order 2).
y-intercepts	The y-intercept is $0(0 - 1)^2(0 + 3)^3 = 0$

An even-degree polynomial function is an **even function** if the exponent of each term of the equation is even. An even function satisfies the property $f(-x) = f(x)$ for all x in the domain of $f(x)$. An even function is symmetric about the y-axis.



An odd-degree polynomial function is an **odd function** if each term of the equation has an odd exponent. An odd function satisfies the property $f(-x) = -f(x)$ for all x in the domain of $f(x)$. An odd function is rotationally symmetric about the origin.



A

1. For each polynomial function below

   i) state the degree and the sign of the leading coefficient

   ii) describe the end behaviour of the graph of the function

   iii) determine the x-intercepts

   a) $f(x) = 3x(x - 2)(x + 3)$

   b) $g(x) = -(x - 2)(x - 4)^2(x - 6)$

   c) $h(x) = (2x - 1)(x + 3)^3$

   d) $p(x) = (4 - x)(x + 1)^2(x - 2)$

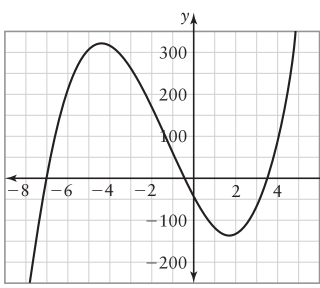
2. For each graph, do the following:

   i) state the x-intercepts

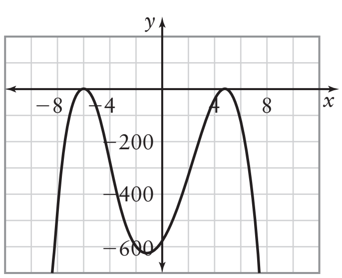
   ii) state the intervals where the function is positive and the intervals where it is negative

   iii) explain whether the graph might represent a polynomial function that has zeros of order 2 or of order 3

   a)



   b)



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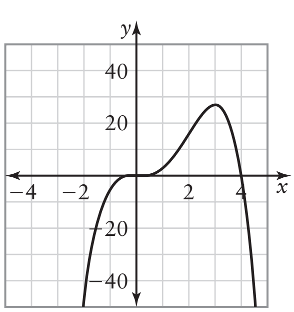


3. **a)** Determine the zeros of each polynomial function below. Indicate whether each is of order 1, 2, or 3.

   i) $f(x) = -x(x - 3)^3$

   ii) $g(x) = (5x - 1)(x + 2)(x - 1)$

   iii) $h(x) = -(x + 4)^2(4x - 3)^2$

   b) Determine algebraically whether each function is even or odd.

   c) Draw a sketch of each function.

☆4. **i)** Determine whether each function is even, odd, or neither. Explain.

   ii) Without graphing, determine whether each polynomial function has line symmetry about the y-axis, point symmetry about the origin, or neither. Explain.

   a) $y = -x^4 + 3x^2$

   b) $y = -6x + 5x^3$

   c) $y = x^4 - x^2 + 4x + 2$

B

5. Determine an equation for each polynomial function described here. State whether the function is even, odd, or neither. Sketch a graph of each.

   a) degree 3, a root at 4 (order 2), a root at -3

   b) degree 4, a root at 2 (order 3), a root at 5

   c) degree 3, roots at $\frac{1}{2}, \frac{3}{4}, -1$

   d) degree 3, starting in quadrant 2, ending in quadrant 4, root at -2, and root at 1 (order 2)

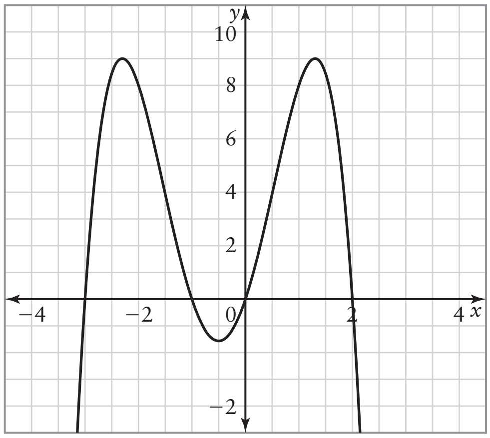
6. **Use Technology** Without graphing, determine whether each polynomial function below has line symmetry, point symmetry, or neither. Verify your response using technology.

   a) $f(x) = -x^4 + 6x^2 + 8$

   b) $g(x) = 8x^5 - 3x^2$

7. Determine an equation for the polynomial function that corresponds to each graph.

a)



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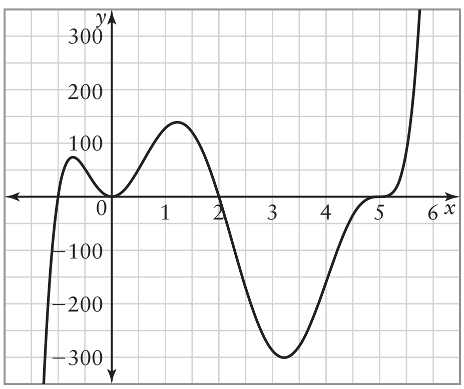


8. Each of the following polynomial functions has x-intercepts at -6, 5, and 0. Determine an appropriate equation for each. Then, sketch a graph of the function.

a) a cubic function with a positive leading coefficient and y-intercept at (0, 0)

b) a quartic function that extends from quadrant 3 to quadrant 4

c) a degree 6 function with a negative leading coefficient

d) a cubic function that extends from quadrant 2 to quadrant 4

☆**9.** **Use Technology** Determine the zeros of each of the following functions. Use graphing technology to verify your answers.

a) $f(x) = (30 - 13x - x^2)(x^2 - 10x + 25)$

b) $g(x) = (x^3 - 4x^2 + 4x)(2x^2 - 7x + 3)$

c) $h(x) = (x^2 - 8x + 15)(27 - 3x^2)$

C

10. Determine the equation of an even quartic function with a negative leading coefficient that has no zeros. Draw a sketch of your function.

