Chapter 2 Polynomial Equations and Inequalities

2.1 The Remainder Theorem

KEY CONCEPTS

- Long division can be used to divide a polynomial by a binomial.
- The result of the division of a polynomial function $P(x)$ by a binomial of the form $x - b$ can be written as $P(x) = (x - b)Q(x) + R$ or $\frac{P(x)}{x - b} = Q(x) + \frac{R}{x - b}$, where $Q(x)$ is the quotient and R is the remainder.
- To check the result of a division, use divisor $\times$ quotient $+$ remainder $=$ dividend.
- The Remainder Theorem states that, when a polynomial function $P(x)$ is divided by $x - b$, the remainder is $P(b)$, and when it is divided by $ax - b$, the remainder is $P\left(\frac{b}{a}\right)$, where a and b are integers and $a \neq 0$.

Example

Use polynomial division (long division) to determine whether the polynomial below is divisible by the given binomial. Verify the result.

a) $P(x) = x^3 - 4x^2 + 2x + 3$; $x - 3$

Solution

a) $(x^3 - 4x^2 + 2x + 3) \div (x - 3)$

$$\begin{array}{r l}
\begin{array}{rr}
 & x^2 - x - 1 \\
x - 3 \,\big)\!\! & \overline{x^3 - 4x^2 + 2x + 3} \\
 & \underline{x^3 - 3x^2} \qquad\qquad \\
 & -x^2 + 2x \qquad \\
 & \underline{-x^2 + 3x} \qquad \\
 & -x + 3 \\
 & \underline{-x + 3} \\
 & 0
\end{array}
\end{array}$$

Divide x^3 by x to get x^2. Write x^2 in the appropriate column in the quotient. Multiply $x - 3$ by x^2.

Subtract. Bring down the next term, $2x$. Divide $-x^2$ by x to get $-x$. Multiply $x - 3$ by $-x$.

Subtract. Bring down the next term, 3. Divide $-x$ by x to get -1. Multiply $x - 3$ by -1.

Subtract.

The remainder is 0.

Therefore, $x^3 - 4x^2 + 2x + 3 = (x - 3)(x^2 - x - 1)$.

Verify that $P(x)$ is divisible by $x - 3$ using the Remainder Theorem.

Since $x - 3 = x - (+3)$, the remainder is $P(3)$.

$$\begin{aligned}
P(3) &= 3^3 - 4(3)^2 + 2(3) + 3 \\
&= 27 - 36 + 6 + 3 \\
&= 0
\end{aligned}$$

Therefore, $x - 3$ is a factor, since $P(3) = 0$.

b) $S(x) = 6x^3 - 4x^2 + 3x - 2;\ 2x + 1$

Solution

b) $(6x^3 - 4x^2 + 3x - 2) \div (2x + 1)$

$$
\begin{array}{r l}
3x^2 - 3.5x + 3.25 & \\
2x + 1\overline{)6x^3 - 4x^2 + 3x - 2} & \text{Divide } 6x^3 \text{ by } 2x \text{ to get } 3x^2\text{. Write } 3x^2 \text{ in the appropriate column. Multiply } 2x + 1 \text{ by } 3x^2. \\
\underline{6x^3 + 3x^2} \qquad\qquad\quad & \\
-7x^2 + 3x \qquad\quad & \text{Subtract. Bring down } 3x\text{. Divide } -7x^2 \text{ by } 2x \text{ to get } -3.5x\text{. Multiply } 2x + 1 \text{ by } -3.5x. \\
\underline{-7x^2 - 3.5x} \qquad\quad & \\
6.5x - 2 & \text{Subtract. Bring down } -2\text{. Divide } 6.5x \text{ by } 2x \text{ to get } 3.25\text{. Multiply } 2x + 1 \text{ by } 3.25. \\
\underline{6.5x + 3.25} & \\
-5.25 & \text{Subtract.}
\end{array}
$$

The remainder is -5.25.

Then, $(2x + 1)(3x^2 - 3.5x + 3.25) - 5.25 = 6x^3 - 4x^2 + 3x - 2$

Verify the result using the Remainder Theorem.

Comparing $2x + 1$ to $ax - b$, gives $a = 2$ and $b = -1$.

The remainder $S\left(\frac{b}{a}\right)$ is $S\left(-\frac{1}{2}\right)$.

$$
\begin{aligned}
S\left(-\tfrac{1}{2}\right) &= 6\left(-\tfrac{1}{2}\right)^3 - 4\left(-\tfrac{1}{2}\right)^2 + 3\left(-\tfrac{1}{2}\right) - 2 \\
&= 6\left(-\tfrac{1}{8}\right) - 4\left(\tfrac{1}{4}\right) + 3\left(-\tfrac{1}{2}\right) - 2 \\
&= -\tfrac{3}{4} - 1 - \tfrac{3}{2} - 2 \\
&= -\tfrac{21}{4}
\end{aligned}
$$

When $S(x) = 6x^3 - 4x^2 + 3x - 2$ is divided by $2x + 1$, the remainder is $-\frac{21}{4}$.

A

1. a) Divide $3x^3 - 2x^2 - 8$ by $x + 5$. Express the result in quotient form.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

d) Verify your answer.

2. a) Divide $3x^4 + 2x^2 - 6x + 1$ by $x + 1$. Express the result in quotient form.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

d) Verify your answer.

3. Perform each division below. Express the result in quotient form. Identify any restrictions on the variable.

   a) $2x^2 - x + 5$ divided by $x + 3$

   b) $x^3 - x - 10$ divided by $x + 4$

   c) $x^3 + x^2 - 4x + 4$ divided by $x - 2$

   d) $3x^4 + 2x^2 - 6x + 1$ divided by x

   e) $4x^3 - 10x^2 + 6x - 18$ divided by $2x - 5$

   f) $2x^3 - x^2 + 8x + 4$ divided by $2x - 1$

   g) $x^5 - 10x^4 + 20x^3 - 5x - 95$ divided by $x + 10$

4. Determine the remainder, R, so that each statement below is true.

   a) $(x^2 - 3x - 7)(x - 2) + R = x^3 - 5x^2 - x - 10$

   b) $(x^2 - 4x - 12)(x + 4) + R = x^3 - 28x - 41$

   c) $(-3x^2 + 13x + 10)(2x - 1) + R = -6x^3 + 29x^2 + 7x - 13$

   d) $(x^2 - 3)(x + 4) + R = x^3 + 4x^2 - 3x - 12$

5. The volume, in cubic centimetres, of a rectangular box can be modelled by the polynomial expression $2y^3 + y^2 - 27y - 36$. Determine possible dimensions of the box if the height, in centimetres, is given by $y + 3$.

