Chapter 3 Rational Functions

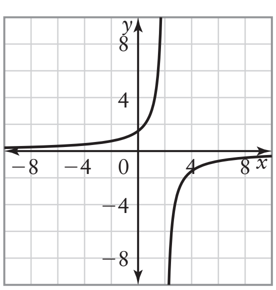
3.1 Reciprocal of a Linear Function

KEY CONCEPTS

Rational functions take the form $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are both polynomial functions and $Q(x) \neq 0$.

Four Critical Properties of a Rational Function

Property	Example $f(x) = -\frac{3}{x-2}$
***X*-intercepts**	Let $f(x) = 0$ and solve for x. $0 = -\frac{3}{x-2}$ gives $0 = -3$, which is not true. $\therefore$ there are no x-intercepts for reciprocal linear functions (where the numerator is a constant).
***Y*-intercepts**	$f(0)$ gives the y-intercept. $f(0) = -\frac{3}{0-2} = \frac{3}{2}$ $\therefore$ the y-intercept is -1.5.
Vertical Asymptotes	Let the denominator be 0 and solve for x. $x - 2 = 0$ $x = 2$ $\therefore$ the graph's vertical asymptote is $x = 2$. Also, note that the domain of the function is $x \neq 2$, $x \in \mathbb{R}$.
Horizontal Asymptotes	When the numerator is a constant, the rational function approaches the x-axis.



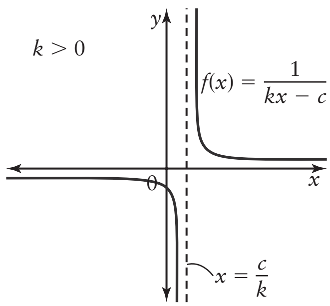
Horizontal Asymptote $y = 0$
No x-intercept

$f(x) = -\frac{3}{x-2}$ — y-intercept (0, 1.5)

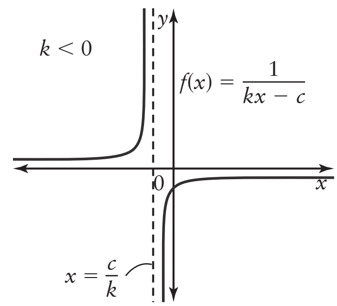
Vertical Asymptote
$x = 2$

The horizontal asymptote of a reciprocal linear function has equation $y = 0$.

If $k > 0$, the left branch of a reciprocal linear function has a negative, decreasing slope, and the right branch has a negative, increasing slope.



If $k < 0$, the left branch of a reciprocal function has a positive, increasing slope, and the right branch has a positive, decreasing slope.



A

1. State an equation representing the vertical and horizontal asymptote of each function.

   a) $f(x) = \dfrac{1}{x - 2}$

   b) $g(x) = \dfrac{3}{x + 7}$

   c) $h(x) = -\dfrac{4}{x - 5}$

   d) $k(x) = \dfrac{2}{9 - x}$

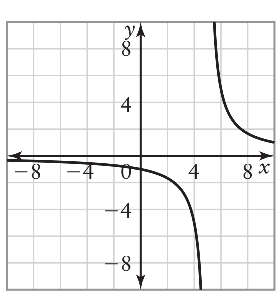
   e) $p(x) = -\dfrac{1}{3x - 4}$

   f) $q(x) = \dfrac{5}{7x + 1}$

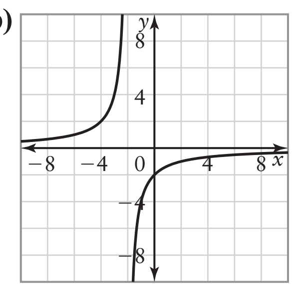
2. Find the y-intercepts of each function in question 1.

3. Determine a possible equation to represent each function.

   a)



   b)



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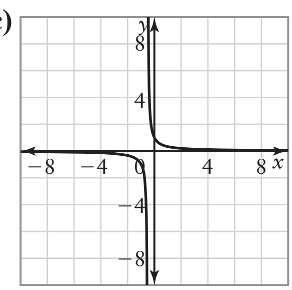
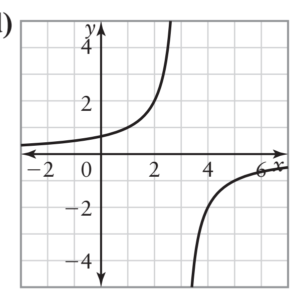


   d)



4. Sketch a graph of each function. State the domain, range, and equations of the asymptotes.

a) $f(x) = \frac{5}{x - 3}$

b) $g(x) = -\frac{1}{x - 4}$

c) $h(x) = -\frac{1}{x + 2}$

d) $k(x) = \frac{3}{x + 1}$

e) $p(x) = -\frac{3}{2x - 1}$

f) $q(x) = \frac{2}{x - 1}$

B

5. Copy and complete the table. Describe the intervals where the slope is increasing and the intervals where it is decreasing in the two branches of the rational function $p(x) = -\frac{3}{2x - 1}$.

Vertical Asymptote	Comparison of Two Pairs of Points to the Left of Vertical Asymptote	Comparison of Two Pairs of Points to the Right of Vertical Asymptote
	Points: Rate of change:	Points: Rate of change:
	Points (closer to asymptote): Rate of change:	Points (closer to asymptote): Rate of change:
Conclusion (Is rate of change increasing or decreasing?)		

☆6. Sketch a graph of each function. Label the y-intercept. State the domain, range, and equations of the asymptotes.

a) $f(x) = \frac{4}{2x - 1}$

b) $g(x) = -\frac{2}{5x + 4}$

c) $h(x) = \frac{1}{3x + 5}$

d) $k(x) = -\frac{3}{1 - 4x}$

☆7. Describe the intervals where the slope is increasing and the intervals where it is decreasing for the functions in question 6.

8. Describe the behaviour of each function in question 6 as x approaches

a) vertical asymptotes

b) $\pm\infty$

☆9. a) Determine an equation of a linear reciprocal function with a vertical asymptote at -4, and a y-intercept at $\frac{1}{2}$.

b) Determine an equation of a linear reciprocal function with a vertical asymptote at $\frac{1}{3}$, and a y-intercept at -6.

