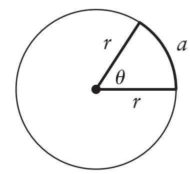
# Chapter 4 Trigonometry

## 4.1 Radian Measure

### KEY CONCEPTS

The radian measure of an angle $\theta$ is defined as the length, $a$, of the arc that subtends the angle divided by the radius, $r$, of the circle.




$\theta = \frac{a}{r}$

For one complete revolution, the length of the arc equals the circumference of the circle, $2\pi r$.

$$\begin{aligned}\theta &= \frac{2\pi r}{r} \\ &= 2\pi\end{aligned}$$

One complete revolution measures $2\pi$ radians.

- Therefore, $2\pi$ rad = 360° or $\pi$ rad = 180°.
- To convert degree measure to radian measure, multiply the degree measure by $\frac{\pi}{180}$ radians.
- To convert radian measure to degree measure, multiply the radian measure by $\left(\frac{180}{\pi}\right)^\circ$.

When angle rotation is positive, the motion is counter-clockwise, and, when the rotation is negative, the motion is clockwise.

**Example – Orbital Motion**

Earth's orbital radius around the Sun is $1.49 \times 10^{11}$ m. If it travels $\frac{5\pi}{8}$ radians, calculate

**a)** this angle in degrees

**b)** the distance Earth has travelled in its orbit

**Solution**

**a)** Since $\pi$ radians equal 180°, multiply the angle by $\frac{180}{\pi}$.

$$\frac{5\cancel{\pi}}{8} \times \frac{180}{\cancel{\pi}} = \frac{900}{8} = 112.5^\circ$$

Notice that the $\pi$s are cancelled out.

**b)** Distance is calculated by determining the arc length.

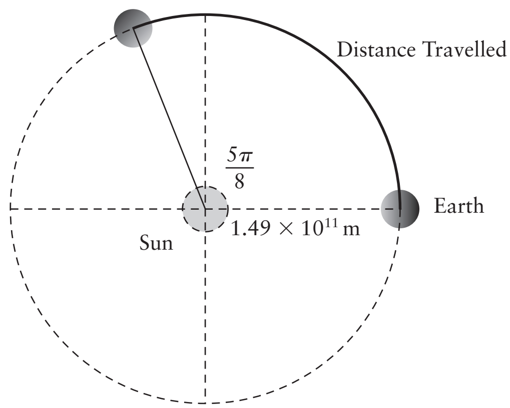
$a = r\theta$

$a = (1.49 \times 10^{11})\dfrac{5\pi}{8}$

$= 2.92 \times 10^{11}$ m

(Note that the calculation of arc length should be done using radian measure.)

Therefore, Earth has travelled $2.92 \times 10^{11}$ m when it moves 112.5°.




## A

1. Convert the following angles to radian measure.
   **a)** 330°
   **b)** 60°
   **c)** 75°
   **d)** 135°
   **e)** 225°
   **f)** 1250°

2. Determine the exact measure for each angle below in degrees.
   **a)** $\pi$
   **b)** $\frac{\pi}{4}$
   **c)** $\frac{5\pi}{6}$
   **d)** $\frac{3\pi}{2}$
   **e)** $\frac{7\pi}{4}$
   **f)** $\frac{\pi}{2}$
   **g)** $\frac{19\pi}{16}$
   **h)** $\frac{2\pi}{9}$

3. Determine the approximate degree measure, to the nearest tenth, for each angle below.
   **a)** 1.02 rad
   **b)** 1.75 rad
   **c)** 3.79 rad
   **d)** 6.00 rad

4. Determine the exact radian measure for each angle.
   **a)** 10°
   **b)** 64°
   **c)** 202.5°
   **d)** 285°

5. The diameter of a circle is 22 cm. If the central angle measures 70°, find the length of the arc.

☆ **6.** An arc of 14.6 cm has a central angle of $\frac{\pi}{4}$ radians. Determine the radius of the circle.

**7.** Find the area of a sector of a circle that has a central angle of $\frac{13\pi}{18}$ and a radius of 12 cm. Round your answer to the nearest tenth.

## B

**8.** Determine the angle formed by the hands of the clock at the following times in terms of degrees and radians.

**a)** 3:00

**b)** 5:30

**c)** 7:20

**d)** 10:50

**9.** How many radians are there in each of the following quantities?

**a)** the second hand of a clock moving 40 s

**b)** a long-distance runner doing 30.2 laps of a track

**c)** Earth's rotation in 8 h

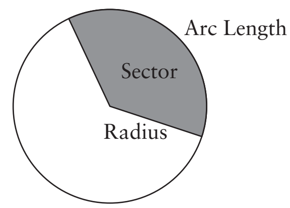
**d)** Earth's orbit in 365 days

☆ **10.** Philadelphia and Ottawa share a common longitude at 79°. The latitude of Philadelphia is 39° and the latitude of Ottawa is 45°. Determine the distance between the cities, given that the radius of Earth is 6336 km.

**11.** Two gears work together. The smaller gear has a radius of 5 cm, and the larger gear has a radius of 12 cm. Determine the number of radians that the larger gear rotates when the smaller gear rotates 300°.

**12.** A pendulum is 40 cm long. If the end of the pendulum swings through a total distance of 13.5 cm, what is the measure, in radians, of the angle through which the pendulum swings?

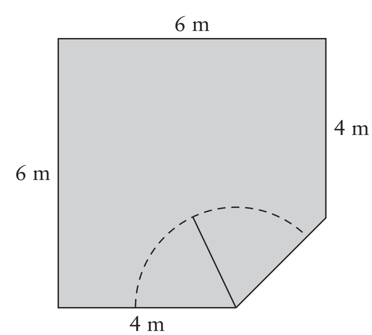
**13.** If the radius of a circle doubles and the measure of the central angle remains the same, what happens to the length of the arc and the area of the sector?




☆ **14.** The diagram below represents a yard that has the shape of a square with one corner cut off. A dog's leash is tied to a fence post at one end of the short "cut-off" side. The leash is 2.2 m long.

**a)** Determine the total area of yard that the dog can reach while on the leash.

**b)** How many times more space will the dog get if the length of the leash is increased by 0.5 m?




**15.** Erik rides his bike 4.5 km every morning.

**a)** If the radius of the tire on his bike is 32 cm, determine the number of radians that the tire will rotate during the entire trip.

**b)** Find the angular velocity in radians per second if the tire turns at 150 rev/min.

> Angular velocity
> $= \frac{\text{Total degrees travelled}}{\text{Time}}$

16. A carousel ride with a diameter of 16 m makes 16 rotations in 1 min.

a) Find its angular velocity in radians per second.

b) If the rider is on the carousel for 7 min, determine the total distance travelled.

17. A satellite with a circular orbit has an angular velocity of 0.0015 rad/s.

a) How long will it take for the satellite to return to its starting position?

b) What is the speed of the orbit if it orbits 1000 km above Earth's surface? (The radius of Earth is 6336 km.)