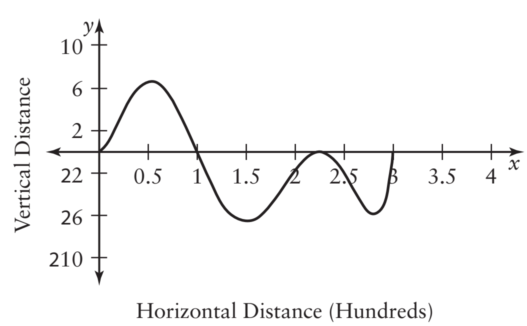
11. Determine an equation for each polynomial function described below. State whether the function is even, odd, or neither. Sketch a graph of each.

a) a quintic function with zeros at -2 (order 3) and 3 (order 2), and that has a y-intercept at 70

b) a quartic function with x-intercepts at -2 (order 2) and 1 (order 2) that passes through the point (0, -12)

c) a quintic function with zeros -3, -2 (order 2), 2 (order 2), that passes through the point (1, -18)

12. The Lazy River Ride at a local amusement park is shown below. Determine a polynomial function that will model the path of the river. Let x be the horizontal distance and y the vertical distance from the starting platform. The starting and ending platforms are at the same level. The starting and ending platforms are 300 m apart.



KEY CONCEPTS

- The graph of a polynomial function of the form $y = a[k(x - d)]^n + c$ can be sketched by applying transformations to the graph of $y = x^n$, where $n \in \mathbb{N}$. The transformations represented by a and k must be applied before the transformations represented by c and d.

- The parameters a, k, d, and c in polynomial functions of the form $y = a[k(x - d)]^n + c$, where n is a non-negative integer, correspond to the following transformations:
  - a corresponds to a vertical stretch or compression and, if $a < 0$, a reflection in the x-axis
  - k corresponds to a horizontal stretch or compression and, if $k < 0$, a reflection in the y-axis
  - d corresponds to a horizontal translation to the left or right
  - c corresponds to a vertical translation up or down

Example

Sketch a graph of the function $y = -\frac{1}{2}(4(x - 2))^3 + 1$.

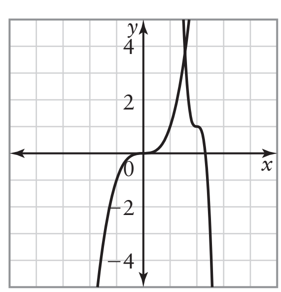
The function $y = x^3$ is
- compressed vertically by a factor of $\frac{1}{2}$ and reflected in the x-axis
- compressed horizontally by a factor of $\frac{1}{4}$
- translated 2 units to the right
- translated 1 unit up

Another way to describe the transformation is to apply a mapping notation to each coordinate: $(x, y) \rightarrow \left(\frac{1}{4}x + 2, -\frac{1}{2}y + 1\right)$. This shows that the stretches and compressions should be applied before the translations.

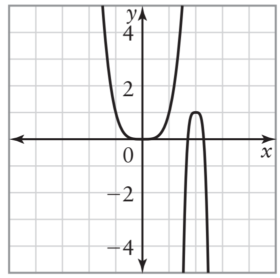
This is a cubic function with a negative leading coefficient. Therefore, it extends from quadrant 2 to quadrant 4.

The function has point symmetry at (2, 1).

The domain is $x \in \mathbb{R}$. The range is $x \in \mathbb{R}$.



Apply the above transformations to the graph of $y = x^4$. Describe the features of the function that are affected by the transformation.



This is a quartic function with a negative leading coefficient. Therefore, it extends from quadrant 3 to quadrant 4.

The function has line symmetry in the line $x = 2$.

The domain is $x \in \mathbb{R}$. The range is $y \leq 1$.

When n is even, the graphs of polynomial functions of the form $y = a[k(x - d)]^n + c$ are even functions and have a vertex at (d, c). The axis of symmetry is $x = d$.

For $a > 0$, the graph opens upward. The vertex is the minimum point on the graph, and c is the minimum value. The range of the function is $\{y \in \mathbb{R}, y \geq c\}$.

For $a < 0$, the graph opens downward. The vertex is the maximum point on the graph, and c is the maximum value. The range of the function is $\{y \in \mathbb{R}, y \leq c\}$.