6. Use the Remainder Theorem to determine the remainder when $2x^3 - 3x^2 - 8x - 3$ is divided by each of the following binomials. Verify your answers using long division.

   a) $x - 3$

   b) $x + 2$

   c) $2x + 1$

   d) $x - 2$

7. Determine the remainder when each of the following polynomials is divided by $x - 2$.

   a) $x^2 - 4x + 13$

   b) $x^3 + 3x^2 - 4x - 12$

   c) $2x^3 + 3x^2 - 17x - 30$

   d) $x^3 + x^2 - 4x + 4$

8. Use the Remainder Theorem to determine the remainder for each division below.

   a) $(x^2 + 2x + 4) \div (x - 2)$

   b) $(2x^3 + 3x^2 - 5x + 2) \div (x + 3)$

   c) $(2x^2 + 5x + 7) \div (2x - 3)$

   d) $(9x^3 - 6x^2 + 3x + 2) \div (3x - 1)$

B

☆9. a) When $P(x) = x^3 + kx^2 - 4x + 2$ is divided by $x + 2$, the remainder is 26; find k.

   b) Determine the remainders when the $P(x)$ is divided by $x + 1$ and $x - 1$.

10. a) Use the Remainder Theorem to determine the remainder when $(3x^3 + 7x^2 - 2x - 11)$ is divided by $(x - 2)$.

   b) Verify your result with long division.

11. a) When $P(x) = kx^4 - 8x^2 - 6$ is divided by $x + 1$, the remainder is -11; find k.

   b) Determine the remainders when $P(x)$ is divided by $x - 1$ and by $x - 3$.

12. Which of the following values for b give a remainder of zero when the given polynomial is divided by $x - b$?

   a) $f(x) = x^4 + 6x^3 - x^2 + 30x$; $b = 2, 3, 0, 1$

   b) $g(x) = 6x^2 + x - 1$; $b = 1, \frac{1}{2}, 2, -\frac{1}{2}, 3$

   c) $h(x) = x^3 + x^2 - 8x - 8$; $b = 2\sqrt{2}, \sqrt{2}, -\sqrt{2}, 1, -1$

☆**13.** When a polynomial $p(x)$ is divided by $x + 3$, the quotient is $x^2 - 3x + 5$ and the remainder is 6. What is the polynomial?

14. When a polynomial $k(x)$ is divided by $x - 2$, the quotient is $x^2 + 4x - 7$ and the remainder is -4. What is the polynomial?

15. At the Niagara Ice Dogs Ticket Office, game tickets have just gone on sale. Sayid is the ticket sales manager and he collects sales results that show the number of tickets sold over the first 12 h of sales. Using the data, and a quadratic regression, the total number of tickets, N, sold over t hours is $N(t) = 0.45t^2 + 2.4t$, where t is measured in hours and $t \in [0, 12]$.

a) Divide $N(t)$ by $t - 2$. State the quotient, $Q(t)$, and the remainder, R. Explain the meaning of $Q(t)$ and R.

b) Does the formula for $N(t)$ seem realistic? Explain.

16. a) Determine the remainder when $2x^3 + 3x^2 - 17x - 30$ is divided by $x - 3$.

b) What information does the remainder provide about $x - 3$?

c) Express $2x^3 + 3x^2 - 17x - 30$ in factored form.

17. Dividing $(2x^3 + 4x^2 - kx + 5) \div (x + 3)$ and $(6y^3 - 3y^2 + 2y + 7) \div (2y - 1)$ leads to the same remainder. Find the value of k.

18. Find the values of a and b such that $ax^3 + bx^2 + 3x - 4$ has a remainder of -2 when divided by $(x - 1)$, and a remainder of 2 when divided by $(x - 2)$.

19. a) Determine the value of c such that when $P(x) = 2x^3 - cx^2 + 4x - 7$ is divided by $x - 2$, the remainder is -3.

b) Use Technology Verify your answer in part a) using a computer algebra system.

20. For what value of k will the polynomial $f(x) = x^3 + 2x^2 + kx + 5$ have the same remainder when it is divided by $x + 1$ and $x - 2$?

21. Use the remainder theorem to determine the remainder when $x^4 + x^3 - 3x + 6$ is divided by $3x - 2$.

C

22. The polynomial $mx^3 + 12x^2 - nx + 2$ has a remainder of 14 when divided by $x - 1$ and a remainder of 68 when divided by $x - 2$. What are the values of m and n?

☆**23.** The polynomial $-x^3 - vx^2 + 2x + w$ has a remainder of 6 when divided by $x + 2$ and a remainder of 119 when divided by $x - 3$. What are the values of v and w?

24. Determine whether the first polynomial in each pair below is divisible by the second.

a) $x^3 + 2x^2 - 5x - 6$, $x^2 + 3x - 1$

b) $x^5 + x^4 - 81x - 81$, $x^2 + 9$

c) $x^4 + 3x^3 - 2x^2 - 3x + 1$, $x^2 + 3x + 1$

25. Determine a and b such that, when $x^4 + x^3 - 7x^2 + ax + b$ is divided by $(x - 1)(x + 2)$, the remainder is 0.

KEY CONCEPTS

- The Factor Theorem states that $x - b$ is a factor of a polynomial $P(x)$ if and only if $P(b) = 0$.
  Similarly, $ax - b$ is a factor of $P(x)$ if and only if $P\left(\frac{b}{a}\right) = 0$.

Example

Is $x - 2$ a factor of $P(x) = x^3 + 2x^2 - 10x + 4$?

Solution

Substitute $x = 2$ into the polynomial expression.

$$\begin{aligned} P(2) &= (2)^3 + 2(2)^2 - 10(2) + 4 \\ &= 8 + 8 - 20 + 4 \\ &= 0 \end{aligned}$$

Therefore, $x - 2$ is a factor of the polynomial $P(x)$.

- The integral zero theorem states that, if $x - b$ is a factor of a polynomial function $P(x)$ with leading coefficient 1 and remaining coefficients that are integers, then b is a factor of the constant term of $P(x)$.

- The rational zero theorem states that, if $P(x)$ is a polynomial function with integer coefficients and $x = \frac{b}{a}$ is a rational zero of $P(x)$, then
  - b is a factor of the constant term of $P(x)$
  - a is a factor of the leading coefficient of $P(x)$
  - $ax - b$ is a factor of $P(x)$

Example

Is $3x - 2$ a factor of $R(x) = 9x^3 + 2x - 4$?

Solution

For $3x - 2$, substitute $x = \frac{2}{3}$ into the polynomial expression.

$$\begin{aligned} R\left(\frac{2}{3}\right) &= 9\left(\frac{2}{3}\right)^3 + 2\left(\frac{2}{3}\right) - 4 \\ &= 9\left(\frac{8}{27}\right) + \left(\frac{4}{3}\right) - 4 \\ &= \frac{8}{3} + \frac{4}{3} - 4 \\ &= 0 \end{aligned}$$

Therefore, $3x - 2$ is a factor of the polynomial $R(x)$.

Example

Factor $2x^3 - 3x^2 - 5x + 6$

Find a value for x such that $P(x) = 0$.

Using the Rational Zero Theorem, let b represent values that are factors of the constant term, which are $\pm 1, \pm 2, \pm 3, \pm 6$.