18. When an angle rotates clockwise from its initial position, it has negative rotation. Determine a positive coterminal angle for each of the following.

a) $-\frac{\pi}{4}$

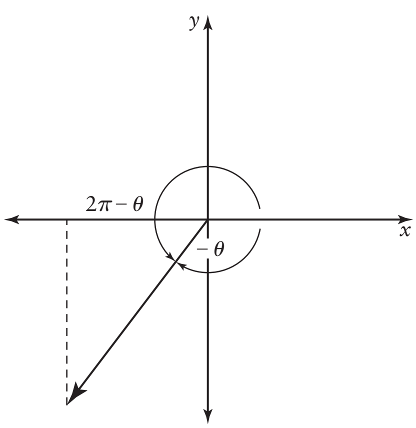
b) $-\frac{5\pi}{6}$

c) $-\frac{2\pi}{3}$

d) $-\frac{9\pi}{5}$

e) $-\frac{11\pi}{3}$

f) $-\frac{14\pi}{15}$




Centripetal acceleration occurs wherever motion is circular and constant. The instantaneous acceleration points toward the centre of the circular motion. The function $a_c = r\omega^2$ determines centripetal acceleration in m/s$^2$, where $r$ is the radius of rotation in metres and $\omega$ is the angular velocity in rad/s.

## C

19. In the Olympic hammer-throw event, an athlete swings the hammer in a circular arc. The speed of the hammer is constant at 1.61 r/s, and the hammer moves in an arc of radius 1.3 m. Determine the centripetal acceleration of the hammer's head.

20. A pulley makes 10 rotations in 3 s.

a) Find the angular velocity in radians per second.

b) If the pulley is 60 cm from its centre of rotation, what is its centripetal acceleration?

c) How far does a point on the circumference travel in 1 s?

An object that moves on a circular path has linear velocity $v = r\frac{\theta}{t}$, where $\frac{\theta}{t}$ is the angular velocity and $r$ is the radius.

☆21. An airplane propeller rotates 20 times per second.

a) Calculate the angular velocity of the propeller in rotations per minute and radians per second.

b) If the propeller has a diameter of 2 m, what is its linear velocity?

22. A satellite rotates around Earth in 90 min.

a) Determine its angular velocity in rotations per day and radians per day.

b) The satellite is 35 000 km above Earth. What is its linear velocity?

23. Low Earth Orbiting (LEO) satellites typically orbit between 320 and 800 km above Earth. They must maintain a speed of 27 000 km/h so that they can stay at a constant distance from Earth.

a) Find the angular velocity needed to maintain a LEO satellite at 320 km above Earth.

b) A LEO satellite has an angular velocity of 3.8 radians per hour. Determine its distance above Earth.

## KEY CONCEPTS

- You can use a calculator to calculate trigonometric ratios for an angle expressed in radian measure by setting the angle mode to radians.

- You can determine the reciprocal trigonometric ratios for an angle expressed in radian measure by first calculating the primary trigonometric ratios and then using the reciprocal key on a calculator.
  The reciprocal ratios are:
  $\csc x = \frac{1}{\sin x}$
  $\sec x = \frac{1}{\cos x}$
  $\cot x = \frac{1}{\tan x}$

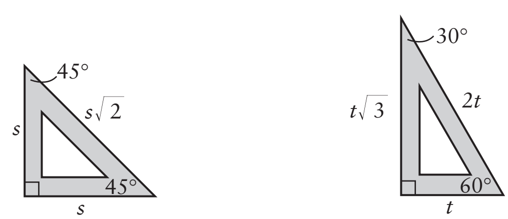
**Example**

To evaluate $\csc \frac{\pi}{3}$, determine $\sin \frac{\pi}{3}$ and invert the ratio:
$\sin \frac{\pi}{3} = \frac{\sqrt{3}}{2}$; therefore, $\csc \frac{\pi}{3} = \frac{2}{\sqrt{3}}$

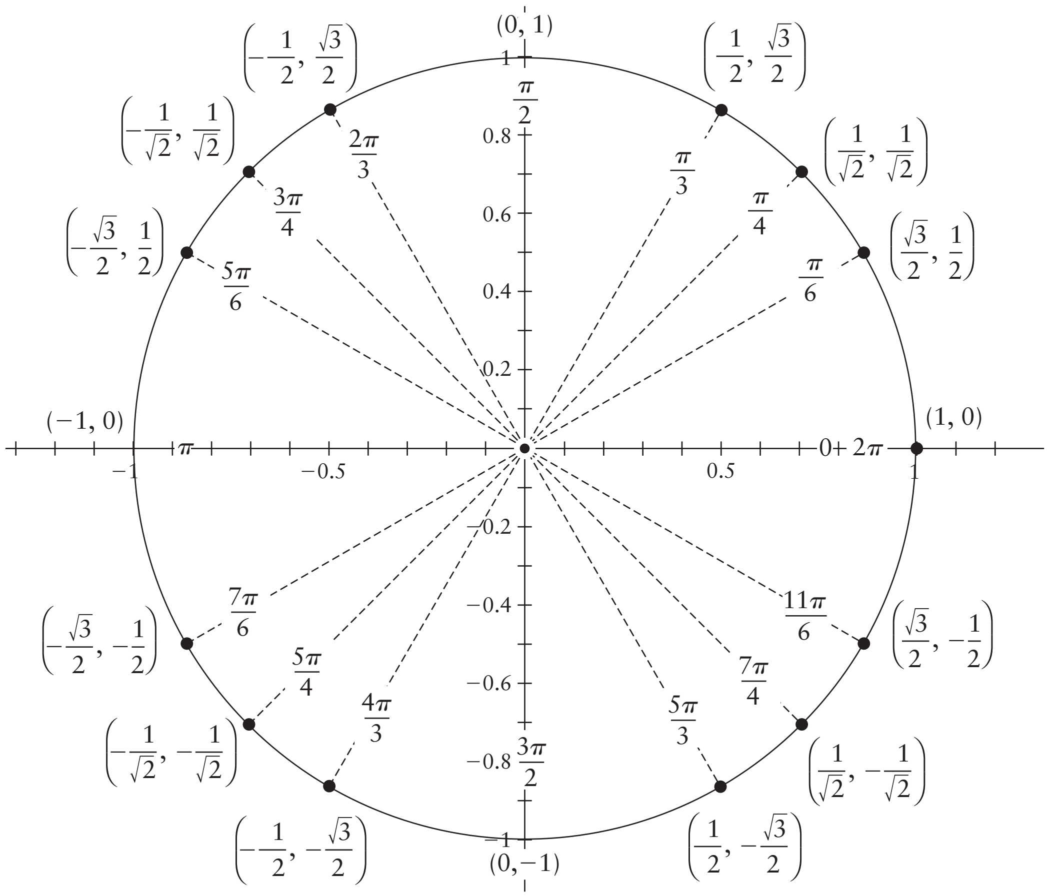
- You can use the unit circle and special triangles to determine exact values for the trigonometric ratios of the special angles 0, $\frac{\pi}{6}$, $\frac{\pi}{4}$, $\frac{\pi}{3}$, and $\frac{\pi}{2}$.