A

1. Compare each polynomial function below with the equation $y = a[k(x - d)]^n + c$. State the values of the parameters a, k, d, c, and the degree n, assuming that the base function is a power function. Describe the transformation that corresponds to each parameter.

a) $y = -5(x - 1)^3$

b) $y = 3x^2 + 5$

c) $y = -(3x - 2)^4$

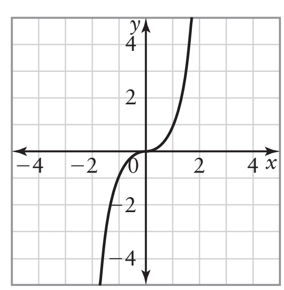
d) $y = \frac{1}{3}(x - 2)^4$

e) $y = \frac{3}{5}[6(x - 1)]^3 - 4$

f) $y = 7(-x)^3 + 1$

2. Determine the domain and range of each function in question 1.

3. Sketch a graph of the following function under the transformation defined by $y = f(x + 2) - 4$.



4. Match each graph with the corresponding function. Justify your choices.

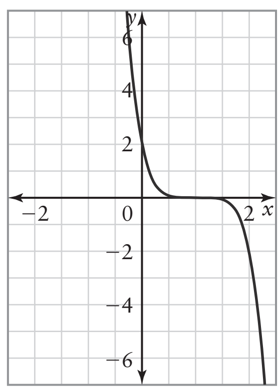
i) $y = -3x^4$

ii) $y = \frac{2}{3}x^3 + 4$

iii) $y = -2(x - 1)^5$

iv) $y = x^4 + 2$

a)



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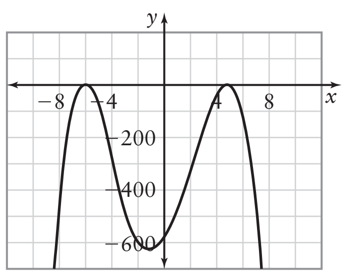

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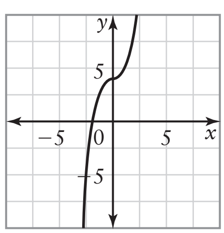

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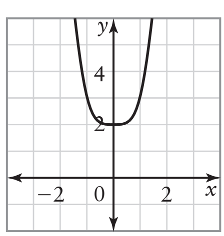


B

5. Describe the transformations that must be applied to the graph of each power function, $f(x)$, to obtain the transformed function below. Write the full equation of the transformed function.

a) $f(x) = x^4$, $y = 5f(-x)$

b) $g(x) = x^3$, $y = -2g(x + 1)$

c) $h(x) = 5(x - 1)^4$, $y = h(-x) + 3$

6. **a)** For each pair of polynomial functions in question 5, sketch the original and transformed functions on the same set of axes.

b) State the domain and range of the functions in each pair. For even functions, give the vertex and the equation of the axis of symmetry.

7. **a)** Use your knowledge of transformations to sketch graphs of the following groups of polynomial functions.

i) $f(x) = x^3$, $g(x) = -x^3$, $h(x) = (-x)^3$

ii) $f(x) = (x + 1)^3$, $g(x) = 2(x + 1)^3 - 1$, $h(x) = -\frac{1}{2}(x + 1)^3 - 3$

iii) $f(x) = x^4$, $g(x) = -x^4$, $h(x) = (-x)^4$

iv) $f(x) = (1 - x)^4$, $g(x) = [2(x + 3)]^4$, $h(x) = -\frac{1}{4}x^4 + 8$

b) For $f(x) = a[k(x - d)]^3 + c$ and $g(x) = a[k(x - d)]^4 + c$, summarize the effects of changing a, k, d, and c in terms of transformations.

☆ 8. a) Given a base function of $y = x^4$, list the parameters of the polynomial function $y = \frac{1}{3}[-2(x + 4)]^4 - 10$.

b) Describe how each parameter in part a) transforms the graph of the function $y = x^4$.

c) Determine the domain, range, vertex, and equation of the axis of symmetry for the transformed function.

d) Describe two possible orders in which the transformations can be applied to the graph of $y = x^4$ in order to sketch the graph of $y = \frac{1}{3}[-2(x + 4)]^4 - 10$.

☆ 9. For each of the transformations described below:

i) Write an equation for the resulting function.

ii) State the domain and range. For even functions, give the vertex and the equation of the axis of symmetry.

a) The function $f(x) = x^3$ is translated 4 units up and 2 units right.

b) The function $f(x) = x^4$ is compressed horizontally by a factor of 5 and translated 3 units to the right.

c) The function $f(x) = x^5$ is stretched vertically by a factor of 2, reflected in the y-axis, and translated 2 units down and 4 units to the left.

d) The function $f(x) = x^6$ is reflected in the x-axis, stretched horizontally by a factor of 3, reflected in the y-axis, and translated 3 units up and 1 unit to the left.

10. i) Describe the transformation that must be applied to the graph of each power function, $f(x)$, to obtain the transformed function.

a) $f(x) = x^2$, $y = -3f(x - 2) - 4$

b) $f(x) = x^4$, $y = 2f(2x + 6)$

c) $f(x) = x^3$, $y = -\frac{1}{2}f\left(\frac{1}{2}(x - 1)\right) - 5$

ii) Write the full equation of each transformed function in part i).

iii) Sketch each base function from part i) and its transformed function from part ii) on the same set of axes.

iv) State the domain and range of each pair of functions in part iii).

11. a) Predict the relationship between the graph of $y = x^4 + x^3$ and the graph of $y = [(x + 3)^4 + (x + 3)^3] - 1$.

b) **Use Technology** Graph each function in part a) to verify the accuracy of your prediction.

c) Determine the x-intercepts of each function in part a). Round your answers to 1 decimal place.

d) Give the approximate domain and range of each function in part a). Round your answers to one decimal place.

C

12. a) The function $f(x) = -3(x - 2)(x + 3)(x + 5)^2$ is translated 2 units to the right and 1 unit up. Write an equation for the transformed function.

b) Suppose the transformed function is then reflected in the y-axis and vertically compressed by a factor of $\frac{1}{3}$. Write an equation for the new transformed function.