Test the values of b for x to find the zeros.

$$\begin{aligned} P(1) &= 2(1)^3 - 3(1)^2 - 5(1) + 6 \\ &= 2 - 3 - 5 + 6 \\ &= 0 \;\; \therefore x - 1 \text{ is a factor.} \end{aligned}$$

Synthetic Division of the polynomial $2x^3 - 3x^2 - 5x + 6$ by $x - 1$ gives the other factors.

Write the coefficients from the polynomial in the first row of the division chart. To the left, write the value of 1 from the factor $x - 1$.

Bring down the first coefficient, 2, to the right of the $\times$ sign.
Multiply 1 (top left) by 2 (right of $\times$ sign) to get 2.

$$\begin{array}{r|rrrr} 1 & 2 & -3 & -5 & 6 \\ + & \downarrow & 2 & -1 & -6 \\ \hline \times & 2 & -1 & -6 & 0 \end{array}$$

Write 2 below -3 in the second column.

Add 2 to -3 to get -1.

Multiply 1 by -1 to get -1. Write -1 in the third column below the -5.

Add -5 and -1 to get -6.

Multiply by 1 by -6. Write -6 in the fourth column below the 6.
Add 6 and -6 to give 0.

This gives $2x^3 - 3x^2 - 5x + 6 = (x - 1)(2x^2 - x - 6)$.
Notice that the coefficients in the second factor come from the result of synthetic division.

Factoring further gives $2x^3 - 3x^2 - 5x + 6 = (x - 1)(x - 2)(2x + 3)$.

A

1. Write the binomial factor that corresponds to each polynomial $P(x)$ below.

a) $P(-4)$

b) $P(2)$

c) $P\left(\frac{2}{5}\right)$

d) $P\left(-\frac{4}{3}\right)$

2. Determine whether $x - 1$ could be a factor of each of the following polynomials.

a) $x^3 + x^2 - x - 1$

b) $x^4 - 3x^3 + 2x^2 - x + 1$

c) $2x^3 - x^2 - 3x - 1$

3. Divide each polynomial below by the given binomial using synthetic division.

a) $(x^2 + 20x + 91) \div (x + 7)$

b) $(x^3 - 9x^2 + 27x - 28) \div (x - 3)$

4. List the values that could be zeros of each polynomial below. Then, factor the polynomial.
   a) $x^3 - 3x^2 + 4x - 2$
   b) $x^3 + 2x^2 + 2x + 1$
   c) $x^3 + 7x^2 + 17x + 15$

5. Factor each polynomial below. Start by grouping terms.
   a) $x^3 - x^2 - 4x + 4$
   b) $x^3 - 2x^2 - 4x + 8$
   c) $x^3 + 3x^2 + 3x + 1$
   d) $x^4 + 2x^3 - x - 2$
   e) $3x^4 - 18x^3 + x - 6$
   f) $x^4 - 1$

6. Determine the values of each polynomial below that could be zeros. Then, factor the polynomial.
   a) $2x^3 - 9x^2 + 10x - 3$
   b) $4x^3 - 7x - 3$
   c) $4x^3 - 8x^2 + x + 3$
   d) $3x^3 - x^2 - 3x + 1$

B

7. Complete each of these division exercises using synthetic division.
   a) $(x^2 - x - 56) \div (x + 7)$
   b) $(x^3 - 9x^2 + 27x - 28) \div (x - 3)$
   c) $(2x^3 - 2x - 3) \div (x - 1)$
   d) $(x^4 - 8x^2 + 16) \div (x + 2)$
   e) $(2x^3 - 7x^2 - 10x + 26) \div (2x - 3)$
   f) $(2x - 1 + 9x^3) \div (3x - 1)$

8. **Use Technology** Factor each of these polynomials, if possible.
   a) $2x^3 - x^2 + x - 2$
   b) $2x^3 + 9x^2 + 3x - 4$
   c) $3x^4 - 6x^3 + 2x - 1$
   d) $4x^4 + 4x^3 - 35x^2 - 36x - 9$
   e) $2x^4 - x^3 + 3x - 1$
   f) $6x^3 - 7x^2 + 1$
   g) $2x^4 - 11x^3 + 12x^2 + x - 4$

9. Determine the value of k so that $x^3 + 5x^2 + kx + 6$ has $x + 2$ as a factor.

10. Determine the value of k so that $kx^3 - 10x^2 + 2x + 3$ has $x - 3$ as a factor.

☆11. Determine the value of k so that $2x^3 - (k + 1)x^2 + 6kx + 11$ has $x - 1$ as a factor.

12. Factor each polynomial.
   ☆a) $x^3 + 2x^2 - 9x - 18$
   b) $4x^3 - 8x^2 + x + 3$
   ☆c) $6x^3 + x^2 - 31x + 10$
   d) $4x^3 + 5x + 21$
   ☆e) $3x^3 + 2x^2 - 19x + 6$
   f) $x^4 + x^3 - 13x^2 - 25x - 12$
   g) $x^4 + 3x^3 - x^2 - 6x$

13. Factor each sum or difference of the cubes below.
   a) $8x^3 - 1$
   b) $512x^3 - 64y^3$
   c) $x^3 - \frac{1}{27}$
   d) $1 + 125x^3$
   e) $\frac{1}{8}x^3 + \frac{1}{64}$
   f) $135x^3 + 625y^3$

C

14. Divide $3x^4 - 3x^3 - 11x^2 + 6x - 1$ by $x^3 + x^2 - 2$.

15. a) Show that $x - 1$ is a factor of $14x^{99} - 65x^{56} + 51$.
   b) Show that $x - 1$ is not a factor of $x^5 + 1$.

☆16. Factor $32x^4 - 128x^3 - 54x^2 + 243x - 108$.

17. Show that $(x - a)$ is a factor of $x^n - a^n$.

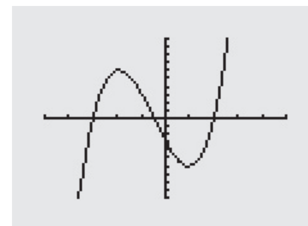
18. Use the Factor Theorem to show that $(x - a)$ is a factor of $(a - x)^3 + (x - b)^3 + (b - a)^3$.

KEY CONCEPTS

- The real roots of a polynomial equation $P(x) = 0$ correspond to the x-intercepts of the graph of the polynomial function $P(x)$.

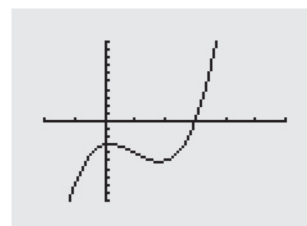
  In factored form, $2x^3 + 3x^2 - 11x - 6$ is $(x - 2)(2x - 1)(x - 3)$.

  The values 2, $-\frac{1}{2}$, and -3 are the roots of the equation $2x^3 + 3x^2 - 11x - 6 = 0$ and are the x-intercepts of the graph of the related function $y = 2x^3 + 3x^2 - 11x - 6$.