**Special Triangles**

The triangles found in a geometry set are a 45°–45°–90° triangle and a 30°–60°–90° triangle. These triangles can be used to construct similar triangles with the same special relationships among the sides.

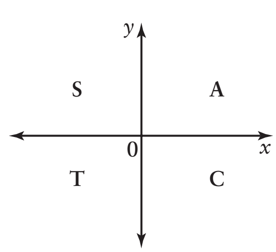


**The Unit Circle**




- The angle of rotation appears on each terminal arm.
- The radius of the circle is 1.
- The coordinates at the circumference of the unit circle determine the exact values of the cosine and sine functions for the given angle of rotation.
- You can use the unit circle along with the CAST rule to determine exact values for the trigonometric ratios of multiples of the special angles.

**CAST Rule**



## A

**1. a)** Use a calculator to evaluate each of these trigonometric ratios, to four decimal places.

**i)** cos 45°

**ii)** tan 80°

**iii)** sin 161°

**iv)** cos 302°

**b)** Use a calculator to evaluate each of these trigonometric ratios, to four decimal places. Reset your calculator to radians.

**i)** cos 0.785 **iii)** sin 2.81

**ii)** tan 1.396 **iv)** cos 5.271

**c)** Note any similarities between the answers to parts a) and b). Explain why they occur.

**2.** Use a calculator to evaluate each trigonometric ratio, to four decimal places.

**a)** $\cos \frac{3\pi}{4}$

**b)** $\sin \frac{5\pi}{8}$

**c)** $\tan \frac{\pi}{6}$

**d)** $\sin \frac{11\pi}{6}$

**3.** Use a calculator to evaluate each trigonometric ratio below, to four decimal places.

**a)** sec 9°

**b)** cot 74°

**c)** csc 200°

**d)** sec 340°

**4.** Use a calculator to evaluate each trigonometric ratio below, to four decimal places.

**a)** cot 0.51

**b)** sec 0.92

**c)** csc 4.17

**d)** cot 6.00

**5.** Use a calculator to evaluate each trigonometric ratio below, to four decimal places.

**a)** $\csc \frac{4\pi}{3}$

**b)** $\cot \frac{\pi}{5}$

**c)** $\sec \frac{2\pi}{9}$

**d)** $\csc \frac{7\pi}{12}$

## B

**6.** Use the unit circle to determine exact values of the primary trigonometric ratios for each angle below.

**a)** $\frac{\pi}{3}$

**b)** $\frac{5\pi}{4}$

**c)** $\frac{7\pi}{6}$

**d)** $\pi$

**e)** $\frac{5\pi}{6}$

**f)** $\frac{7\pi}{4}$

☆**7.** Use the unit circle to determine exact values of the six trigonometric ratios for each angle below.

**a)** $\frac{\pi}{6}$

**b)** $\frac{5\pi}{3}$

**c)** $\frac{3\pi}{4}$

**d)** $\frac{5\pi}{6}$

**e)** $\frac{11\pi}{6}$

**f)** $\frac{2\pi}{3}$

☆**8.** The angle $\theta$ is in standard position with a terminal arm in the given quadrant. For each function, find the values of the remaining five functions for $\theta$.

**a)** $\sin \theta = -\frac{1}{5}$; quadrant IV

**b)** $\tan \theta = 2$; quadrant III

**c)** $\csc \theta = -2$; quadrant III

**d)** $\sec \theta = \sqrt{3}$; quadrant I

**9.** Use the unit circle or special triangles to determine all possible trigonometric ratios that have the following values:

**a)** $\frac{1}{\sqrt{3}}$

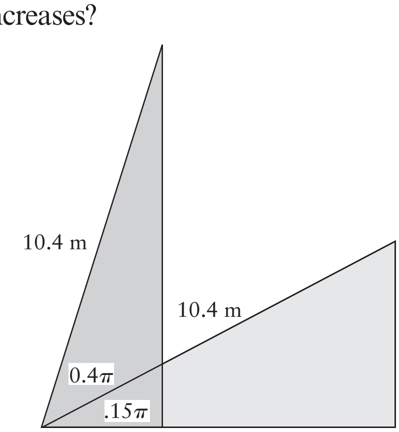
**b)** $-\frac{\sqrt{3}}{2}$

**c)** $\frac{2}{\sqrt{3}}$

**d)** $-\sqrt{2}$

**10.** A kite is fastened to the ground by a string that is 45 m long. If the angle of elevation of the kite is $\frac{4\pi}{9}$, determine the kite's height above the ground.

☆**11.** Solve for the missing sides of the two right-angled triangles below. What are the horizontal and vertical displacements between the two triangles as the angle increases?




**12. a)** Determine an exact value for each expression.

**i)** $\cos \frac{2\pi}{3} \cos \frac{5\pi}{6} + \sin \frac{2\pi}{3} \sin \frac{5\pi}{6}$

**ii)** $\sin \frac{5\pi}{6} \cos \frac{4\pi}{3} - \cos \frac{5\pi}{6} \sin \frac{4\pi}{3}$

**iii)** $\dfrac{\tan \frac{7\pi}{3} - \tan \frac{5\pi}{3}}{1 + \tan \frac{7\pi}{3} \tan \frac{5\pi}{3}}$

**b)** Use a calculator to verify your answers to part a).

**13.** Determine an expression using trigonometric ratios of special angles that simplifies to an answer of zero. You must use three different angles and three different ratios.

## C

**14.** An angle in standard position coincides with the line $y = -1.5x$ and lies in the second quadrant. Find the six trigonometric functions of the angle.

**15.** Brett is enjoying a sunset at his cottage. As he sits outside on the deck, the angle of elevation to the sun changes from $\frac{\pi}{5}$ to $\frac{\pi}{6}$. If the sun is 146 million kilometres away from Earth, determine the vertical displacement of the sun over the given angle displacement. (Assume that the sun's distance from Earth remains constant.)

☆**16. a)** Determine an exact value for each expression below.

**i)** $\dfrac{\sec \frac{\pi}{4} \cos \frac{2\pi}{3}}{\tan \frac{\pi}{6} \csc \frac{3\pi}{4}}$

**ii)** $\sin \frac{5\pi}{4} - \cos \frac{11\pi}{6} \cot \frac{\pi}{3}$

**b)** Use your calculator to check your answers to part a).

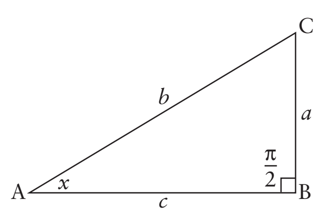
**17.** Determine the area of the intersection of the two triangles in question 11.