10. The Jones family is travelling to the beach. The youngest child, Samantha, determines that the distance from their home is 110 km.

a) How long will the trip take if Mr. Jones averages 75 km/h? 88 km/h? 105 km/h?

b) Write a function that describes the time it takes to make this trip as a function of the speed of the vehicle. Identify the meaning of the variables.

c) Graph the function determined in part b). Show asymptotes.

d) What does this graph tell you about the time it will take to travel, depending on the speed of the car?

11. **a)** Sketch a graph that satisfies each of the following characteristics.

i) as $x \to \frac{1}{2}$, $y \to \infty$

ii) as $x \to -\frac{1}{2}$, $y \to -\infty$

iii) as $x \to \pm\infty$, $y \to 0$

b) Determine a possible equation for the graph in part a).

12. The pressure inside a cylinder is inversely proportional to the volume of the gas inside it. When the volume of gas is 50 cm^3, the pressure is 400 kPa.

a) Write a function to represent the pressure as a function of the volume.

b) Sketch a graph of this function.

c) Calculate the pressure for a volume of 75 cm^3.

d) As the volume increases, what happens to the rate of change of pressure?

13. Investigate a variety of functions of the form $f(x) = \frac{b}{x} + 2$, where $b > 0$.

a) What is the effect on the graph as the value of b is varied?

b) Use the results from your investigation to sketch a graph of each function.

i) $f(x) = \frac{1}{x} + 2$

ii) $f(x) = \frac{3}{x} + 2$

iii) $f(x) = \frac{5}{x} + 2$

14. Analyse the key features (domain, range, vertical asymptotes, and horizontal asymptotes) of $f(x) = \frac{1}{\sin x}$, and then sketch the function.

C

15. Analyse the key features (domain, range, vertical asymptotes, and horizontal asymptotes) of each function, and then sketch the graph of each function.

a) $f(x) = \frac{1}{x + 1} + 2$

b) $g(x) = \frac{2x}{x + 1}$

c) $h(x) = \frac{4}{|3x + 1|}$

16. **Use Technology** Use graphing technology to verify the graphs for the functions in question 15.

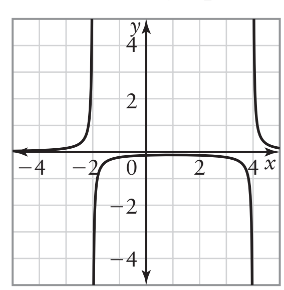
17. Find a positive number such that the sum of the number and its reciprocal is minimized.

18. How is a linear function related to its reciprocal?

19. Describe the similarities and differences of the functions $f(x) = \frac{1}{1 - x}$ and $g(x) = \frac{1}{x - 1}$.

20. Describe the similarities and differences of the functions $f(x) = \frac{1}{x + 2}$ and $g(x) = \frac{1}{(x - 2)(x + 2)}$.

21. Determine a possible equation to represent the following function. Give reasons for your choice. Why do you think there are more parts to this function? Why has the number of asymptotes increased?



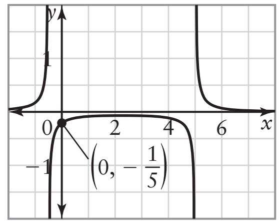
KEY CONCEPTS

- Reciprocal functions can be analysed using key features: asymptotes, intercepts, slope (positive or negative, increasing or decreasing), domain, range, and positive and negative intervals.
- Reciprocals of quadratic functions with two zeros have three parts, with the middle one reaching a maximum or minimum point. This point is equidistant from the two vertical asymptotes.
- The behaviour near asymptotes is similar to that of reciprocal linear functions.
- All of the behaviours listed above can be predicted by analysing the roots of the quadratic relation in the denominator.

Key Features of the Rational Function

Property	Example $f(x) = \dfrac{1}{2x^2 - 9x - 5}$
***X*-intercepts**	Let $f(x) = 0$ and solve for x. $\dfrac{1}{2x^2 - 9x - 5} = 0$ $1 = 0(2x^2 - 9x - 5)$ $1 \neq 0$ $\therefore$ there are no x-intercepts. For a rational function, only the numerator determines x-intercepts.
***Y*-intercepts**	The y-intercept is determined by setting $x = 0$. Then, $f(0) = \dfrac{1}{2(0)^2 - 9(0) - 5} = -\dfrac{1}{5}$. $\therefore$ the y-intercept is $-\dfrac{1}{5}$.
Vertical Asymptotes	Let the denominator be 0 and solve for x. Factor the denominator. $(2x + 1)(x - 5) = 0$ $x = -\dfrac{1}{2}, x = 5$ $\therefore$ the graph's vertical asymptotes are $x = -\dfrac{1}{2}, 5$. Also, note that the domain of the function is $x \neq -\dfrac{1}{2}, 5, x \in \mathbb{R}$.
Horizontal Asymptotes	If the degree of the numerator is less than the degree of the denominator, then the horizontal asymptote is the x-axis, $y = 0$.

Further Analysis of the Rational Function



The vertical asymptotes are $x = -\frac{1}{2}$ and $x = 5$.

The maximum of the parabolic branch of the function (the middle section) is determined by finding the midpoint of the two vertical asymptotes. Then, the local maximum occurs at $x = 2\frac{1}{4}$.

The behaviour of the graph can be summarized by the table below.

Interval	$x < -\frac{1}{2}$	$-\frac{1}{2} < x < 2\frac{1}{4}$	$x = 2\frac{1}{4}$	$2\frac{1}{4} < x < 5$	$x > 5$
Sign of $f(x)$	+	−	−	−	+
Sign of Slope	+	+	0	−	−
Change in Slope	+	−		−	+

The graph

- is positive and increasing with an increasing rate of change for $x < -\frac{1}{2}$
- is negative and increasing with a decreasing rate of change for $-\frac{1}{2} < x < 2\frac{1}{4}$
- is negative and decreasing with an increasing rate of change for $2\frac{1}{4} < x < 5$
- is positive and decreasing with a decreasing rate of change for $x > 5$
- reaches a local maximum on the interval $-\frac{1}{2} < x < 5$ at $x = 2\frac{1}{4}$