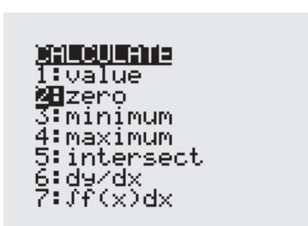
- The x-intercepts of the graph of a polynomial function correspond to the real roots of the related polynomial equation.

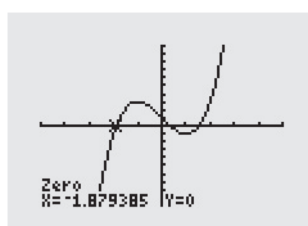
  The function $y = (x - 3)(x^2 + 1)$ has only one real zero, so the equation $(x - 3)(x^2 + 1) = 0$ has one real root. The x-intercept of the graph is 3.

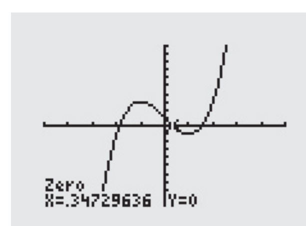


- If a polynomial equation is factorable, the roots are determined by factoring the polynomial, setting its factors equal to zero, and solving each factor.

- If a polynomial equation is not factorable, the roots can be determined from the graph using technology.

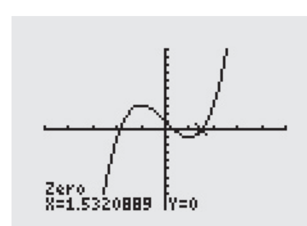






From the graph of $y = x^3 - 3x + 1$, there are three x-intercepts, one near -2, another near 0, and a third near 2. Use the ZERO operation.

The three roots of the equation are -1.9, 0.3, and 1.5, to one decimal place.



A

1. Solve.

a) $x(x - 1)(x + 3) = 0$

b) $(x + 2)(x + 5)^2 = 0$

c) $(2x - 5)(x + 3)(4x - 1) = 0$

d) $(2x - 7)(3x - 4)(x + 6) = 0$

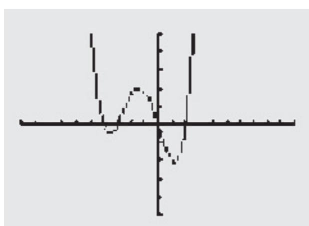
e) $(x - 3)(2x + 1)(15x - 2) = 0$

f) $(x + 8)(x - 9)(3x - 1) = 0$

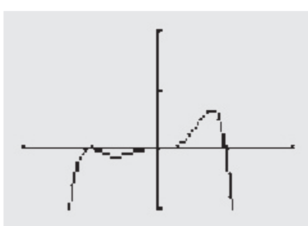
g) $4(x - 2)(5x - 4)(5x + 4) = 0$

2. Use each graph below to determine the integral roots of the corresponding polynomial equation.

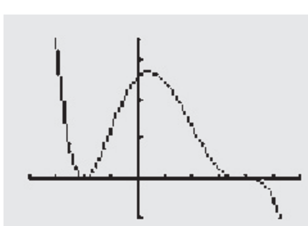
a) Window variables: $x \in [-10, 10]$, $y \in [-50, 50]$, Yscl = 5



b) Window variables: $x \in [-2, 2]$, $y \in [-0.5, 1]$, Yscl = 0.5



c) Window variables: $x \in [4, 6]$, $y \in [100, 350]$, Yscl = 100



3. Determine the real roots of each polynomial equation.

a) $(x^2 - 81)(x^2 + 6x + 9) = 0$

b) $(x^2 - 2x - 15)(x^2 - 4x + 4) = 0$

c) $(3x^2 - 7x + 2)(x^2 + 10x + 16) = 0$

d) $(4x^2 - 36)(x^2 - 2x + 1) = 0$

e) $(4x^2 - 8x - 5)(7x^2 - 28) = 0$

f) $(x^2 + 4)(x^2 - 100) = 0$

g) $(3x^2 - x - 4)(6x^2 + x - 1) = 0$

4. Determine the x-intercepts of the graph of each of the following polynomial functions.

a) $y = 16x^3 - 49x$

b) $f(x) = x^3 + 12x^2 + 36x$

c) $g(x) = x^3 - 16x^2 + 63x$

d) $h(x) = 3x^4 - 363x^2$

e) $k(x) = x^4 - 27x$

f) $p(x) = x^4 - 64$

g) $q(x) = x^4 - 12x^2 + 27$

B

☆**5.** Determine the equation of a cubic function whose graph passes through $(-2, 0)$ and has a local minimum at $(1, 0)$.

6. Determine an equation for a quartic function whose graph passes through $(-4, 0)$, $(2, 0)$, $(1, 0)$, and $(2, 0)$.

7. Solve by factoring.

a) $x^3 - x^2 - 16x - 20 = 0$

b) $x^3 - 3x^2 - 4x + 12 = 0$

c) $x^3 - 3x^2 - 16x + 48 = 0$

d) $x^3 - 4x^2 + x + 6 = 0$

e) $x^3 + 8x^2 + 19x + 12 = 0$

f) $x^4 + x^3 - 13x^2 - 25x - 12 = 0$

g) $x^4 - x^3 - 11x^2 + 9x + 18 = 0$

8. Compare the zeros of $f(x) = x^3 + 3x^2 - 6x - 8$ and $g(x) = -x^3 - 3x^2 + 6x + 8$.

9. Solve the following by factoring:

a) $x^2(x + 1) = 12 + 8x$

b) $x^3(x - 1) = 11x^2 - 9(x + 2)$

c) $x(x^2 - 4) = -3(x^2 - 4)$

d) $x^5 + 3x^4 - 5x^3 - 15x^2 + 4x + 12 = 0$

10. Use Technology Solve the following by factoring. Round your answers to one decimal place.

a) $x^2 - 4x + 2 = 0$

b) $x^3 - 5x + 2 = 0$

c) $x^3 - 7x^2 + 9x + 2 = 0$

d) $x^3 + 4x^2 + 7x + 6 = 0$

e) $x^3 + 8 = 0$

f) $6x^4 - 10x^3 - 10x^2 - 32x - 8 = 0$

11. Use Technology During the first 100 h of an experiment, the growth rate of a bacteria population at time t hours is $p(t) = -0.0041t^3 + 0.02t^2 + 0.5t + 1$ bacteria per hour.

a) What is the growth rate at 10 h? at 25 h?

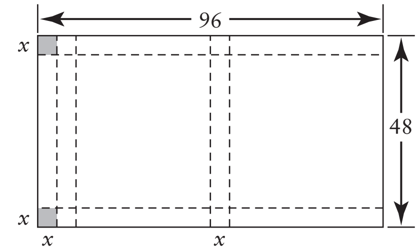
b) At what time is the growth rate 20 bacteria per hour? What does this mean?

c) At what time is the growth rate 0?

d) At what time is the growth rate −50 bacteria per hour? What does this mean?

e) At approximately what time does the highest growth rate occur?

12. A box with a lid is to be made from a 96 cm by 48 cm sheet of metal. It is cut and *folded* as shown in the figure. If the box must be at least 12 cm high, is it possible to cut squares from the corners so that the box has a volume of 9600 cm^3? If so, find the size of the squares to be cut. If not, explain why not.