## KEY CONCEPTS

- You can use a right triangle to derive equivalent trigonometric expressions that form the cofunction identities, such as $\sin x = \cos\left(\frac{\pi}{2} - x\right)$.
- You can use the unit circle along with transformations to derive equivalent trigonometric expressions that form other trigonometric identities, such as $\cos\left(\frac{\pi}{2} + x\right) = -\sin x$.
- Given a trigonometric expression of a known angle, you can use equivalent trigonometric expressions to evaluate trigonometric expressions of other angles.
- You can use graphing technology to demonstrate that two trigonometric expressions are equivalent.

| **Trigonometric Identities Featuring $\frac{\pi}{2}$** | | | |
|---|---|---|---|
| **Cofunction Identities** | | | |
| $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ | $\cos x = \sin\left(\frac{\pi}{2} - x\right)$ | $\sin\left(x + \frac{\pi}{2}\right) = \cos x$ | $\cos\left(x + \frac{\pi}{2}\right) = -\sin x$ |
| $\tan x = \cot\left(\frac{\pi}{2} - x\right)$ | $\cot x = \tan\left(\frac{\pi}{2} - x\right)$ | $\tan\left(x + \frac{\pi}{2}\right) = -\cot x$ | $\cot\left(x + \frac{\pi}{2}\right) = -\tan x$ |
| $\csc x = \sec\left(\frac{\pi}{2} - x\right)$ | $\sec x = \csc\left(\frac{\pi}{2} - x\right)$ | $\csc\left(x + \frac{\pi}{2}\right) = \sec x$ | $\sec\left(x + \frac{\pi}{2}\right) = -\csc x$ |

**Summary of Cofunction Identities**

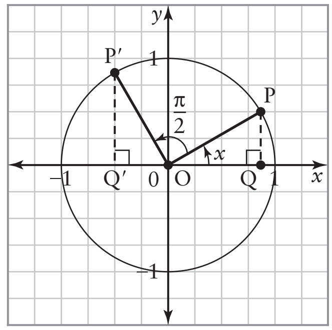


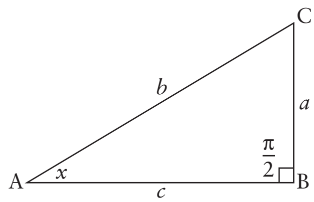

$\angle C = \frac{\pi}{2} - x$

$\sin x = \frac{a}{b}$ and $\cos\left(\frac{\pi}{2} - x\right) = \frac{a}{b}$

Therefore, $\sin x = \cos\left(\frac{\pi}{2} - x\right)$

Similar expressions can be derived for the other five trigonometric ratios.




After a rotation of $\frac{\pi}{2}$, or 90°, counter-clockwise about the origin, the coordinates of the point $(x, y)$ become $(-y, x)$.
The coordinates of P are $(\cos x, \sin x)$.
The coordinates of P′ are $\left[\cos\left(x + \frac{\pi}{2}\right), \sin\left(x + \frac{\pi}{2}\right)\right]$.
Because the angle has rotated counter-clockwise $\frac{\pi}{2}$,
$\cos\left(x + \frac{\pi}{2}\right) = -\sin x$, and $\sin\left(x + \frac{\pi}{2}\right) = \cos x$.

**A**

**1.** Given that $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, use an equivalent trigonometric expression to show that $\cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}$.

**2.** Given that $\sec \frac{\pi}{6} = \frac{2}{\sqrt{3}}$, use an equivalent trigonometric expression to show that $\csc \frac{\pi}{3} = \frac{2}{\sqrt{3}}$.

**3.** Given that $\tan \frac{\pi}{3} = \sqrt{3}$, use an equivalent trigonometric expression to show that $\cot \frac{\pi}{6} = \sqrt{3}$.

**4.** Given that $\cos \frac{2\pi}{9} = 0.766$, use an equivalent trigonometric expression to show that $\sin \frac{13\pi}{18} = 0.766$.

**5.** Given that $\sec \frac{5\pi}{4} = -\csc x$, first express $\frac{5\pi}{4}$ as a sum of $\frac{\pi}{2}$ and an angle, and then apply a trigonometric identity to determine the measure of angle $x$.

☆**6.** Given that $\sin \frac{2\pi}{3} = \cos y$, first express $\frac{2\pi}{3}$ as a sum of $\frac{\pi}{2}$ and an angle, and then apply a trigonometric identity to determine the measure of angle $y$.

**7.** Given that $\tan \frac{7\pi}{8} = -\cot z$, first express $\frac{7\pi}{8}$ as a sum of $\frac{\pi}{2}$ and an angle, and then apply a trigonometric identity to determine the measure of angle $z$.

8. Given that $\sin \frac{2\pi}{5} = \cos x$, first express $\frac{2\pi}{5}$ as a difference between $\frac{\pi}{2}$ and an angle, and then apply a cofunction identity to determine the measure of angle $x$.

9. Given that $\csc \frac{3\pi}{10} = \sec y$, first express $\frac{3\pi}{10}$ as a difference between $\frac{\pi}{2}$ and an angle, and then apply a cofunction identity to determine the measure of angle $y$.

☆10. Given that $\sin \frac{\pi}{12} \doteq 0.2588$, use equivalent trigonometric expressions to evaluate the following, to four decimal places.

**a)** $\cos \frac{5\pi}{12}$

**b)** $\cos \frac{7\pi}{12}$

11. Given that $\sec \frac{4\pi}{21} \doteq 1.21$, use equivalent trigonometric expressions to evaluate the following, to two decimal places.

**a)** $\csc \frac{13\pi}{42}$

**b)** $\csc \frac{29\pi}{42}$

## B

12. Given that $\cos a = -\sin \frac{3\pi}{7}$ and that $a$ lies in the second quadrant, determine the measure of angle $a$, to two decimal places.

13. Given that $\csc b = -\sec \frac{3\pi}{5}$ and that $b$ lies in the first quadrant, use a co-function identity to determine the measure of angle $b$, to two decimal places.

☆14. Given that $\sec x = -\csc 0.57$ and that $x$ lies in the second quadrant, determine the measure of angle $x$, to two decimal places.

15. Given that $\tan y = \cot 1.52$ and that $y$ lies in the first quadrant, determine the measure of angle $y$, to two decimal places.

16. Given that $\sin z = \cos 0.84$ and that $z$ lies in the first quadrant, determine the measure of angle $z$, to two decimal places.

17. **Use Technology**

**a)** Verify graphically that

**i)** $\sin\left(\frac{\pi}{2} - x\right) = \cos x$

**ii)** $\cos\left(\frac{\pi}{2} - x\right) = \sin x$

**b)** Graph $\sin x$ and $\sin\left(\frac{\pi}{2} - x\right)$. Explain the differences you see in the functions. How do these functions verify the co-function identities graphed in part a)?

18. Using the trigonometric identities for $(\pi - x)$ and $(x + \pi)$ and given that $\cos \frac{\pi}{14} \doteq 0.9749$, find

**a)** $\cos \frac{13\pi}{14}$

**b)** $\cos \frac{15\pi}{14}$

19. Use a graph or unit circle to verify the following identities:

**a)** $\sin(-x) = -\sin x$

**b)** $\cos(-x) = \cos x$

**c)** $\tan(-x) = -\tan x$

Explain why you think this relationship exists. Do you think the same relationship exists for the reciprocal trigonometric identities?