A

1. Determine the equations of the vertical asymptotes for each function. Then, state the domain.

a) $f(x) = \dfrac{1}{(x-3)(x+4)}$

b) $g(x) = -\dfrac{2}{(x+3)^2}$

c) $h(x) = \dfrac{1}{x^2 + 8x + 12}$

d) $k(x) = -\dfrac{4}{x^2 - 9}$

2. Copy and complete the tables to describe the behaviour of the function as x approaches each key value.

a) $f(x) = \dfrac{1}{(x-1)(x+3)}$

$x \to$	$f(x) \to$
1^-	
1^+	
-3^-	
-3^+	
$-\infty$	
$+\infty$	

b) $g(x) = \dfrac{1}{(2x+1)(x-2)}$

$x \to$	$f(x) \to$
$-\frac{1}{2}^-$	
$-\frac{1}{2}^+$	
2^-	
2^+	
$-\infty$	
$+\infty$	

3. For each rational function, determine

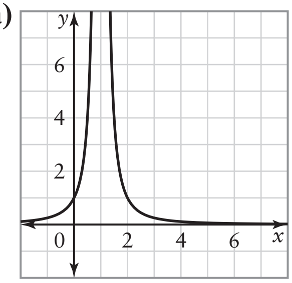
i) y-intercepts

ii) x-intercepts

iii) horizontal asymptotes

iv) vertical asymptotes

a)



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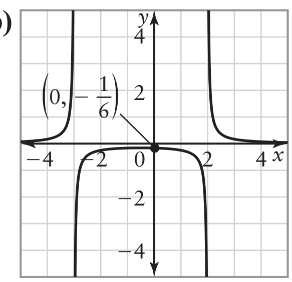

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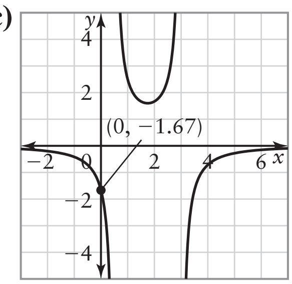


B

4. Determine a possible equation for each function in question 3.

5. Make a summary table with the headings shown for each graph in question 3.

Intervals	
Sign of Function	
Sign of Slope	
Change in Slope	

☆**6.** For each of the following functions

i) state equations for the vertical and horizontal asymptotes

ii) determine the x-intercepts and y-intercepts

iii) describe the increasing and decreasing intervals

iv) sketch a graph of the function

v) state the domain and range

a) $f(x) = \dfrac{1}{x^2 - 4x - 21}$

b) $g(x) = -\dfrac{1}{x^2 - 4x + 4}$

c) $h(x) = \dfrac{4}{x^2 - 25}$

7. Write an equation for a function that is the reciprocal of a quadratic and that has the following properties:

i) horizontal asymptote $y = 0$

ii) vertical asymptotes $x = -2$ and $x = 7$

iii) $y > 0$ on the intervals $(-\infty, -2)$ and $(7, +\infty)$

☆**8. a)** For the function below, determine

i) horizontal asymptotes

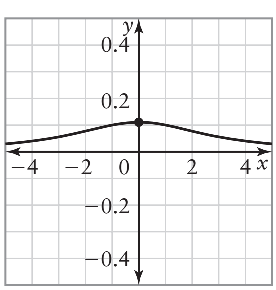
ii) x-intercepts and y-intercepts

iii) domain and range

iv) intervals of increase and decrease

b) Explain why there are no vertical asymptotes.

c) Determine a possible equation for the graph.



☆**9.** The function $g(x) = \dfrac{1}{f(x)}$ describes the quadratic reciprocal graphed in question 8.

a) Determine the corresponding quadratic function $f(x)$ and sketch its graph.

b) Compare the properties of the two graphs.

10. a) Sketch a function that satisfies each of the following:

i) as $x \to 4^+$, $y \to \infty$

ii) as $x \to 4^-$, $y \to -\infty$

iii) as $x \to -4^-$, $y \to +\infty$

iv) as $x \to -4^+$, $y \to -\infty$

v) as $x \to \pm\infty$, $y \to 0$

b) Determine a possible equation for the function described in part a).

C

11. As blood moves from the heart through the major arteries out to the capillaries and back through the veins, the systolic pressure continuously drops. The pressure is given by the function $P(t) = \dfrac{25t^2 + 125}{t^2 + 1}$, where P is measured in millimetres of mercury and t is measured in seconds.

a) Determine the domain and range of the function.

b) Where are the horizontal and vertical asymptotes?

c) Sketch the graph of the function.

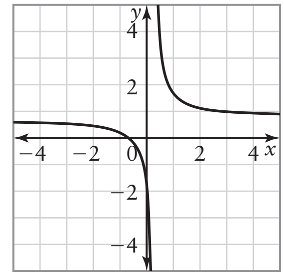
d) Determine the rate of change of the blood pressure at each second for $0 < t < 10$. What does this tell you?

KEY CONCEPTS

Four Critical Properties of a Rational Function

Property	**Example $f(x) = \frac{3x + 2}{4x - 1}$**
***X*-intercepts**	Let $f(x) = 0$ and solve for x. $\frac{3x + 2}{4x - 1} = 0$ gives $3x + 2 = 0$. $\therefore x = -\frac{2}{3}$. For a rational function, only the numerator determines x-intercepts.
***Y*-intercepts**	The y-intercept is determined by setting $x = 0$. Then, $f(0) = \frac{3(0) + 2}{4(0) - 1} = -2$. $\therefore$ the y-intercept is -2.
Vertical Asymptotes	Let the denominator be 0 and solve for x. $4x - 1 = 0$ $x = \frac{1}{4}$ $\therefore$ the graph's vertical asymptote is $x = \frac{1}{4}$. Also, note that the domain of the function is $x \neq \frac{1}{4}$, $x \in \mathbb{R}$.
Horizontal Asymptotes	If the degree of the numerator is equal to the degree of the denominator, then the horizontal asymptote is $\frac{a}{c}$ (a and c are the leading coefficients of the numerator and denominator of the rational expression). Since the degrees are equal, the horizontal asymptote is $y = \frac{3}{4}$.