☆**13.** At the turn of the twentieth century, the wolf population north of the Algonquin Park area experienced a rapid increase because hunters had reduced the numbers of natural predators. The food supply was not sufficient to support the increased population, and the population declined. The wolf population for the period from 1905 to 1930 can be modelled by $f(x) = -0.125x^5 + 4.125x^4 + 3500$, where x is the number of years since 1905.

a) Graph the function.

b) What value represents the population in 1905?

c) Use the model to determine the population in 1920.

d) According to this model, when did the wolf population become zero?

C

☆**14.** Determine the equation of a cubic function whose graph passes through (1, 0), (0, −1), (−1, −4), and (2, 5).

15. Factor $f(x) = x^5 + 4x^4 - 10x^3 - 40x^2 + 9x + 36$.

16. Find all real and complex roots of the given polynomial functions.

a) $x^3 - x^2 + 5x = 0$

b) $2x^3 - 9x^2 = 0$

c) $x^4 - 16x^2 = 0$

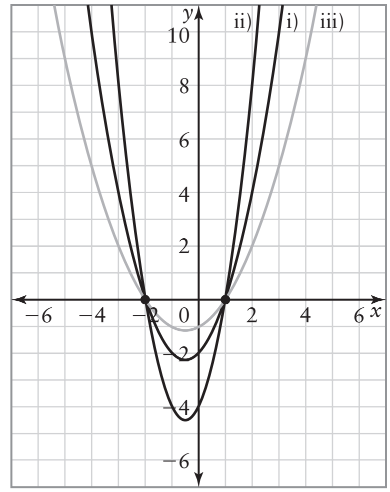
d) $(x^2 + 1)(x^2 + 9) = 0$

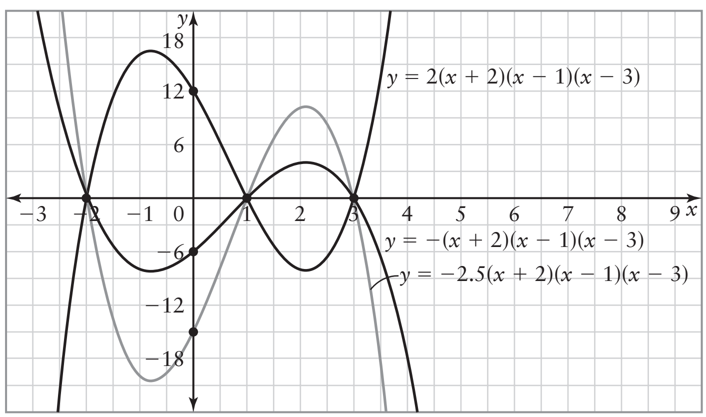
e) $x^4 - 4x^2 + 1 = 0$

A complex number is a number that can be written in the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$.

KEY CONCEPTS

- A family of functions is a set of functions with the same characteristics.
- Polynomial functions with graphs that have the same x-intercepts belong to the same family.
- A family of polynomial functions with zeros $a_1, a_2, a_3, \ldots, a_n$ can be represented by an equation of the form $y = k(x - a_1)(x - a_2)(x - a_3) \ldots (x - a_n)$, where $k \in \mathbb{R}$, $k \neq 0$.
- An equation for a particular member of a family of polynomial functions can be determined if a point on the graph is known.





- You can use any point on a function, and the zeros, to determine the equation of a particular function within a family.
- Notice that, if you multiply all the integer values in each equation, the result is the y-intercept.

  Example: Given $y = -2.5(x + 2)(x - 1)(x - 3)$, multiply $-2.5 \times 2 \times -1 \times -3$ to give -15, which is the y-intercept.
- Determine an additional equation that belongs to the same family by changing the multiple in front of the factors.

- Determine an additional equation that belongs to the same family by finding a point on the new function. Substitute the point into the function.

  Example: If the point (2, 16) lies on the graph, then

  $$\begin{aligned} 16 &= a(2 + 2)(2 - 1)(2 - 3) \\ &= a(4)(1)(-1) \\ &= -4a \\ a &= -4 \end{aligned}$$

  Then, a new equation is $y = -4(x + 2)(x - 1)(x - 3)$.

A

1. The zeros of a quadratic function are 5 and 8.

a) Determine an equation for the family of quadratic functions with these zeros.

b) Write equations for two functions that belong to this family.

c) Determine an equation for the member of the family that passes through the point (7, 6).

2. A cubic function is given by $y = k(x + 2)(x - 3)(x - 8)$.

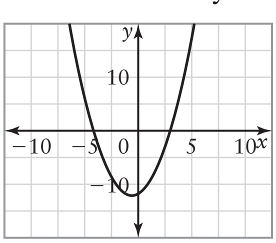
a) What are the zeros of this family of cubic functions?

b) Write equations for two functions that belong to this family.

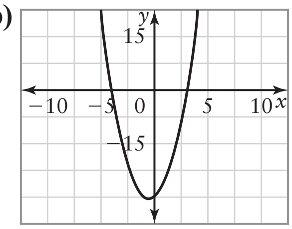
c) Determine an equation for the member of the family that has a y-intercept at 192.

3. Examine the following graphs to determine functions that belong to the same family. Write an equation that describes each family of functions.

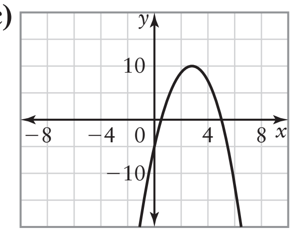
a)



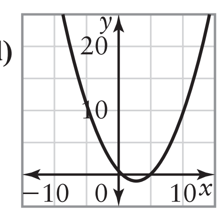
b)



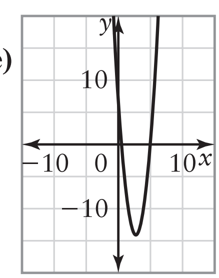
c)



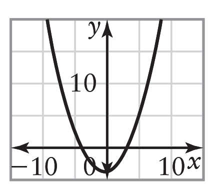
d)



e)



f)



4. Which of the following functions belong to the same families? Explain. Sketch a graph of the functions in each family to verify your answer.

a) $y = 0.3(x - 2)(x + 4)(x + 6)$

b) $y = -2(x - 5)(2x - 1)(1 + 3x)$

c) $y = 4(x - 2)(x + 3)^2$

d) $y = -(3x + 1)(x - 5)(1 - 2x)$

e) $y = -4(x + 6)(x - 2)(x + 4)$

f) $y = 0.5(x + 3)^2 (x - 2)$

g) $y = 4(2x - 1)(1 + 3x)(x - 5)$

h) $y = (5x - 10)(x^2 + 10x + 24)$

5. Write an equation for a family of polynomial functions with the following zeros.

a) $3, -2$

b) $0, 1, 5$

c) $-3, 1, 6$

d) $-\sqrt{3}, 0, \frac{2}{3}, \sqrt{3}$

B

6. The zeros of a cubic function are -5, -1, and 2.

a) Determine an equation for the family of cubic functions with these zeros.

b) Write equations for two functions that belong to this family.

c) Determine an equation for the member of the family that passes through (3, 12).