20. **Use Technology**

Use graphing technology to verify that $\cos(3\pi - x) = -\cos x$, where $x$ lies in the first quadrant.

☆**21.** Co-function angle identities are sometimes called correlated angle identities. Investigate the relationship among other correlated angles. Complete each statement below by determining an appropriate trigonometric function. The first one has been completed.

**a)** $\sin \alpha = \cos\left(\alpha - \frac{\pi}{2}\right)$

$= -\sin(\alpha - \pi)$

$= -\cos\left(\alpha - \frac{3\pi}{2}\right)$

**b)** $\cos \alpha =$ ______ $\left(\alpha - \frac{\pi}{2}\right)$

$=$ ______ $(\alpha - \pi)$

$=$ ______ $\left(\alpha - \frac{3\pi}{2}\right)$

**c)** $\tan \alpha =$ ______ $\left(\alpha - \frac{\pi}{2}\right)$

$=$ ______ $(\alpha - \pi)$

$=$ ______ $\left(\alpha - \frac{3\pi}{2}\right)$

**d)** $\csc \alpha =$ ______ $\left(\alpha - \frac{\pi}{2}\right)$

$=$ ______ $(\alpha - \pi)$

$=$ ______ $\left(\alpha - \frac{3\pi}{2}\right)$

**e) sec** $\alpha =$ ______ $\left(\alpha - \frac{\pi}{2}\right)$

$=$ ______ $(\alpha - \pi)$

$=$ ______ $\left(\alpha - \frac{3\pi}{2}\right)$

**f) cot** $\alpha =$ ______ $\left(\alpha - \frac{\pi}{2}\right)$

$=$ ______ $(\alpha - \pi)$

$=$ ______ $\left(\alpha - \frac{3\pi}{2}\right)$

**22. a)** Repeat the exercise in question 21, using sums instead of differences.

For example,

$\sin \alpha =$ ______ $\left(\alpha + \frac{\pi}{2}\right)$

$=$ ______ $(\alpha + \pi)$

$= \left(\alpha + \frac{3\pi}{2}\right).$

**b)** How do these identities compare with the identities investigated in question 21?

**23.** Using the trigonometric identities for $\left(\frac{3\pi}{2} - x\right)$ and $\left(x + \frac{3\pi}{2}\right)$ and given that $\tan \frac{2\pi}{9} \doteq 0.8391$, find

**a)** $\cot \frac{23\pi}{18}$

**b)** $\cot \frac{31\pi}{18}$

**24. a)** Determine an exact value of $a$ such that $\sec\left(3a - \frac{\pi}{4}\right) = -\csc\left(4a - \frac{\pi}{4}\right)$.

**b)** Check your answer.

## C

**25.** Simplify.

**a)** $\sin\left(\frac{\pi}{2} + x\right) - \cos\left(\frac{3\pi}{2} - x\right) + \sin\left(\frac{3\pi}{2} - x\right)$

**b)** $\dfrac{\cos(x + \pi)\cos\left(\frac{\pi}{2} + x\right)}{\cos(\pi - x)} - \dfrac{\sin\left(\frac{3\pi}{2} - x\right)}{\sec(\pi + x)}$

**26.** Determine whether or not

$$\frac{\sin\left(x + \frac{\pi}{2}\right)}{\sin x} = \tan x.$$

## KEY CONCEPTS

A compound angle expression is a trigonometric expression that depends on two or more angles.

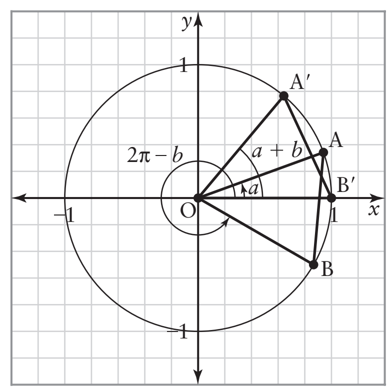
- You can develop compound angle formulas using algebra and the unit circle.
- Once you have developed one compound angle formula, you can develop others by applying equivalent trigonometric expressions.
- The compound angle, or addition and subtraction, formulas for sine and cosine are

  $\sin(x + y) = \sin x \cos y + \cos x \sin y$

  $\sin(x - y) = \sin x \cos y - \cos x \sin y$

  $\cos(x + y) = \cos x \cos y - \sin x \sin y$

  $\cos(x - y) = \cos x \cos y + \sin x \sin y$
- You can apply compound angle formulas to determine exact trigonometric ratios for angles that can be expressed as sums or differences of special angles.




**Addition Formula for Cosine**

The unit circle can be used to show that the formula $\cos(x + y) = \cos x \cos y - \sin x \sin y$ is valid for all angles. Consider the unit circle shown.

**Subtraction Formula for Cosine**

The subtraction formula for cosine can be derived from the addition formula for cosine.

| | |
|---|---|
| $\cos(x + y) = \cos x \cos y - \sin x \sin y$ | |
| $\cos(x + (-y)) = \cos x \cos(-y) - \sin x \sin(-y)$ | Substitute $-y$ for $y$. |
| $\cos(x - y) = \cos x \cos(2\pi - y) - \sin x \sin(2\pi - y)$ | From the unit circle, angle $-y$ is the same as angle $(2\pi - y)$. |
| $\cos(x - y) = \cos x \cos y - \sin x(-\sin y)$ | $\cos(2\pi - y) = \cos y$; $\sin(2\pi - y) = -\sin y$ |
| $\cos(x - y) = \cos x \cos y + \sin x \sin y$ | |

**Addition Formula for Sine**

Recall the cofunction identities $\sin x = \cos\left(\frac{\pi}{2} - x\right)$ and $\cos x = \sin\left(\frac{\pi}{2} - x\right)$.
Apply these and the subtraction formula for cosine.

| | |
|---|---|
| $\sin(x + y) = \cos\left[\frac{\pi}{2} - (x + y)\right]$ | Apply a cofunction identity. |
| $= \cos\left[\left(\frac{\pi}{2} - x\right) - y\right]$ | Regroup the terms in the argument. |
| $= \cos\left(\frac{\pi}{2} - x\right)\cos y + \sin\left(\frac{\pi}{2} - x\right)\sin y$ | Apply the subtraction formula for cosine. |
| $= \sin x \cos y + \cos x \sin y$ | Apply cofunction identities. |

**Subtraction Formula for Sine**

The subtraction formula for sine can be derived from the addition formula for sine, following the approach used for the subtraction formula for cosine.

$\sin(x + -y)) = \sin x \cos(-y) + \cos x \sin(-y)$ Substitute $-y$ for $y$.

$\sin(x - y) = \sin x \cos y + \cos x(-\sin y)$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

**Compound-Angle Formulas for Tangent**

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}$$

$$\tan(x - y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}$$

**Double-Angle Formulas**

Addition and Subtraction formulas can be applied to develop the following double-angle formulas.