$f(x) = \dfrac{3x + 2}{4x - 1}$



For a rational function $f(x) = \dfrac{ax + b}{cx + d}$,

- the y-intercept is $\dfrac{b}{d}$
- the x-intercept is $-\dfrac{b}{a}$
- the vertical asymptote is $-\dfrac{d}{c}$
- the horizontal asymptote is $\frac{a}{c}$, where a and c are the leading coefficients of the binomials in the numerator and denominator, and the degree of each is the same

A rational function of the form $f(x) = \dfrac{ax + b}{cx + d}$ has the following key features:

- The equation of the vertical asymptote can be found by setting the denominator equal to zero and solving for x, provided the numerator does not have the same zero.
- The equation of the horizontal asymptote can be found by dividing each term in both the numerator and the denominator by x, and by investigating the behaviour of the function as $x \to \pm\infty$.
- The coefficient b acts to stretch the curve, but it has no effect on the asymptotes, domain, or range.
- The coefficient d shifts the vertical asymptote.
- The two branches of the graph of the function are equidistant from the point of intersection of the vertical and horizontal asymptotes.

A

1. Determine the equation of the vertical asymptote for each function.

a) $f(x) = \dfrac{3x}{x - 7}$

b) $g(x) = \dfrac{5x}{6 - 9x}$

c) $h(x) = \dfrac{3x - 1}{x + 4}$

d) $k(x) = -\dfrac{3x - 7}{x + 11}$

2. What is the domain of each function in question 1?

3. Determine the equation of the horizontal asymptote for each function.

a) $f(x) = \dfrac{x}{x + 5}$

b) $g(x) = \dfrac{5x}{6 - x}$

c) $h(x) = -\dfrac{x + 3}{x - 3}$

d) $k(x) = -\dfrac{3x - 2}{6 - 4x}$

4. What is the range of each function in question 3?

B

5. Sketch the graph of each function, and then summarize the increasing and decreasing intervals.

a) $f(x) = \frac{3x}{x - 7}$

b) $g(x) = -\frac{x}{5x - 1}$

c) $h(x) = \frac{2x - 1}{x + 2}$

d) $k(x) = -\frac{2 - 3x}{x - 5}$

6. For each rational function, determine

i) x-intercepts

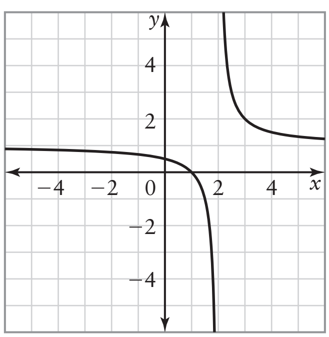
ii) y-intercepts

iii) horizontal asymptotes

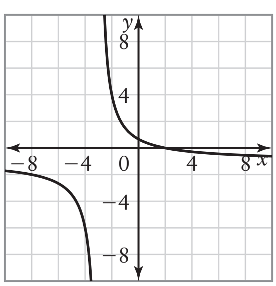
iv) vertical asymptotes

v) domain and range

a)



b)



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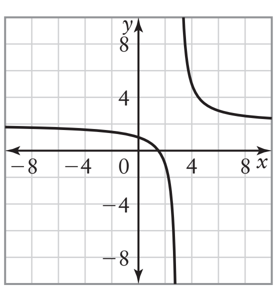


7. Determine an equation for each graph in question 6.

8. Determine an equation for the horizontal asymptote of each function. Then, determine an equation for the vertical asymptote and graph the function. State the domain and range. Summarize the key features of each function.

a) $f(x) = \frac{x}{x + 3}$

b) $g(x) = \frac{4x}{x + 1}$

c) $h(x) = -\frac{x - 1}{x + 6}$

d) $k(x) = \frac{5x + 3}{2x - 1}$

☆**9.** Write an equation of a rational function that has a graph with all of the indicated properties.

i) x-intercept at 5

ii) y-intercept at $-\frac{5}{8}$

iii) vertical asymptote has equation $x = -\frac{8}{3}$

iv) horizontal asymptote has equation $y = \frac{1}{3}$

☆**10. a)** Use long division to rewrite the function $f(x) = \frac{6x - 3}{2x + 1}$ as the sum of a constant and a rational function.

b) Explain how this method could be used to graph rational functions.

c) Use this method to sketch a graph of $f(x)$.

11. Use your method from question 10 to graph each function.

a) $f(x) = \frac{55x - 27}{5x + 14}$

b) $g(x) = -\frac{27x - 9}{9x + 4}$

12. Troy and Annie are graphing functions at the chalkboard. Troy is graphing $f(x) = \frac{7x - 2}{x}$ and Annie is graphing $g(x) = 7 - \frac{2}{x}$.

a) Determine the domain of each of the functions.

b) Identify the horizontal and vertical asymptotes for each of the functions.

c) Use Technology Sketch the graphs of each function and check using a graphing calculator.

d) Explain why these functions produce the same graph, and describe how to determine the important key features of each equation.

☆**13.** Consider the function $y = \frac{3x}{x + 6}$.

a) Determine an equation for the vertical asymptote.

b) State the domain.

c) Determine an equation for the horizontal asymptote.

d) State the range.

e) Sketch the graph of the function.

f) Summarize the increasing and decreasing intervals.

g) Compare the slopes of the tangents at the points where

i) $x = -5$ and $x = 15$

ii) $x = -7$ and $x = -15$

C

14. The average cost per year in electricity for a widescreen television is approximately $18.

a) Assume that a new television costs $1299. Determine the total annual cost for a television that lasts for 10 years. (Assume the only costs associated with the television are its purchase cost and electricity.)

b) Develop a function that gives the annual cost of the television as a function of the number of years that the television is owned.

c) Sketch a graph of the function. What is an appropriate window?

d) Determine the asymptotes of this function.

e) Explain the meaning of the horizontal asymptote in terms of the television.

f) If a company offers a television that costs $1600, but says that it will last at least 20 years, is the television worth the difference in cost?

g) Describe the rate of change of the cost as the number of years increases.

15. A beach must be roped off to cover a rectangular area of 500 m^2. Since the side at the water's edge must be accessible, only three sides of the beach should be enclosed.

a) Determine an appropriate equation to determine the minimum amount of rope in terms of the width of the enclosed area.

b) Graph the function. What is the domain?

c) What dimensions require the least amount of rope?

☆**16.** Describe how the graph of $y = \frac{3}{x - 5} - 8$ can be obtained from the graph of $y = \frac{1}{x}$.