7. a) Determine an equation for the family of quartic functions with zeros -4, -3, 1, and 6.

b) Write equations for two functions that belong to this family.

c) Determine an equation for the family member whose graph passes through the point$(-1, 21)$.

d) Sketch a graph of the functions in parts b) and c).

☆8. a) Determine an equation for the family of quartic functions with zeros $-\frac{3}{2}, 0, \frac{1}{2}$, and 2.

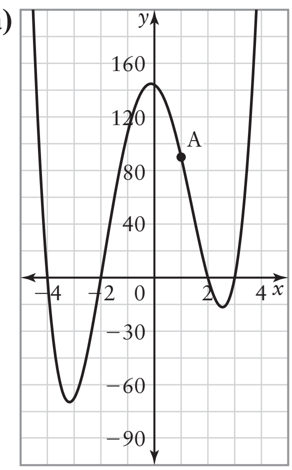
b) Write equations for two functions that belong to this family.

c) Determine an equation for the family member whose graph passes through the point $(-1, 4.5)$.

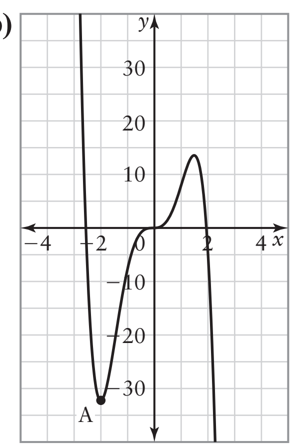
d) Sketch a graph of the functions in parts b) and c).

☆9. Determine equations for the functions in the graphs below.

a)



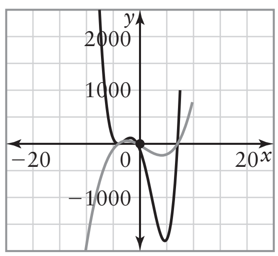
b)



10. a) Determine an equation, in simplified form, for the family of quartic functions with zeros 5 (order 2) and $-1 \pm 2\sqrt{2}$.

b) Determine an equation for the family member whose graph passes through the point (3, 9.6).

11. Determine equations for the families of equations shown in the graph below. Describe the similarities in the graphs and in the equations.



$$f(x) = (x + 4)^2(2x + 1)(x - 7)$$

$$f(x) = (x + 4)(2x + 1)(x - 7)$$

☆**12. a)** Determine an equation for a family of functions with zeros at $1 \pm \sqrt{5}$, -3, and $\frac{1}{3}$.

b) Determine an equation for the family member whose graph passes through the point $(-2, -7)$.

☆**13.** Determine the equation for a family of functions with the same zeros as given in question 12, but with a higher degree. Explain how your equation differs from the equation in question 12.

C

14. a) Write an equation for a family of odd functions with three x-intercepts, two of which are -1.5, and one of which is 7.

b) What is the least degree this family of functions can have?

c) Determine an equation for the member of this family that passes through the point $(-3, 11.25)$.

d) Determine an equation for the member of this family that is a reflection in the y-axis of the function in part c).

e) Determine an equation that is a reflection in the x-axis of the function in part c).

15. Consider the function $f(x) = 4x^4 + (k - 16)x^3 + (17 - 4k)x^2 + 4(k - 1)x + 4$.

a) Use the Factor Theorem to determine an integral root of $f(x)$.

b) Use Technology Investigate how the function changes as k is varied.

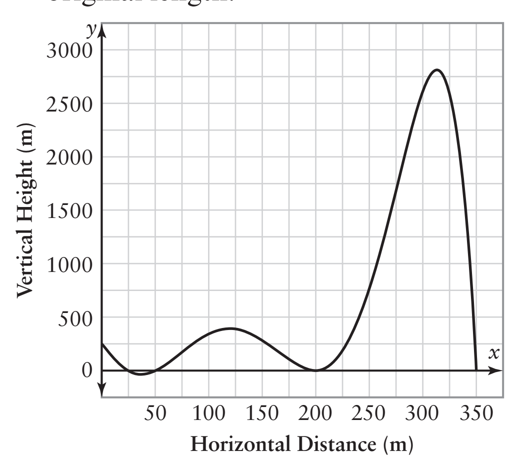
c) For what values of k does $f(x)$ have three or more real roots?

d) Choose a value for k that gives the function one real root (order 2). Can you create a family of equations that have the same roots (both real and complex)? Explain why or why not.

16. The graph below represents a section of the track of a roller coaster.

a) Write an equation for the family of functions that models the section of the track.

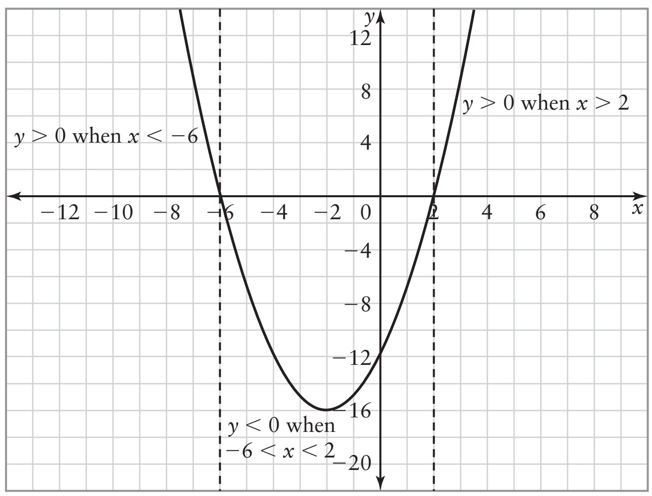
b) Add an additional section of track to extend the roller coaster to twice its original length.



KEY CONCEPTS

- A polynomial inequality results when the equal sign in a polynomial equation is replaced with an inequality symbol.
- The real zeros of a polynomial function, or x-intercepts of the corresponding graph, divide the x-axis into intervals that can be used to solve a polynomial inequality.

Interval	$x < -6$	$-6 < x < 2$	$x > 2$
Sign of Function	$+$	$-$	$+$

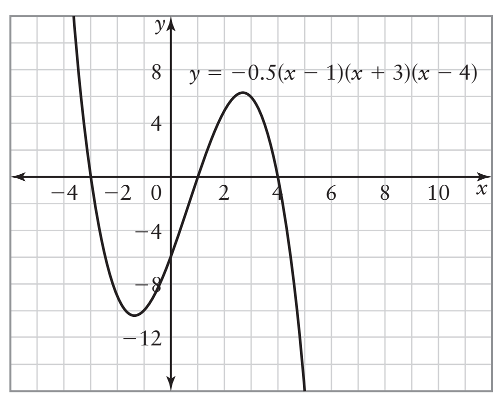


- A CAS may be used to solve a polynomial inequality numerically by determining the roots of the polynomial equation and then testing values in each interval to see if they make the inequality true.
- Polynomial inequalities may be solved graphically by determining the x-intercepts and then using the graph to determine the intervals that satisfy the inequality.