$\sin 2x = 2\sin x \cos x$

$\tan 2x = \dfrac{2\tan x}{1 - \tan^2 x}$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = 2\cos^2 x - 1$

$\cos 2x = 1 - 2\sin^2 x$

**A**

**1.** Use the appropriate compound-angle formula to express each of the following as a single trigonometric function, and then determine an exact value for each.

**a)** $\sin\frac{\pi}{2}\cos\frac{\pi}{4} - \cos\frac{\pi}{2}\sin\frac{\pi}{4}$

**b)** $\sin\frac{2\pi}{3}\cos\pi + \cos\frac{2\pi}{3}\sin\pi$

**c)** $\cos\frac{\pi}{6}\cos\frac{3\pi}{4} - \sin\frac{\pi}{6}\sin\frac{3\pi}{4}$

**d)** $\cos\frac{\pi}{12}\cos\frac{7\pi}{4} + \sin\frac{\pi}{12}\sin\frac{7\pi}{4}$

**2.** Determine an exact value for $\cos\frac{\pi}{12}$ using a sum or difference formula for cosine. (*Hint:* Find two angles that have a sum or difference of $\frac{\pi}{12}$.)

**3.** Apply a compound-angle formula to each expression below, and then determine an exact value for each.

**a)** $\sin\left(\frac{\pi}{2} + \frac{\pi}{12}\right)$

**b)** $\sin\left(\frac{\pi}{2} - \frac{\pi}{12}\right)$

**c)** $\cos\left(\frac{3\pi}{4} + \frac{\pi}{6}\right)$

**d)** $\cos\left(\frac{3\pi}{4} - \frac{\pi}{6}\right)$

☆4. Determine an exact value for $\sin \frac{13\pi}{36}$ using a sum or difference formula for sine.

5. Determine an exact value for $\tan \frac{19\pi}{24}$ using a sum or difference formula for tangent.

6. Use an appropriate compound angle formula to determine an exact value for each of the following:

**a)** $\cos \frac{7\pi}{12}$

**b)** $\sin \frac{11\pi}{12}$

**c)** $\sin \frac{\pi}{12}$

**d)** $\cos \left(-\frac{\pi}{12}\right)$

**B**

☆7. Use sum or difference identities to determine an exact value for each trigonometric function below.

**a)** $\sin \frac{23\pi}{12}$

**b)** $\cos \frac{13\pi}{12}$

**c)** $\tan \frac{23\pi}{12}$

**d)** $\csc \frac{5\pi}{12}$

8. If $x$ and $y$ are acute angles such that $\sin x = \frac{1}{2}$ and $\sin y = \frac{1}{4}$,

**a)** determine the exact values for $\cos x$ and $\cos y$

**b)** use the results from part a) to determine an exact value for each of the following:

**i)** $\sin (x + y)$

**ii)** $\sin (x - y)$

**iii)** $\cos (x + y)$

**iv)** $\cos (x - y)$

9. If $x$ and $y$ are angles in the second quadrant such that $\sin x = \frac{2}{9}$ and $\cos y = -\frac{4}{5}$,

**a)** determine the exact value for $\cos x$ and $\sin y$

**b)** use the results from part a) to determine an exact value for each of the following:

**i)** $\sin (x + y)$

**ii)** $\sin (x - y)$

**iii)** $\cos (x + y)$

**iv)** $\cos (x - y)$

☆10. Find the exact value of each of the following if $0 \le x \le \frac{\pi}{2}$ and $0 \le y \le \frac{\pi}{2}$:

**a)** $\sin (x - y)$ if $\cos x = \frac{3}{5}$ and $\sin y = \frac{24}{25}$

**b)** $\sin (x + y)$ if $\sin x = \frac{5}{13}$ and $\sin y = \frac{12}{13}$

**c)** $\cos (x + y)$ if $\cos x = \frac{2}{3}$ and $\sin y = \frac{10}{13}$

**d)** $\cos (x - y)$ if $\cos x = \frac{3}{5}$ and $\cos y = \frac{4}{5}$

**e)** $\tan (x + y)$ if $\cot x = \frac{6}{5}$ and $\sec y = \frac{3}{2}$

**f)** $\csc (x - y)$ if $\sec x = \frac{4}{3}$ and $\tan y = \frac{13}{5}$

11. Find and simplify an expression for $\sin \left(\frac{\pi}{3} - \alpha\right)$.

12. Verify that $\csc \left(\frac{3\pi}{2} + \theta\right) = -\sec \theta$, using a compound-sum formula.

13. Express $\cos \frac{13\pi}{4}$ as a trigonometric function of an angle in quadrant I.

**14.** The angle $2x$ lies in the third quadrant such that $\cos 2x = -\frac{6}{15}$.

**a)** Sketch the location of angle $2x$.

**b)** Determine the quadrant of angle $x$.

**c)** Find an exact value for $\cos x$.

**d)** Use a calculator to determine the measure of $x$, in radians.

**e)** Use a calculator to verify your answer in part c).

**15.** Angle $b$ lies in the second quadrant such that $\cos b = -\frac{3}{5}$.

**a)** Determine an exact answer for $\sin b$ and $\tan b$.

**b)** Determine an exact answer for $\cos 2b$.

**c)** Determine an exact answer for $\sin 2b$.

**d)** Determine an exact answer for $\tan 2b$.

**e)** Use a calculator to determine an approximate measure for $b$, in radians, to two decimal places.

**f)** In which quadrant does angle $2b$ lie? Justify your answer.

**16. a)** Use the half-angle formula $\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$ to find $\sin \frac{\pi}{12}$.

**b)** Check your answer to part a) by using another method to find $\sin \frac{\pi}{12}$.

## C

**17.** An angle, $x$, lies in the second quadrant, and $\tan x = -\frac{4}{3}$. Determine $\sin 2x$ and $\cos 2x$.

**18.** Find $\sin 2x$ and $\cos 2x$ for each of the following:

**a)** $\sin x = -\frac{3}{5}$ for $\pi \le x \le \frac{3\pi}{2}$

**b)** $\cos x = -\frac{1}{\sqrt{2}}$ for $\frac{\pi}{2} \le x \le \pi$

**c)** $\csc x = 4$ for $0 \le x \le \frac{\pi}{2}$

**d)** $\tan x = -\frac{3}{2}$ for $\frac{\pi}{2} \le x \le \pi$

**19.** Find an expression for $\sin 3x$ by applying an addition formula for the sine function. Express your answer in terms of multiples and powers of $\sin x$.

**20.** Find an expression for $\cos 3x$ by applying an addition formula for the cosine function. Express your answer in terms of multiples and powers of $\cos x$.