13. **Use Technology**

a) Describe the transformations that must be applied to the graph of $y = -x^5 - x^3 + x$ to obtain the graph of $y = (2x)^5 + (2x)^3 - (2x)$.

b) Sketch each graph using technology.

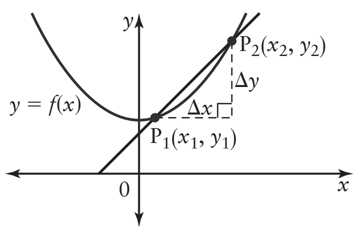
☆14. **Use Technology**

a) Graph the function $y = 2(x - 1)^3 + 2(x - 1)^2 - 2(x - 1) + 4$.

b) What transformations may have been applied to the original function? What was the function before the transformations were applied?

KEY CONCEPTS

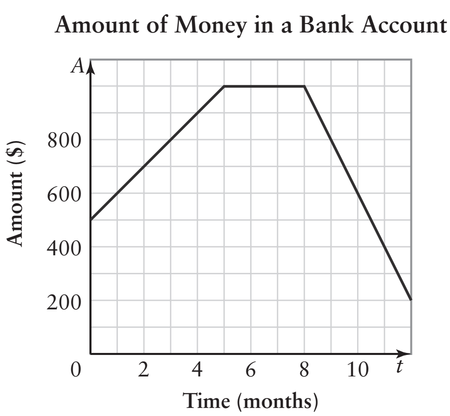
- A rate of change is a measure of how quickly one quantity (the dependent variable) changes with respect to another quantity (the independent variable).
- Average rates of change
  - represent the rate of change over a specified interval
  - correspond to the slope of a secant between two points $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ on a curve



$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}$$

- An average rate of change can be determined by calculating the slope between two points given in a table of values or by using an equation.

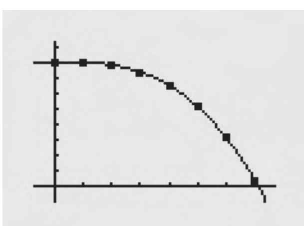
Calculating Average Rate of Change

Average Rate of Change from a Graph	The points that correspond to month 0 and month 5 are (0, 500) and (5, 1000).
	$\text{Average rate of change} = \frac{\text{change in amount}}{\text{change in time}} = \frac{1000 - 500}{5 - 0} = \frac{500}{5} = 100$ The amount of money in the account is increasing on average by \$100 per month.

Average Rate of Change from a Table of Values

A new antibacterial spray is tested on a bacterial culture. The table shows the population, P, of the bacterial culture t minutes after the spray is applied.

t (min)	*P*
0	800
1	799
2	782
3	737
4	652
5	515
6	314
7	37



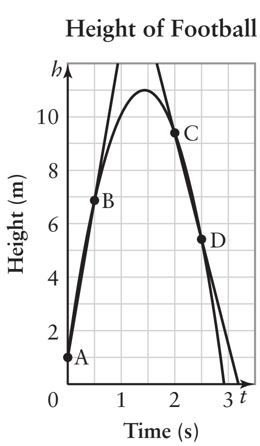
From the table, the points are (0, 800) and (7, 37).

$$\text{Average rate of change} = \frac{\Delta P}{\Delta t}$$
$$= \frac{37 - 800}{7 - 0}$$
$$= \frac{-763}{7}$$
$$= -109$$

During the entire 7 min, the number of bacteria decreases on average by 109 bacteria per minute.

Average Rate of Change from an Equation

A football is kicked into the air such that its height, h, in metres, after t seconds can be modelled by the function $h(t) = -4.9t^2 + 14t + 1$.



Use the equation to determine the endpoints corresponding to each interval.

For [0, 0.5]:

Substitute $t = 0$ to find the height at 0 s.

$$h(0) = -4.9(0)^2 + 14(0) + 1$$
$$= 1$$

Substitute $t = 0.5$ to find the height at 0.5 s.

$$h(0.5) = -4.9(0.5)^2 + 14(0.5) + 1$$
$$= 6.775$$

The points that correspond to 0 s and 0.5 s are (0, 1) and (0.5, 6.775).

$$\text{Average rate of change} = \frac{\Delta h}{\Delta t}$$
$$= \frac{6.775 - 1}{0.5 - 0}$$
$$= 11.55$$

The average rate of change of the height of the football from 0 s to 0.5 s is 11.55 m/s.

A

1. Which of the following are examples of average rates of change?

   a) The average height of the players on a baseball team is 2.1 m.

   b) The class average on the last math test was 75%.

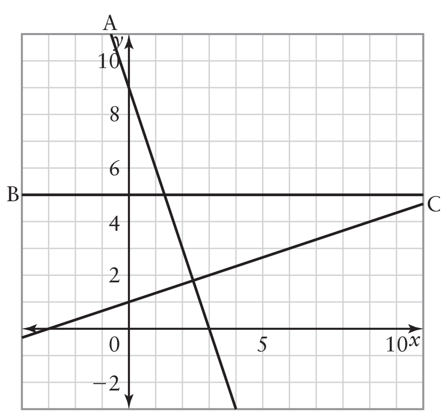
   c) The value of the Canadian dollar increased from \$0.81 US to \$1.04 U.S. in 8 months.

   d) The track athlete crossed the 100-m finish line at 12 km/h.

   e) The temperature of water in the pool increased by 8°C over a period of 12 h.

   f) Last February, approximately 100 cm of snow fell over a 48-h period.