17. Use Technology The concentration of a drug in the bloodstream is given by the equation $C(t) = \frac{5t}{0.01t^2 + 3.3}$, where t is the time, in minutes, and C is the concentration, in micrograms per millilitre.

a) Graph the function using technology.

b) Determine the maximum concentration and when it will occur.

c) Determine the effect of changing the values of the coefficients in the equation.

KEY CONCEPTS

- To solve rational equations algebraically, start by factoring the expressions in the numerator and denominator to find asymptotes and restrictions.
- Next, multiply both sides by the factored denominators, and simplify to obtain a polynomial equation. Then, solve using techniques from Chapter 2.
- For rational inequalities, note the following:
  - It can often help to rewrite with the right side equal to 0. Then, use test points to determine the sign of the expression in each interval.
  - If there is a restriction on the variable, you may have to consider more than one case. For example, if $\frac{a}{x - k} < b$, case 1 is $x > k$ and case 2 is $x < k$.

Example

Solve $\frac{x^2 - 3x - 4}{x^2 + 11x + 30} \geq 0$.

Factor the rational expression: $\frac{(x - 4)(x + 1)}{(x + 5)(x + 6)} \geq 0$

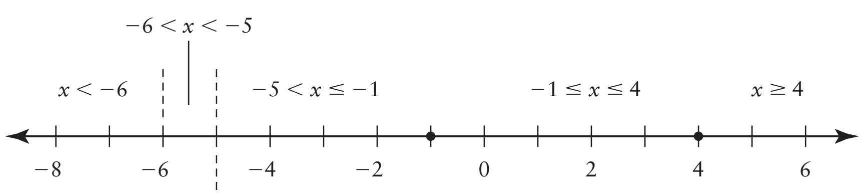
Solution

The x-intercepts are the factors of the numerator. Therefore, $x = 4, -1$.

The restrictions occur at $x = -5, -6$.

The function may change sign at the x-intercepts or at the vertical asymptotes. Thus, these values will be used to determine the intervals of the function.

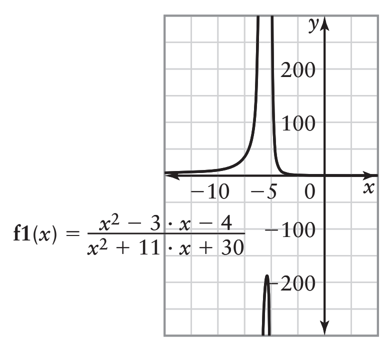
The number line illustrates the intervals to test in the function.



Interval	Test Value for x	Sign of Factors of $\frac{(x-4)(x+1)}{(x+5)(x+6)}$	Sign of Function $\frac{(x-4)(x+1)}{(x+5)(x+6)}$
$x < -6$	$x = -7$	$\frac{(-)(-)}{(-)(-)}$	+
$-6 < x < -5$	$x = -5.5$	$\frac{(-)(-)}{(-)(+)}$	−
$-5 < x < -1$	$x = -3$	$\frac{(-)(-)}{(+)(+)}$	+
-1	$x = -1$	$\frac{(-)(0)}{(+)(+)}$	0
$-1 < x < 4$	$x = 0$	$\frac{(-)(+)}{(+)(+)}$	−
4	$x = 4$	$\frac{(0)(+)}{(+)(+)}$	0
$x > 4$	$x = 5$	$\frac{(+)(+)}{(+)(+)}$	+

The inequality is satisfied over those intervals where the sign of the function is positive (+) or zero (0).

For the inequality $\frac{x^2 - 3x - 4}{x^2 + 11x + 30} \geq 0$, the solution is $x < -6$, $-5 < x \leq -1$, or $x \geq 4$.