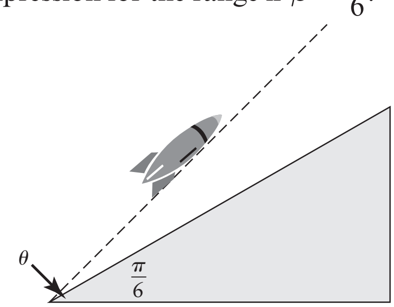
**21.** The half-angle formula for sine is $\sin \frac{x}{2} = \pm\sqrt{\frac{1 - \cos x}{2}}$. Use this formula to find an exact value for each of the following:

**a)** $\sin \frac{\pi}{12}$

**b)** $\sin \frac{3\pi}{8}$

**22.** Use compound-angle formulas to prove $\cos x \cos y = \frac{1}{2}[\cos (x + y) + \cos (x - y)]$.

☆**23.** A rocket is launched with velocity $v$ at an angle of $\theta$ to the horizontal from the base of a hill that makes an angle of $\beta$ with the horizontal. The range of the rocket, measured along the hill's slope, is given by $R = \frac{2v^2 \cos \theta \sin (\theta - \beta)}{g \cos^2 \beta}$. Determine an expression for the range if $\beta = \frac{\pi}{6}$.



## KEY CONCEPTS

- A trigonometric identity is an equation with trigonometric expressions that is true for all angles in the domain of the expressions on both sides.
- One way to show that an equation is not an identity is to determine a counter-example.
- To prove that an equation is an identity, treat each side of the equation independently and transform the expression on one side into the exact form of the expression on the other side.

**Examples**

Prove that $\csc 2x = \dfrac{\csc x}{2\cos x}$.

**Solution**

**L.S.** $= \csc 2x$

$= \dfrac{1}{\sin 2x}$ Use a reciprocal identity.

$= \dfrac{1}{2\sin x\cos x}$ Use the double-angle formula.

$= \dfrac{1}{2} \times \dfrac{1}{\sin x} \times \dfrac{1}{\cos x}$ Write as separate rational expressions.

$= \dfrac{1}{2} \times \csc x \times \dfrac{1}{\cos x}$ Use a reciprocal identity.

$= \dfrac{\csc x}{2\cos x}$ Combine into one term.

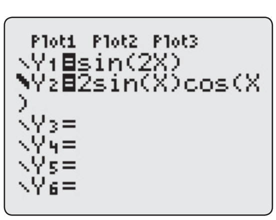
**R.S.** $= \dfrac{\csc x}{2\cos x}$

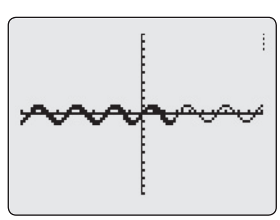
**L.S.** $=$ **R.S.**

Therefore, $\csc 2x = \dfrac{\csc x}{2\cos x}$ is an identity.

Another way to verify an identity is to graph both sides of the equation.

For example, show that $\sin 2x = 2\sin x\cos x$.



- The basic trigonometric identities are the Pythagorean identity, the quotient identity, the reciprocal identities, and the compound angle formulas. You can use these identities to prove more complex identities.

**Pythagorean Identity**

$\sin^2 x + \cos^2 x = 1$

**Quotient Identity**

$\tan x = \dfrac{\sin x}{\cos x}$

**Reciprocal Identities**

$\csc x = \dfrac{1}{\sin x}$ $\quad \sec x = \dfrac{1}{\cos x}$ $\quad \cot x = \dfrac{1}{\tan x}$

**Compound-Angle Formulas**

$\sin(x + y) = \sin x \cos y + \cos x \sin y$

$\sin(x - y) = \sin x \cos y - \cos x \sin y$

$\cos(x + y) = \cos x \cos y - \sin x \sin y$

$\cos(x - y) = \cos x \cos y + \sin x \sin y$

- Trigonometric identities can be used to simplify solutions to problems that result in trigonometric expressions. This is important in understanding solutions for problems in mathematics, science, engineering, economics, and other fields.

## A

**1.** Simplify each expression.

**a)** $\cos x \csc x \tan x$

**b)** $\cos x \cot x + \sin x$

**c)** $\dfrac{\cot x}{\cos x}$

**d)** $(\sin x + \cos x)^2 + (\sin x - \cos x)^2$

**2.** Determine whether each of the following represents an identity.

**a)** $\sin^2 x = (1 - \cos x)(1 + \cos x)$

**b)** $\cos x \sin x \tan x + \cos^2 x = 1$

**c)** $\sec x + \csc x = 1$

**d)** $\sin x - \cos x = \dfrac{1}{\csc x - \sec x}$

**e)** $(\csc x - \cot x)^2 = \dfrac{1 - \cos x}{1 + \cos x}$

**f)** $\dfrac{\sec x - \cos x}{\tan x} = \sin x$

**3.** **Use Technology** Use graphing software to check your answers for question 2.

**4.** For the equations that are not identities in question 2, modify them so that they will become identities.

## B

☆**5.** Prove that

**a)** $\tan^2 x + \sec^2 x = 1$

**b)** $\cos x \tan^2 x = \sin x \tan x$

**c)** $\tan 2x - \sin 2x = 2 \tan 2x \sin^2 x$

**d)** $\sin 2x = \tan x(1 + \cos 2x)$

**e)** $\sec 2x = \dfrac{\sec x \csc x}{2}$

**f)** $\sin^2 x + \cos^2 x + \tan^2 x = \sec^2 x$

**g)** $\dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin x \cos x} = 1 - \tan x$

6. Prove that each of the following is NOT a trigonometric identity by determining a counterexample. *Hint*: Find an angle that makes the equation false.

**a)** $\sin x \cos x = \tan x$

**b)** $\sin x + \cos x = 1$

**c)** $\tan^2 x + \cot^2 x = 1$

**d)** $\sec^2 x - 1 = \dfrac{\cos x}{\csc x}$

7. Apply a compound angle formula to verify the identity $\sin\left(\frac{\pi}{2} + \theta\right) = \cos\theta$.

8. Apply a compound-angle formula to determine an identity for each of the following:

**a)** $\cos\left(\frac{3\pi}{2} - x\right)$

**b)** $\sin\left(\frac{3\pi}{2} - x\right)$

**c)** $\sin\left(\frac{3\pi}{2} + y\right)$

**d)** $\cos\left(\frac{3\pi}{2} + y\right)$

9. Explain why it is not possible to determine identities for $\tan\left(\frac{3\pi}{2} - x\right)$ and $\tan\left(\frac{3\pi}{2} + y\right)$ using a compound angle formula. Propose an alternate method of determining identities for these expressions.

☆10. Apply double-angle formulas to verify the identity $\sin\left(x - \frac{4\pi}{3}\right) = \dfrac{\sqrt{3}\cos x - \sin x}{2}$.

11. Apply double-angle formulas to verify the identity $\cos\left(2x + \frac{\pi}{2}\right) = -\sin 2x$.

12. Prove that $\cos\left(\frac{3\pi}{4} - x\right) - \sin\left(\frac{3\pi}{4} + x\right) = \dfrac{-1}{\sqrt{2}}(\cos x - \sin x)$.