2. Identify whether or not the average rate of change for pairs of points along each graph is constant and positive, constant and negative, zero, or non-constant. Justify your responses.



3. Determine the average rate of change for two points on each line segment in question 2.

4. Find the average rate of change of the function *f* over the given interval.

   a) $f(x) = 4 - x^2$, [0, 2]

   b) $f(x) = 0.2x^4 - 3x^2 - 5x + 6$, [−1, 4]

   c) $f(x) = x^3 - 3x^2 - 5x + 1$, [−1, 3]

   d) $f(x) = -\sqrt{3x^3 - 5x}$, [0, 3]

5. The new-housing price index in the St. Catharines-Niagara area rose from 120.5 in 2003 to 150.1 in 2007. Determine the average rate of change of the new-housing price index over this time period.

B

☆6. The table below shows the new-housing price index in the St. Catharines-Niagara area for each year from 2003 to 2007.

Year	Index
2003	120.5
2004	128.8
2005	137.8
2006	144.2
2007	150.1

Source: Statistics Canada, CANSIM Table 327-0005

   a) Determine the average rate of change of the new-housing price index for each pair of consecutive years from 2003 to 2007.

   b) Compare the values found in part a). Which value is the greatest? Which is the least? What is the significance of these values?

   c) Compare the values found in part a) with your calculation in question 5. Explain any similarities or differences.

☆7. Water is draining from a large tank. After *t* min, there are $150\ 000 - 7500t + t^2$ litres of water in the tank.

   a) Determine the average rate at which the water empties from the tank in the interval between 5 and 10 min.

   b) Determine the average rate at which the water empties from the tank in the interval between 9 and 10 min.

   c) Estimate the rate at which the water runs out after exactly 10 min.

8. a) Find the average rate of change of the area of a circle with radius r as r changes from

   i) 2 to 2.5

   ii) 2 to 2.3

   iii) 2 to 2.25

   iv) 2 to 2.1

   v) 2 to 2.01

   b) Estimate the rate of change at the instant when radius is 2.

   c) How is your answer in part b) related to the circumference of a circle of radius 2?

9. The following graph shows the value of a stock over a two year period.

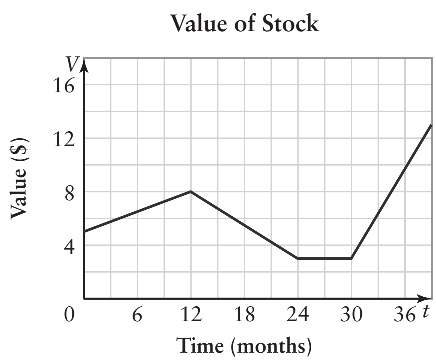
   a) What does the graph tell you about the average rate of change over

   i) the first year

   ii) the second year

   iii) the third year

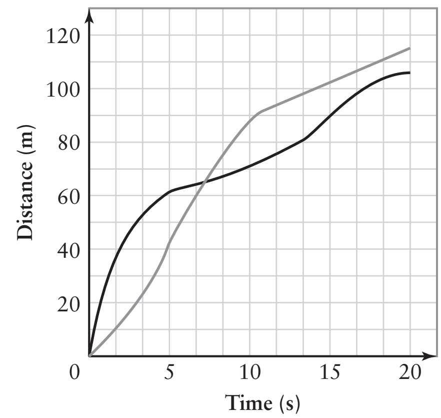
   b) Determine the average rate of change over the intervals described in part a).



10. Marshall and Teagan raced their bikes on a straight road, beginning from a dead stop. The distance (in metres) each bike has covered in each time during the first 20 s is shown in the graph.

   a) What is the average speed of each bike during this 20-s interval?

   b) Prove that Marshall travelled at a higher average speed than Teagan from $t = 4$ to $t = 10$.



11. The cost, C, of making one unit of a product at any time x, in years, can be modelled by the function $C(x) = 4500 + 1530x - 0.04x^3$, $x \in [0, 200]$.

   a) Determine algebraically the average rate of change of the cost from

   i) year 0 to year 4

   ii) year 4 to year 7

   iii) year 7 to year 9

   b) Interpret your answers from part a).

12. If a ball is dropped from the top of a 120-m cliff, its height, h, in metres, after t seconds can be modelled by $h(t) = 120 - 4.9t^2$.

   a) Find the average rate of change of the height of the ball with respect to time over the intervals

   i) 1 s to 4 s

   ii) 4 s to 6 s

   iii) 6 s to 7 s

   b) What does the average rate of change represent in this situation?

   c) Interpret the significance of your answers in part a).

13. A census is taken of the permanent population, P, of Collingwood, Ontario every five years.

Year	Population
1991	13 500
1996	15 596
2001	16 039
2006	17 290

a) Determine the average rate of change of the population for

i) 1991 to 1996

ii) 1996 to 2001

iii) 2001 to 2006

iv) the entire fifteen-year period

b) What factors do you think might have led to the varied results?

C

14. Suppose you have the flu. The data below show your temperature during the first day of the flu. As you might guess, the faster your temperature rises, the worse you will feel.

a) At what average rate does your temperature rise during the entire day?

b) During what 2-h period do you expect to feel the worst?

c) Find two time intervals during which you feel about the same (that is, during which your temperature is rising at the same average rate).