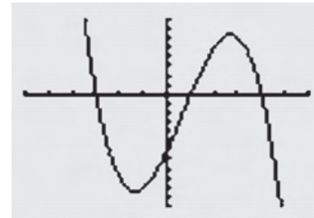
Example

Solving an inequality graphically on the TI-84+:

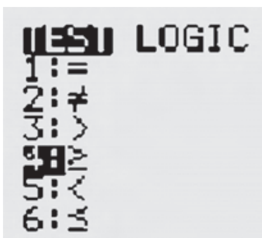
The zeros of the function
$f(x) = -0.5(x + 3)(x - 1)(x - 4)$
are -3, 1, and 4.

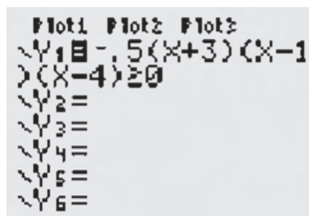


1. Graph the function. Note that the zeros of the function are $x = -3$, $x = 1$, and $x = 4$.



2. • Return to the equation.
   • Press Y =.
   • Position the cursor at the end of the equation and choose the TEST function (2[nd] MATH).
   • Choose the $\geq$ symbol.
   • Graph the inequality by choosing the ZDecimal operation from the ZOOM Menu.

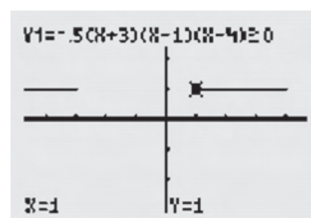




When an inequality is true, the test function plots a point at 1. Otherwise, the plot is set to zero. The intervals where the inequality is true are represented by the horizontal bars at 1.

Use the TRACE key to move the cursor to the end points of each interval.

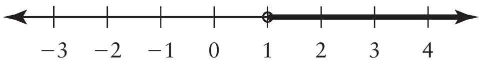
This test shows that the solution to this inequality is $x \leq -3$, $1 \leq x \leq 4$.

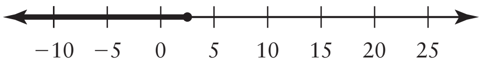


Here is the TI-84+ test for the inequality $-0.5(x + 3)(x - 1)(x - 4) < 0$. The solution is $-3 < x < 1$, $x > 4$. Notice that there are no equal signs in this solution. Also, the solution to this inequality includes the intervals that are NOT in the $\geq$ inequality.

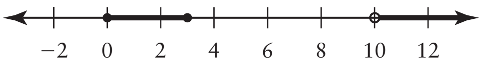
A

1. Write inequalities for the values of x as shown.

a) 

b) 

c) 

d) 

2. Write the intervals into which the x-axis is divided by each set of x-intercepts below.

   a) $-2, 0$

   b) $-3, 1, 2$

   c) $1, 5, 6, 10$

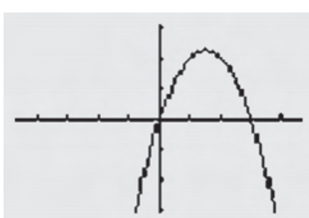
   d) $-14, -12, -7.5, 21$

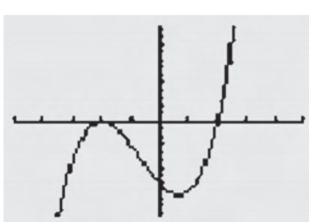
3. For each graph, write

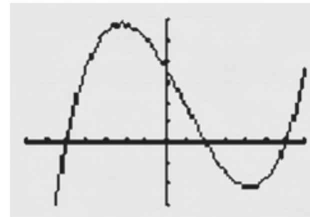
   i) the x-intercepts

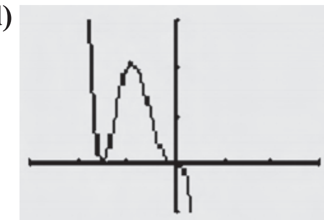
   ii) the intervals of x for which the graph is positive

   iii) the intervals of x for which the graph is negative

a) 

b) 

c) 

d) 

4. **a)** Sketch the graph of a quadratic polynomial function $y = f(x)$, such that $f(x)$ is negative when $x \in (-\infty, -5)$ or $x \in (10, \infty)$ and $f(x)$ is positive when $x \in [-5, 10]$.

   b) Write an inequality statement for the quadratic function described in part a).

☆5. **a)** Sketch the graph of a quartic polynomial function $y = f(x)$, such that $f(x) \leq 0$ when $x \in (-\infty, -5)$ or $x \in (2, 3)$ or $x \in (4, \infty)$ and $f(x) > 0$ when $x \in (-5, 2)$ or $x \in (3, 4)$.

   b) Write an inequality statement for the quartic function described in part a).

B

6. Solve each polynomial inequality below by graphing the polynomial function.

   a) $x^2 - 13x + 30 \leq 0$

   b) $x^2 + 20x + 96 > 0$

   c) $x^3 - 3x^2 - 4x + 12 \leq 0$

   d) $x^3 - 3x^2 - 16x + 48 \geq 0$

   e) $x^3 - 4x^2 + x + 6 < 0$

   f) $x^3 + 8x^2 + 19x + 12 \geq 0$

   g) $x^4 - x^3 - 11x^2 + 9x + 18 < 0$

7. **Use Technology** Solve each polynomial inequality. Use a CAS or a TI-84+ graphing calculator if available.

a) $2x^3 - 3x^2 - 5x + 6 \geq 0$

b) $3x^3 - x^2 - 6x + 2 < 0$

c) $2x^3 - 5x^2 + 1 \geq 0$

d) $4x^3 - 8x^2 + x + 3 \leq 0$

e) $2x^3 - x^2 - 15x + 18 > 0$

f) $4x^3 + 16x^2 + 9x - 9 \geq 0$

g) $6x^3 + x^2 - 31x + 10 < 0$

8. Solve each polynomial inequality by first finding the approximate zeros of the related polynomial function. Round answers to two decimal places.

a) $x^2 - 6x - 2 > 0$

b) $-2x^2 - 17x + 3 > 0$

c) $x^3 + 4x^2 - 3x - 16 \geq 0$

d) $4x^2 - 12x + 13 < 0$

e) $-x^3 + 5x^2 + 10x + 2 > 0$

f) $x^3 - 2x^2 + 5 > 0$

g) $x^4 - 3x^2 + 2x - 2 < 0$

9. a) Sketch the graph of a cubic polynomial function $y = f(x)$ such that $f(x) > 0$ when $x < -4.5$ and $f(x) \leq 0$ when $x \geq -4.5$.

b) Explain what the above information tells you about the cubic function. Why is it possible to determine more than one function that satisfies this criterion?