A

1. Determine the x-intercepts of each function.

a) $f(x) = \dfrac{-3x}{2x + 5}$

b) $g(x) = \dfrac{6x - 5}{x^2 - 6x + 4}$

c) $h(x) = \dfrac{x - 5}{x^3 + 7x^2 + 2x}$

d) $k(x) = \dfrac{x^2 - 3x - 18}{x^2 - x - 6}$

2. Solve algebraically. Check each solution.

a) $\dfrac{x - 3}{10} = 4x$

b) $\dfrac{3}{x} - 2 = \dfrac{5}{x}$

c) $\dfrac{3}{x + 2} - \dfrac{1}{x} = \dfrac{1}{5x}$

d) $\dfrac{10}{x + 4} = \dfrac{15}{4(x + 1)}$

e) $\dfrac{1}{x} = \dfrac{x + 3}{2x^2}$

f) $\dfrac{x - 2}{x} = \dfrac{x - 4}{x - 6}$

g) $\dfrac{x}{x - 2} + \dfrac{1}{x + 4} = \dfrac{2}{x^2 - 6x + 8}$

3. Use Technology Use graphing technology to solve each equation. Express answers to two decimal places.

a) $\dfrac{1 + 3x}{x} = \dfrac{x^2}{x - 2}$

b) $\dfrac{x^2 - 3x + 2}{x - 3} = \dfrac{x}{x + 2}$

c) $\dfrac{5x - 1}{1 + 2x} = \dfrac{x + 4}{x - 3}$

d) $\dfrac{3x - 1}{x^2 - 2} = \dfrac{4x^2 - 3}{x}$

4. Solve each inequality without using technology. Illustrate the solution on a number line.

a) $\dfrac{3x + 1}{2x - 4} > 0$

b) $\dfrac{x - 3}{x + 3} \leq 5$

c) $\dfrac{2x - 1}{5x + 3} \geq 0$

d) $\dfrac{1}{x - 1} < \dfrac{-1}{x + 2}$

e) $\dfrac{x - 8}{x} \leq 3 - x$

f) $\dfrac{2}{x + 3} \geq \dfrac{1}{x - 1}$

g) $\dfrac{x^2 - x - 6}{x - 3} > 1$

h) $\dfrac{x^2 - x - 2}{x^2 + x - 2} > 3$

5. Use Technology Solve each inequality using intervals. Check using technology.

a) $\dfrac{(2x - 3)(x + 4)}{(x - 5)(x - 1)} \leq 0$

b) $\dfrac{(x - 2)(x - 1)}{(x - 3)(x - 4)^2} < 0$

c) $\dfrac{x^2 - 16}{x^2 - 4x - 5} \leq 0$

d) $\dfrac{x^2 - 8x - 48}{x^2 + 6x} > 0$

e) $\dfrac{x^2 + 58x - 120}{x^2 - 12x - 28} \geq 0$

f) $\dfrac{2x^2 + x - 1}{x^2 - 4x - 4} \leq 0$

B

6. Write a rational equation that cannot have $x = -2$ or $x = \frac{1}{2}$ as a solution. Explain your reasoning.

7. Determine the intervals where $\dfrac{2}{x + 1} > x$.

8. Solve and check.

a) $\frac{12}{x} + x - 8 = 0$

b) $\frac{1}{3x} + \frac{6x - 9}{3x} = \frac{3x - 3}{4x}$

c) $\frac{10}{x^2 - 1} + \frac{2x - 5}{x - 1} = \frac{2x + 5}{x + 1}$

d) $\frac{-4}{x - 1} = \frac{7}{2 - x} + \frac{3}{x + 1}$

e) $1 = \frac{1}{1 - x} + \frac{x}{x - 1}$

f) $\frac{7x}{3x + 3} - \frac{5}{4x - 4} = \frac{3x}{2x + 2}$

☆**9.** Solve.

a) $\frac{2}{3x} + \frac{5}{6x} > \frac{3}{4}$

b) $5 + \frac{1}{x} > \frac{16}{x}$

c) $1 + \frac{5}{x - 1} \le \frac{7}{6}$

d) $\frac{1}{2x + 1} + \frac{1}{x + 1} > \frac{8}{15}$

e) $\frac{x^2 + 3x + 2}{x^2 - 9} \le 0$

f) $\frac{(-2x - 10)(3 - x)}{(x^2 + 5)(x - 2)^2} < 0$

☆**10.** Prove that $\frac{x^2 - 2x + 5}{x^2 - 4x + 4} < 0$ has no real solutions.

11. Compare the solutions $\frac{x^2 - 6x + 9}{x - 4} \le \frac{2x - 1}{x + 3}$ and $\frac{x^2 - 6x + 9}{x - 4} \ge \frac{2x - 1}{x + 3}$.

12. Rachael runs 2 km to her bus stop, and then rides 4.5 km to school. On average, the bus is 45 km/h faster than Rachael's average running speed. If the entire trip takes 25 min, how fast does Rachael run?

☆**13.** The function $T(d) = \frac{518}{d - 10}$ gives the maximum time a diver can remain underwater and still surface at a steady rate with no decompression stops when the diver's depth is greater than 10 m. Assume that $T(d)$ represents time in minutes and d represents the diver's depth in metres. Determine the maximum depth the diver can go if the diver wants to dive for 60 min.

14. The ratio of $x + 3$ to $x - 5$ is greater than 40%. Solve for x.

☆**15.** Solve $x + \frac{x^2 - 5}{x^2 - 1} = \frac{x^2 + x + 2}{x + 1}$.

16. Explain why $x + \frac{3}{x - 3} = 3 + \frac{3}{x - 3}$ has no solution.

17. Jordan has a sister who is three years older than he is, and a brother who is two years younger than he is. How old must Jordan be in order that the ratio of his sister's age to his brother's age is less than 2?

C

18. Determine all real solutions algebraically.

a) $\frac{2x^5 - 10x + 5}{x^3 + x^2 - 12x} = 0$

b) $\frac{x^3 - 4x + 1}{x^2 + x - 6} \ge 0$

19. Find two rational expressions that have a sum of $\frac{9 - 9x}{x^2 - 9}$.

20. If $\frac{3x}{5y} = 11$, find the value of $\frac{3x - 5y}{5y}$.

KEY CONCEPTS

Special Cases of Rational Functions

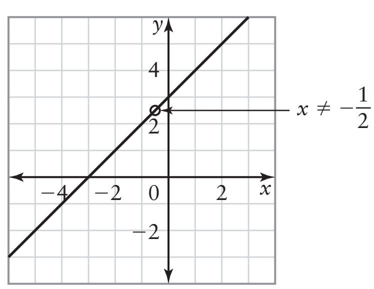
- When solving a problem, it is important to read carefully to determine whether a function is being analysed or an equation or inequality is to be solved.
- A full analysis will involve four components:
  - numeric (tables, ordered pairs, calculations)
  - algebraic (formulas, solving equations)
  - graphical
  - verbal (descriptions)
- When investigating special cases of functions, factor and reduce where possible. Indicate the restrictions on the variables in order to identify hidden discontinuities.
- When investigating new types of rational functions, consider what is different about the coefficients and the degree of the polynomials in the numerator and denominator. These differences could affect the stretch factor of the curve and the equation of the asymptotes, and they could cause other discontinuities.

Special Case: Discontinuities

A rational function such as $f(x) = \frac{2x^2 + 7x + 3}{2x + 1}$ appears to be linear because when it is factored and simplified, the denominator becomes 1.

$$f(x) = \frac{2x^2 + 7x + 3}{2x + 1} = \frac{\cancel{(2x + 1)}(x + 3)}{\cancel{(2x + 1)}} = x + 3$$

The restriction $x \neq -\frac{1}{2}$ leads to a "hole" or discontinuity in the graph.

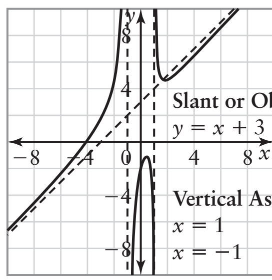


Special Case: Oblique Asymptotes

$y = \dfrac{x^3 + 3x^2 - 4x + 2}{x^2 - 1}$

The graph of the function has vertical asymptotes at $x = \pm 1$. It also has a "slant asymptote" which is found by long division.

Factoring by long division, $y = \dfrac{x^3 + 3x^2 - 4x + 2}{x^2 - 1} = x + 3 + \dfrac{-3x + 5}{x^2 - 1}$. The slant or oblique asymptote is $y = x + 3$.



A

☆**1.** For an Ideal Gas, pressure, P, in kilopascals (kPa); volume, V, in litres (L); and temperature, T, in degrees Kelvin, are related by the equation $PV = 8.314nT$, where n is the number of moles of gas.