Hours (t)	Temperature (°C)
0	98.1
2	98.1
4	98.5
6	98.9
8	99.25
10	100
12	100.7
14	101.8
16	102
18	102.2
20	102.4
22	102.45
24	102.5

☆**15.** The distance, d, in metres, that it takes a vehicle to stop from a speed, s, in km/h can be modelled by the function $d(s) = 0.01s^2 - 0.25s + 10$.

a) What does the average rate of change represent for this situation?

b) Determine the average rate of change in the speed of the vehicle for each time interval.

i) 20 to 30 km/h

ii) 40 to 50 km/h

iii) 60 to 70 km/h

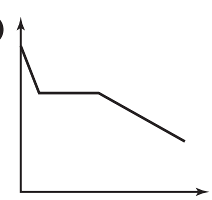
iv) 80 to 90 km/h

c) Describe how the average rate of change changes as the speed increases.

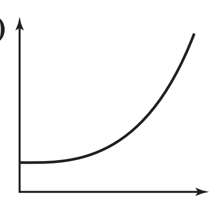
d) Graph the data modelled by the function. Add secant lines to your graph that show the average rates of change you calculated in part b).

16. Describe the average rate of change of each of the following functions. Sketch a graph of the average rate of change.

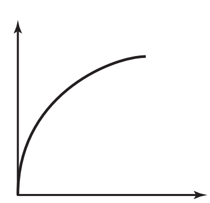
a)



b)



c)



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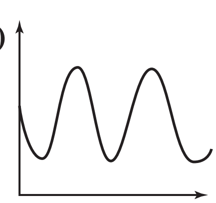
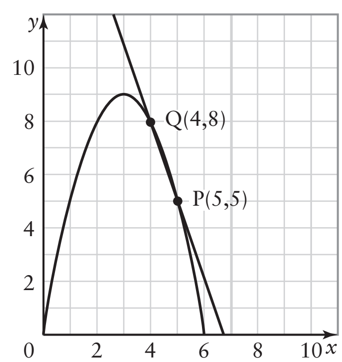


KEY CONCEPTS

- An instantaneous rate of change corresponds to the slope of a tangent to a point on a curve.
- An approximate value for an instantaneous rate of change at a point may be determined using
  - a graph, either by estimating the slope of a secant passing through that point or by sketching the tangent and estimating the slope between the tangent point and a second point on the approximate tangent line
  - a table of values, by estimating the slope between the point and a nearby point in the table
  - an equation, by estimating the slope using a very short interval between the tangent point and a second point found using the equation

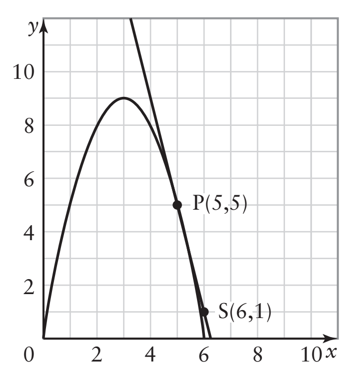
Example—Determine the instantaneous rate of change of $y = -x^2 + 6x$ at $x = 5$.



Graph and Two Points

Estimate instantaneous rate of change from the graph by finding the slope of a secant passing through the given point and another point close to (5, 5) on the curve.

$$m_{PQ} = \frac{8 - 5}{4 - 5} = -3$$



Points on the Tangent Line

Estimate the slope of a tangent at a point, P, on a graph by sketching an approximate tangent line through that point and then selecting a second point on that line. Select the point (6, 1). Label it S.

$$m_{PS} = \frac{1 - 5}{6 - 5} = -4$$

x	y
3	9
3.5	8.75
4	8
4.5	6.75
5	5
5.5	2.75
6	0

Table of Values

To estimate an instantaneous rate of change from a table, calculate the average rate of change over a short interval by using points in the table that are closest to the tangent point.

$$\begin{aligned}\text{Average rate of change} &= \frac{\Delta y}{\Delta x}\\ &= \frac{5 - 6.75}{5 - 4.5}\\ &= -3.5\end{aligned}$$

$y = -x^2 + 6x$

Equation

Determine the average rate of change over shorter and shorter intervals.

Interval	Δy	Δx	$\frac{\Delta y}{\Delta x}$
$4.5 \le x \le 5$	$y(5) - y(4.5)$ $= -(5)^2 + 6(5)$ $- [-(4.5)^2 + 6(4.5)]$ $= -25 + 30 + 20.25 - 27$ $= -1.75$	0.5	$\frac{-1.75}{0.5}$ $= -3.5$
$4.9 \le x \le 5$	$y(5) - y(4.9)$ $= -(5)^2 + 6(5)$ $- [-(4.9)^2 + 6(4.9)]$ $= -25 + 30 + 24.01$ $- 29.4$ $= -0.39$	0.1	$\frac{-0.39}{0.1}$ $= -3.9$
$4.99 \le x \le 5$	$y(5) - y(4)$ $= -(5)^2 + 6(5)$ $-[-(4.99)^2 + 6(4.99)]$ $= -25 + 30 + 24.9001$ $- 29.94$ $= -0.0399$	0.01	$\frac{-0.0399}{0.01}$ $= -3.99$

As the time intervals decrease, the average rate of change (which corresponds to the slope of a secant line) becomes closer to (or approaches) −4.