10. Solve.

☆a) $-x^3 - 3x - 17 < 0$

b) $x^3 - 7x^2 + 9x + 2 \geq 0$

☆c) $x^3 - x^2 - 16x - 20 > 0$

d) $-4x^3 - 6x^2 + x \leq 0$

☆e) $x^4 + x^3 - 13x^2 - 25x - 12 \leq 0$

f) $5x^4 - 3x - 10 < 0$

11. Holly and Matt play in an Ultimate Disc Golf League. On a windy day, and throwing against the wind, the height, in metres, of the flying disc, t seconds after it leaves Matt's hand, is determined by the function $h(t) = 0.69t^2 - 3.26t^2 + 3.33t + 1.2$. How many seconds after it is thrown must Holly catch the flying disc to ensure that it does not hit the ground?

☆12. A rectangular prism has dimensions $(2x - 1)$, $(x - 3)$, and $(3x - 4)$.

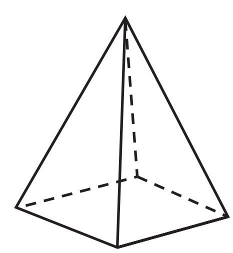
a) What are the restrictions on x?

b) Determine the range of values that x has in order for the volume of the prism to be at least four cubic units.

13. A square-based pyramid has a height that is twice its base length.

a) Determine an equation to represent the volume of the pyramid, in terms of the base length.

b) What are the domain and range of the function if the base must be at least 1 cm^2 and the volume cannot exceed 60 cm^3?



14. The solutions below correspond to inequalities involving a cubic function. For each solution, write two possible cubic polynomial inequalities, one with the less-than symbol (<) and the other with the greater-than symbol (>).

a) $-2.5 < x < 3.5$ or $x > 5$

b) $x < -2\sqrt{2}$ or $-\sqrt{2} < x < \sqrt{2}$

15. Describe the solution to the inequalities as shown in the following graphs. Assume that the x-axis scale = 1, and give estimates of the solutions where appropriate.

a)

b)

c)

16. Create two polynomial inequalities (of different degrees) that would lead to the solutions shown in question 15.

17. a) Describe the solution to an inequality (either > or <) whose polynomial function has

i) a double root

ii) no real roots

iii) complex roots

b) Give an example of each polynomial inequality.

18. Use Technology
Solve each polynomial inequality by first finding the approximate zeros of the related polynomial function. Round answers to two decimal places.

a) $2x^2 - 5x + 1 \geq 0$

b) $2x^3 + x^2 - 3x - 1 < 0$

c) $-4x^3 - 2x + 5 > 0$

d) $x^3 + 2x^2 - 4x - 6 \leq 0$

e) $3x^4 - 5x^2 - 4x + 5 < 0$

19. Solve. Round answers to one decimal place.

a) $3x^3 - 2x^2 - 12x - 12 > 0$

b) $2x^3 + x^2 + 3x - 2 < 0$

c) $-x^3 + 10x - 5 \leq 0$

d) $-2x^4 + 6x^3 - x^2 + 3x - 10 \geq 0$

20. The solutions given below correspond to an inequality involving a quartic function. Write a possible quartic polynomial inequality.
$x < -\frac{5}{2}$ or $\frac{3}{2} < x < \frac{5}{2}$ or $x > 7$

C

☆**21.** Solve the inequality
$-x^8 - 3x^7 + 4x^6 - 8x^2 + 24x - 32 < 0$.

22. Solve $2x^3 - 6x^2 + x - 1 \leq 2x - 3$.

23. A bungee jumper models her descent and subsequent bounces according to the function $h(x) = 2.3x^4 - 14.1x^3 + 29.6x^2 - 24.9x + 9.1$. What adjustments could she make to this model so that she does not hit the rocks or water below? Are there any other considerations that might improve the model?

Chapter 2: Challenge Questions

C1. Is $x + b$ a factor of each of the polynomials given below? Explain.

a) $x^4 + b^4$

b) $x^5 + b^3x^2$

c) $x^9 - 3b^2x^7 + 5bx^8 - b^9$

d) $x^{12} - 10b^4x^8 - 6b^7x^5 + 5b^{10}x^2$

C2. Show that $x^4 + a^4 = (x^2 - \sqrt{2}ax + a^2)(x^2 + \sqrt{2}ax + a^2)$, and find all roots of $x^2 + 16 = 0$.

C3. A rectangle is inscribed inside the semicircle $y = \sqrt{100 - x^2}$. What are the dimensions of the rectangle with maximum area?

C4. Suppose the cubic polynomial $p(x) = ax^3 + bx^2 + cx + d$, where $a, b, c, d \in \mathbb{R}$ and $a \neq 0$, has zeros r_1, r_2, and r_3.

a) Show that $p(x) = a(x - r_1)(x - r_2)(x - r_3)$, $x \in \mathbb{R}$.

b) Show that $r_1 + r_2 + r_3 = -\frac{b}{a}$ and $r_1 r_2 r_3 = -\frac{d}{a}$.

c) What other expression involving the roots can you evaluate?

C5. Find an equation (of greater or equal degree) whose roots are double the roots of each of the following:

a) $x^2 - 4x - 6 = 0$

b) $x^3 - 6x^2 + 9x = 0$

C6. The zeros of $ax^2 + 57x + 14 = 0$ are reciprocals of the roots to $x^2 + bx + 2 = 0$. Find a and b.

C7. Descartes Rule of Signs says that the number of positive real zeros of a polynomial function is the same as the number of changes in sign of the coefficients of the terms, or is less than this by an even number. The number of negative real zeros of $P(x)$ is equal to the number of changes in sign of the coefficient of $P(-x)$, or less than this by an even number.
Use Descartes Rule to determine the number of positive real zeros for each function below. Then, find the real roots.

a) $f(x) = 8x^3 - 6x^2 - 23x + 6$

b) $g(x) = 2x^3 + 3x^2 - 8x + 3$

C8. Solve $\dfrac{2x^2 + 6x - 8}{2x^2 + 5x - 3} < 1$.
What are the restrictions on x?

C9. Solve $-3x + 4 < 5x + 9 \leq 2x - 3$.
Express the solution on a number line.

C10. Solve by considering all cases. Show each solution on a number line.

a) $(x + 2)(x - 3) \geq 0$

b) $(2x + 1)(x - 2) < 0$

C11. Solve using intervals. Show each solution on a number line.

a) $(x + 4)(3x - 5) > 0$

b) $(3x + 2)(x - 1) \leq 0$

Chapter 2: Checklist

By the end of this chapter, I will be able to:

- Apply the Remainder Theorem to determine the remainder when a polynomial is divided by a binomial
- Apply the Factor Theorem to factor polynomials in one variable of degree greater than two
- Determine the equation of a family of polynomial functions that satisfy given conditions
- Solve polynomial equations using a variety of strategies
- Describe the connection between the real roots of a polynomial equation and the x-intercepts of the graph of the corresponding function
- Solve linear and factorable polynomial inequalities and represent the solutions on a number line
- Explain the difference between the solution to a polynomial equation and a polynomial inequality
- Solve polynomial inequalities algebraically and using technology