To convert Celsius to Kelvin, add 273.

a) A 50-L cylinder is filled with argon gas to a pressure of 10 130.0 kPa at 30°C. How many moles of argon gas are in the cylinder?

b) What volume is needed to store 0.050 mol of helium gas at 202.6 kPa and 400K?

c) Sketch a graph of pressure versus volume for 1 mol of a gas assuming that the temperature is fixed.

d) What are the asymptotes of the graph?

e) What will happen to the pressure if the temperature is held fixed and the volume the gas occupies increases?

2. Sketch a graph of each function. Describe each special case.

a) $f(x) = \dfrac{-2x}{x(x + 3)}$

b) $g(x) = \dfrac{x^2 - 1}{x^2 - 7x - 8}$

c) $h(x) = \dfrac{x^2 - 6x}{x^2 - 8x + 12}$

d) $k(x) = \dfrac{x^2 - 9x + 20}{4x^2 - 21x + 5}$

e) $m(x) = \dfrac{x^2 + 2x - 8}{x + 4}$

f) $p(x) = \dfrac{x^2}{x^4 + 3x^2}$

g) $q(x) = \dfrac{x^3 + x^2 - 2}{4x^3 - 7x + 3}$

B

☆**3.** According to the Law of Universal Gravitation, the attractive force (F) between two bodies is proportional to the product of their masses (m_1 and m_2), and inversely proportional to the square of the distance (r) between them. This force is represented by the equation $F(r) = \frac{Gm_1m_2}{r^2}$, where G is the gravitational constant 6.67×10^{-11}N.

a) Determine the force of gravitational attraction between Earth ($m = 5.98 \times 10^{24}$ kg) and an 80-kg student if the student is in an airplane 12 000 m above Earth's surface, a total distance of 6.39×10^6 m from Earth's centre.

b) Determine the force of gravitational attraction between two students standing 1 m apart if each student weighs 85 kg.

c) Sketch a graph of F if G, m_1, and m_2 are held constant. What are the restrictions on the function?

d) Describe what happens to F as the distance between the objects increases.

e) Compare your answers from parts a) and b). What can you conclude about the force of gravity in smaller objects?

4. A delivery company models its cost, C, in dollars per kilometre for a trip with the equation $C(x) = \frac{500}{4.50 + x}$, where x is the number of kilometres over its minimum trip of 4.5 km. Describe the change in the cost model represented by each of the following, and draw a graph for each.

a) $C(x) = \frac{700}{4.50 + x}$

b) $C(x) = \frac{500}{3.50 + x}$

c) $C(x) = \frac{700}{5.50 + x}$

☆**5.** Consider the function $y = \frac{x^3 - 2x^2 - 5x + 6}{x^2 + 3x + 2}$.

a) Determine the vertical asymptotes.

b) Determine the x-intercepts and y-intercepts.

c) Does the function have any additional discontinuities? If so, explain.

d) Does the function have a horizontal or oblique asymptote? Explain how you determined this.

e) Graph the function.

6. Give a complete analysis of each function by determining key properties (see question 5). Graph each function.

a) $f(x) = \frac{4x^2 - 5}{2x^3 - 3x^2 + x}$

b) $g(x) = \frac{x^3 - 4x^2 + 6x + 5}{x - 2}$

c) $h(x) = \frac{x^3 - 1}{x^2 - 4}$

d) $k(x) = \frac{4x^2 + 4x - 3}{2x - 5}$

e) $m(x) = \frac{x - 4}{x^3 + 2x^2 - 23x - 60}$

f) $n(x) = \frac{2x^4 + 7x^3 + 7x^2 + 2x}{x^3 - x + 50}$

7. The kinetic energy of a moving particle is defined as the total work it can do in kilograms until it stops. For a particle at rest with mass m, the function $K = \frac{1}{2}mv^2$ finds in Joules the kinetic energy of the particle, or the work required to reach a velocity, v, in metres per second.

a) Determine an equation for mass of the particle based on its velocity and kinetic energy.

b) Sketch a graph of the function determined in part a) if the particle's kinetic energy is held constant.

c) Determine the mass (in kilograms) of a particle moving at 6.3 m/s with kinetic energy of 30 joules (J).

8. Creating a chemical solution requires precise measurements and mixing. For example, a chemist may have 35 L of a 12-mol hydrochloric acid (HCl) solution. The solution is diluted with 0.5-mol HCl solution in order to decrease the concentration. The concentration of the mixture can be modelled by the function $C(x) = \frac{420 + 0.5x}{35 + x}$, where x is the number of litres of 0.5-mol solution added.

a) Determine the vertical and horizontal asymptotes of $C(x)$. Explain their meaning in the context of the problem.

b) Write the function $C(x)$ as a transformation of the graph of $\frac{1}{x}$.

c) Write a function that models the concentration of a mixture that will dilute 50 L of a 12-mol solution by adding a 3-mol solution to it.

d) How many litres of the 3-mol solution must be added to the mixture described in part c) to create an 8-mol solution?

9. The volume of a rectangular prism with a square base is fixed at 500 cm^3.

a) Determine the surface area of the prism as a function of the length of the side of the square.

b) Graph the surface area function.

c) Describe how the surface area changes as the side length approaches zero.

d) If the surface area needs to be at least 50 cm^2, what does the side length need to be?

10. The current in an electric circuit is given by the formula $I = \frac{t + 2}{15 - t}$, where t is the time, in seconds.

a) Describe what happens to the circuit as t approaches 15 s.

b) Determine any discontinuities in the graph.

☆**11.** When Rosalyn empties her pool for the winter, she knows that the time required to empty it varies inversely as the rate, r, of pumping.

a) Write an equation that represents this situation where k is the constant.

b) Last fall, Rosalyn emptied her pool in 45 min at a rate of 1000 L/min. She now owns a new pump that can empty the pool at a rate of 900 L/min. How long will it take to empty the pool using this new pump?

☆**12.** Use long division to prove that the function $f(x) = \frac{x^3 - 2x^2 - 5x + 10}{x + 2}$ has a parabolic asymptote.

C

13. Graph $f(x) = \frac{1}{x^2}$ after it has been translated 2 units to the right and down 5 units. What are its asymptotes? What is the equation of the transformed graph?

14. Determine the inverse of the function $f(x) = \frac{2}{x} + 3$. Is the inverse a function? Describe the similarities and differences.

15. Recall the Universal Gravitation law in question 3. Suppose that two objects attract each other with a gravitational force of 16 N.

a) If the distance between the two objects is doubled, what is the new force of attraction between the two objects?

b) If the mass of both objects is doubled, and if the distance between the objects remains the same, what would be the new force of attraction between the two objects?