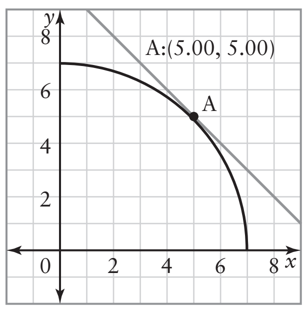
A

1. On each graph, a tangent has been drawn at the point where $x = 5$.

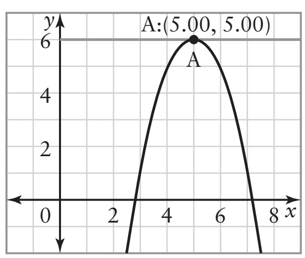
   i) At each of the indicated points on the graph, is the instantaneous rate of change positive, negative, or zero? Explain.

   ii) Determine the instantaneous rate of change of y with respect to x, at $x = 5$.

a)



b)



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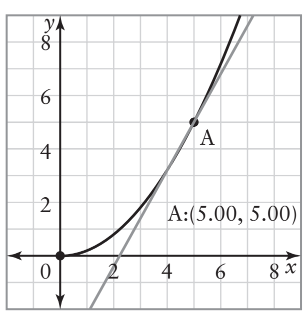


2. For each data set below, calculate the average rate of change of y between each consecutive pair of values for x.

a)

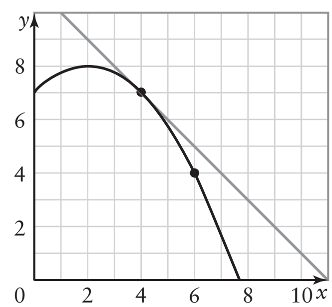
x	y
0	1
1	3
2	9
3	27
4	81
5	243
6	729

b)

x	y
50	7
100	6
150	4.6
200	2.9
250	0.9
300	−1.3
350	−3.8

B

☆3. Use two different methods to estimate the instantaneous rate of change at the point (4, 7) in the following graph.



4. For water draining from a container, the height, in millimetres, of the water as a function of time, in seconds, is $h = 0.00185(250 - t)^2$.

   a) Calculate the average rate of change from $t = 50$ s to $t = 100$ s.

   b) Estimate the instantaneous rate of decrease in height for each time:

   i) 0 s

   ii) 60 s

   iii) 120 s

   iv) 180 s

   c) Use Technology Graph the function. How does the graph support your answers in parts a) and b)?

☆**5.** A population increased over the past 50 years and is modelled by the function $P(t) = 0.2t^2 + 500$.

a) Copy and complete the table.

Interval	ΔP	Δt	$\frac{\Delta P}{\Delta t}$
$9 \le t \le 10$			
$9.9 \le x \le 10$			
$9.99 \le x \le 10$			
$10 \le t \le 10.1$			
$10 \le t \le 10.01$			
$10 \le t \le 10.001$			

b) What do you notice about the intervals before 10 years and the intervals after 10 years?

c) Estimate the instantaneous rate of change at 10 years. What does this value represent?

6. The data below represent the number of births in Canada from 2000 to 2006.

Year	Births
2000	336 912
2001	327 107
2002	328 155
2003	330 523
2004	337 762
2005	338 894
2006	343 517

Source: Statistics Canada, CANSIM Table 051-0013

a) Determine the average rate of change of births annually between 2000 and 2006.

b) Estimate the birth rate change in 2001 and in 2005.

c) How do your answers in part b) compare with your answer in part a)?

7. The function $T(d) = -0.1d^3 + 1.2d^2 - 5.4d + 12$ models the thickness, T, of the ice on a lake over a period of d days. The graph of the function is provided below.

a) Use the graph to determine the warmest day. Justify your answer.

b) Use the graph to determine the rate of change between days 1 and 2.

c) Use the equation to determine the instantaneous rate of change at day 1.

d) Write an expression for the average rate of change between $t = 1$ and $t = 1 + h$.

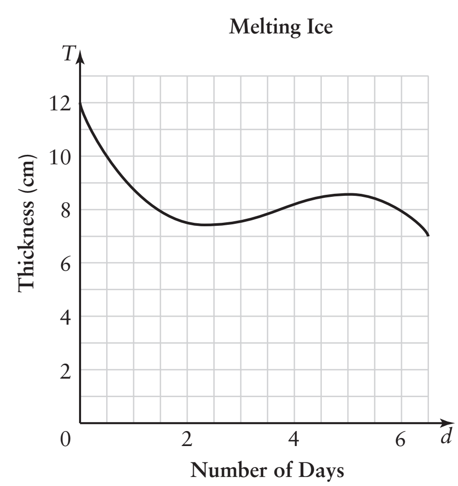
e) Use the expression in part d) to determine the average rate of change of the population when

i) $h = 0.1$

ii) $h = 0.01$

iii) $h = 0.001$

f) Estimate the instantaneous rate of change of the thickness after one day.



8. Determine the instantaneous rate of change of $f(x) = -4x^2 + 16x + 12$ at $x = 3$ and use it to find the equation of the tangent line. Graph $f(x)$ and the tangent line on the same graph.

9. The profit, in dollars, of a manufacturer selling video-game systems is given by the equation $P(x) = -0.05x^2 + 350x - 45000$, where x is the number of systems sold.

a) Find the instantaneous rate of change at $x = 1000$.

b) What are the units of the rate of change?

c) What does the instantaneous rate of change at $x = 1000$ tell you about the manufacturer's profit?