C1. A car travels at a constant speed and burns $g(x)$ litres of gas per kilometre, where x is the speed of the car in kilometres per hour and $g(x) = \frac{1280 + x^2}{320x}$.

a) If fuel costs \$1.29 per litre, find the cost function $C(x)$ that expresses the cost of the fuel for a 200-km trip as a function of the speed.

b) What driving speed will make the cost of fuel equal to \$300?

c) What driving speed will minimize the cost of fuel for the trip?

C2. a) Find a rational function with domain $x \geq 0$ that has the same graph as $f(x) = \frac{x - 3}{|x| - 2}$.

b) Find a rational function with domain $x < 0$ that has the same graph as $f(x) = \frac{x - 3}{|x| - 2}$.

c) Explain why $f(x) = \frac{x - 3}{|x| - 2}$ has two vertical asymptotes. Confirm your answer by graphing the function.

C3. A cylindrical can with volume 60 cm^3 is to be designed to minimize the amount of material on the outside of the can. The can must be at least 1 cm high and at least 4 cm in diameter. Determine the optimal dimensions of the can.

C4. Ohm's Law states that the potential difference (voltage) across an ideal conductor is proportional to the current through it. The constant of proportionality is called the "resistance," R. The internal resistance of the circuit can be measured by the formula $E = IR$, where I is the current, in amperes (A); $R = x + 6.1$ ohms; x is the ohms of the resistor; and 6.1 is the internal resistance of the circuit.

a) Find I if $E = 1.6$ volts.

b) What is the domain of the function you found in part a)?

c) What is the maximum value of I?

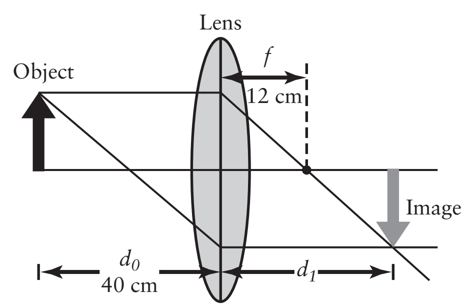
d) Graph the function in part a). Is there a vertical asymptote?

e) What happens to I when the resistance is very large?

C5. The lens equation is $\frac{1}{f} = \frac{1}{d_i} + \frac{1}{d_0}$, where f is the focal length, d_i is the distance between the lens and the image, and d_0 is the distance between the lens and the object.

a) Write a rational equation to find the distance from the lens to the image.

b) If an object is 40 cm from the lens and the focal length is 12 cm, find the distance, d_i.



C6. Determine an equation for a rational function with a horizontal asymptote at 2, vertical asymptotes at ± 3, and y-intercept at –4.

C7. Wind chill describes the additional cooling that occurs when skin is exposed to wind. The equation is $W(v) = -0.17v^3 + 18.4v^2 - 584v - 239$ for $0 \leq v \leq 45$, where v is the wind speed in metres per second. The units for W are megajoules per square metres per hour. Determine the speed of the wind that gives a minimum wind-chill factor.

C8. A manufacturer sells x items per week at a price of $p(x) = 250 - 0.01x$ cents per item. It costs $C(x) = 60x + 10\ 000$ cents to produce the x items. How many items should be produced to maximize profit? What range of items should be produced to ensure the manufacturer breaks even?

C9. Henry starts to walk south at a speed of 1.5 m/s, while Marion starts at the same point and walks east at a speed of 2 m/s. At what rate is the distance between Henry and Marion increasing 1 min later?

C10. Electric potential energy between two point charges with magnitude q_1 and q_2, in coulombs, is calculated according to the equation $E_e = \frac{kq_1q_2}{r}$, where k is known as Coulomb's constant, $9 \times 10^9\ N \cdot m^2 / kg^2$, and r is the distance in metres between the charges.

a) What will happen to the graph of electric potential energy versus charge separation when the sign of one of the charges changes?

b) In what instances will electric potential energy be maximized?

C11. The frequency, N, of the sound heard by an observer can be measured by the Doppler Effect. N is given by function $N(u) = \frac{nv}{u + v}$, where n is the source of sound frequency moving away from an observer with a speed of u, and v is the speed of sound in the air. At 20°C, the speed of sound in air is 343.7 m/s. Determine the rate of change at which the observer loses the sound when its source is 50 Hz and it is moving away at a speed of 35 m/s.

C12. The cost, C, in thousands of dollars, of running a car wash can be modelled by the function $C(n) = \frac{3}{45 - n}$, where n is the average daily number of vehicles that use the car wash. The rate of change of the cost, in thousands of dollars, is given by $R(n) = \frac{3}{(45 - n)^2}$.

a) Sketch a graph of each function.

b) Determine the domain and range of each function and explain their meaning.

c) Calculate the rate of change at $n = 25$ by finding the slope of the tangent to the function $C(n)$.

d) Compare the rate of change from part c) to the rate of change function $R(n)$ when $n = 25$.

C13. The quotient of the cube of a number n and a number that is 4 less than the square of n can be modelled by the rational function $f(n) = \frac{n^3}{n^2 - 4}$.

a) Find the equation of the oblique asymptote to the graph of $y = f(n)$.

b) Determine vertical asymptotes, and x- and y-intercepts of $y = f(n)$.

c) Determine an approximate minimum value of the quotient when $n > 2$.

d) Determine an approximate maximum value of the quotient when $n < 2$.

C14. Given the function $g(x) = \frac{-5x}{x + 2}$, determine the coordinates of all points on the function where the slope of the tangent equals the slope of a secant line that passes through the points A $(2, -2.5)$ and B $(-1, 5)$.

Chapter 3: Checklist

By the end of this chapter, I will be able to:

- Determine the equations of the vertical and horizontal asymptotes of simple rational functions
- Determine the x- and y-intercepts of simple rational functions
- Determine the domain and range of simple rational functions
- Determine the rate of change at selected points of simple rational functions
- Sketch and label graphs of simple rational functions
- Solve rational equations using the properties of polynomial equations
- Solve simple rational inequalities using the properties of polynomial equations
- Determine and graph special cases of simple rational functions through investigation
- Solve contextual problems involving simple rational functions and equations