10. The displacement, s, in metres, of a particle moving back and forth in a straight line can be modelled by the function $s(t) = 4t^2 - 10t + 13$ where t is measured in seconds.

a) Find the average rate of change of the distance with respect to time from 1 s to 4 s.

b) Estimate the instantaneous rate of change of the displacement of the particle after 1 s.

c) Sketch the curve and the tangent at $t = 1$.

d) Interpret the average rate of change and the instantaneous rate of change for this situation.

11. The population, P, of a small town is modelled by the function $P(t) = -2t^3 + 55t^2 + 15t + 22\,000$, where $t = 0$ represents the beginning of this year.

a) Write an expression for the average rate of change of population from $t = 10$ to $t = 10 + h$.

b) Use the expression in part a) to determine the average rate of change of the population when

i) $h = 3$

ii) $h = 5$

c) Use the expression in part a) to estimate the instantaneous rate of change of the population after 10 years.

d) **Use Technology** Graph $P(t)$.

e) Using the graph from part d), would it be justified for a large department store to open 10 years from now in this town? Explain.

f) If the store was not opened 10 years from now, would it be justified to open it 30 years from now? Explain.

C

12. Rachael is a short-distance runner. Her position on the track is given by $s = 10.5t - 0.75t^2$ for $t \in [0, 14]$, where s is in metres and t is in seconds.

a) Sketch the position-time graph of the runner. Why do you think her direction changes?

b) Find Rachael's velocity at five points over the course of the race. Use these points to draw a velocity-time graph.

c) Describe Rachael's motion. Indicate position, direction of motion, and velocity.

☆**13.** A fish dives underwater and follows a path modelled by the function $d(t) = -(t - 1)^3 - 3(t - 1)^2 + 0.5(t - 1) - 2$, where d is the depth in metres after t seconds.

a) Draw a sketch of the graph and describe the path of the fish.

b) Copy and complete the table.

Interval	Δd	Δt	$\frac{\Delta d}{\Delta t}$
$3 \le t \le 4$			
$3.5 \le t \le 4$			
$3.9 \le t \le 4$			
$3.99 \le t \le 4$			

c) Use the table to determine the fish's velocity at $t = 4$ s. Where is the fish at this moment?

Chapter 1: Challenge Questions

C1. If $f(x) = x^2 + 1$ and $g(x) = x - 2$, find the average rate of change of $f[g(x)]$ as x changes from -1 to 1.

C2. Doctors can measure cardiac output in potential heart-attack patients by monitoring the concentration of dye after the known amount is injected in a vein near the heart. In a normal heart, the concentration of the dye is given by $f(x) = -0.006x^4 + 0.140x^3 - 0.053x^2 + 1.79x$, where x is the time in seconds.

a) Graph $f(x)$.

b) Find all the zeros of this function. Explain the meaning of the zeros in the context of the problem.

c) Determine the rate of change of the concentration of the dye for the first 10 s after injection.

C3. At liftoff, a space shuttle has a constant acceleration, a, of approximately 5 metres per second squared. The function $d(t) = \frac{1}{2}at^2$ can be used to determine the distance from Earth for each time interval, t, in seconds after liftoff.

a) Find its distance from Earth after 10 s, 30 s, and 1 min.

b) Study the pattern of answers to part a). If the time the space shuttle is in flight triples, how does the distance from Earth change? Explain.

C4. Instantaneous rate of change is the rate of change at a particular instant. Find the instantaneous rate of change of $f(t) = -3t^2 + 12t + 9$ at $t = 3$ and use it to determine the equation of the tangent line. Graph $f(t)$ and the tangent line on the same graph.

C5. A rectangular pool measures 6 m by 8 m. The pool is increased on all sides by the same amount. If the area is increased by 40 m^2, find the new dimensions of the pool.

C6. The profit, in dollars, a company makes selling a new line of MP3 Players is given by the equation $P(x) = -0.03x^2 + 225x - 24000$, where x is the number of players sold.

a) Find the instantaneous rate of change of the profit at $x = 1000$ players sold.

b) What are the units of the rate of change?

c) What does this tell you about the company's profit?

C7. Two soccer players start at opposite sides of an 80-m field. One runs at 4 m/s and the other runs at 5 m/s. If they run back and forth for 15 min, how many times will they pass each other?

C8. Find the point of intersection of $f(x) = 5 + \sqrt{x + 1}$ and $g(x) = 2x + 1$ by solving a quadratic equation.

C9. A function that represents the volume of a cardboard box is $V(x) = -0.65x^3 + 4x^2 + 3x$, where x is the width of the box. Determine the width that will maximize the volume. What are the restrictions on the width?

Chapter 1: Checklist

By the end of this chapter, I will be able to:

- Identify polynomial expressions and polynomial functions
- Represent polynomial functions numerically, graphically, and algebraically
- Describe key features of the graphs of polynomial functions
- Distinguish polynomial functions from exponential and sinusoidal functions
- Identify the connection between the factored form of a polynomial function and the x-intercepts of the corresponding graph
- Sketch the graph of a polynomial function using the key features given the factored form of the equation
- Describe the transformation associated with the roles of the parameters a, k, c, and d in polynomial functions of the form $y = a[k(x - d)]^n + c$
- Determine an equation of a polynomial given a set of conditions
- Identify and distinguish properties of even and odd polynomial functions
- Understand the connection between average rate of change and the slope of a secant, and instantaneous rate of change and the slope of a tangent
- Apply numerical and graphical methods to calculate and interpret average and instantaneous rate of change in real-world applications that involve polynomial functions