13. **a)** Apply double-angle formulas to verify the identity $\sin 4x = 2\sin 2x \cos 2x$.

**b)** Conjecture an identity for $\sin 6x$ and $\sin 8x$. Do you think these identities will hold for all $\sin kx$, where $k$ is a positive integer? Explain why or why not.

14. Use trigonometric identities to find all solutions for $x$ below, given $0 \le x \le 2\pi$.

**a)** $\sin 2x + \cos x = 0$

**b)** $\sin 2x + \cos 2x = 0$

**c)** $(\sin x - \cos x)^2 = 1$

**d)** $\sin x \cos x + \dfrac{1}{2} = 0$

**e)** $\tan x \sin 2x = 2\sin^2 x$

**f)** $\csc^2 \dfrac{x}{2} = 2\sec x$

15. **a)** Use a compound angle formula to prove that $\tan(\pi - x) = -\tan x$.

**b) Use Technology** Use graphing technology to illustrate the identity.

16. Prove that $\sin(\pi - x) - \tan(\pi + x) = \dfrac{\sin x(\cos x - 1)}{\cos x}$.

17. **a)** Prove that $\dfrac{\sin 2x}{1 - \cos 2x} = \cot x$.

**b) Use Technology** Illustrate the identity by graphing with technology.

18. Prove that $\dfrac{2\csc 2x \tan x}{\sec x} = \sec x$.

19. **a) Use Technology** Use graphing technology to determine whether it is reasonable to conjecture that $\sin^4 x - \cos^4 x = 2\sin^2 x + 1$ is an identity.

**b)** If it appears to be an identity, prove the identity. If not, determine a counter-example.

## C

☆ **20.** The strength of the magnetic field in a wire can be modelled by $B = \frac{F \csc \theta}{I\ell}$, where $F$ is the force on the wire, $I$ is the current, $\ell$ is the length of the wire, and $\theta$ is the angle the wire makes with the magnetic field. Determine an identity for this expression by finding an equivalent trigonometric expression.

**21.** The intensity of light passing through a lens can be found using the formula $I = I_0 - \frac{I_0}{\csc^2 \theta}$, where $I_0$ is the intensity of light coming in, $I$ is the intensity of the light that emerges or goes out of the lens, and $\theta$ is the angle between.

**a)** Find and simplify an equivalent expression for $I = I_0 - \frac{I_0}{\csc^2 \theta}$.

**b)** Suppose light passes through a polarized lens at angle $\frac{\pi}{4}$ to the original lens. How much less intense than the original light will the light now be?

**22.** Prove that $\sin (x + y) \sin (x - y) = \sin^2 x - \sin^2 y$.

**23.** The equation $\sin x \sin y = \frac{1}{2}[\cos(x - y) - \cos(x + y)]$ is called a Product to Sum formula.

**a)** Show that this formula is an identity.

**b)** Develop other Product to Sum formulas by determining a product that is equal to the following:

**i)** $\frac{1}{2}[\cos(x + y) + \cos(x - y)]$

**ii)** $\frac{1}{2}[\sin(x + y) - \sin(x - y)]$

**iii)** $\frac{1}{2}[\sin(x + y) + \sin(x - y)]$

**24.** Develop Sum to Product formulas by determining a sum of two trigonometric functions that is equal to each of the following.
(*Hint*: Use the identities developed in question 28 as well as half-angle identities.)

**a)** $\sin \frac{x + y}{2} \sin \frac{x - y}{2}$

**b)** $\sin \frac{x + y}{2} \cos \frac{x - y}{2}$

**c)** $\cos \frac{x + y}{2} \sin \frac{x - y}{2}$

**d)** $\cos \frac{x + y}{2} \cos \frac{x - y}{2}$

**25.** Prove each of the following:

**a)** $2 \sin 5x \cos 4x - \sin x = \sin 9x$

**b)** $\frac{\sin x + \sin y}{\cos x - \cos y} = -\cot \frac{x - y}{2}$

**c)** $\frac{\sin x - \sin 3x}{\cos x + \cos 3x} = -\tan x$

## Chapter 4: Challenge Questions

1. A lighthouse keeper sits at her post 30 metres above the water and sees a sailboat sailing directly towards her. As she watches, the angle of depression to the boat changes from 15° to 60°. How far has the boat travelled during that time?

2. The area of a circular sector that has been cut in a hay field is approximately 26 metres squared. What are the possible measures for the radius and central angle of the sector?

3. A child spins a hoop about her foot at an angular velocity of 4.1 radians per second. Determine the linear velocity of a point on the outside of the hoop if the diameter of the hoop is 85 centimetres from the centre of the rotating object.

4. The formula $C = 2\pi r \cos L$, where $r$ is the radius of the Earth, and $L$ is the latitude, determines the distance around Earth along a given latitude. How does the distance along a given latitude change as you go from the equator (0°) to each of the poles (90°)?

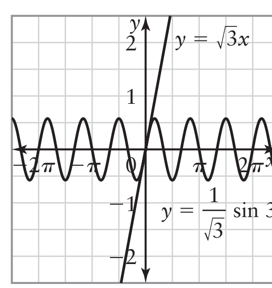
5. A highway curve is banked so that vehicles can safely negotiate the turn. To determine an appropriate banking angle $\theta$ of a car making a turn of radius $r$ metres at a velocity of $v$ metres per second, engineers use the equation $\tan \theta = \frac{v^2}{gr}$. The variable $g$ is the acceleration due to gravity, which is a constant at $9.8 \text{m/s}^2$. A new exit ramp is designed with a radius of 210 metres. If the speed limit on the curve is 40 km/h, at what angle should the curve be banked? What adjustments should be made to the speed if the angle is increased?

6. The hypotenuse of a right-triangle is 10 centimetres in length. Find the measures of the angles in the triangle that will maximize the perimeter.

7. In calculus, you will be introduced to the difference quotient, $\frac{f(x+h) - f(x)}{h}$.
   a) Let $f(x) = \cos x$. Write and expand an expression for the difference quotient.
   b) Let $h = 0.1$. Evaluate the function.
   c) Graph the function you found in part b). What graph has a similar graph?

8. A position function of a particle that moves along the $x$-axis is given by $x = 2\pi t + \cos 2\pi t$, $0 \le t \le 5$.
   a) Graph the function using graphing technology.
   b) When is the particle moving left and right?
   c) When is the particle at rest?
   d) Determine the average rate of change over the first five seconds.
   e) Estimate the velocity of the particle for each second over the given interval.

9. What angle does the tangent line to the curve $y = \frac{1}{\sqrt{3}} \sin 3x$ at the origin make with the $x$-axis?




10. A voltage $V$ being supplied to an electrical circuit at time $t$ is $V = 100 \sin 50t + 50 \sin 100t$. Find the maximum and minimum values of $V$ over one period.

## Chapter 4 Checklist

By the end of this chapter, I will be able to:

- Convert degree measure to radian measure, in exact and approximate formats
- Convert radian measure to degree measure, in exact and approximate formats
- Determine arc lengths
- Determine angular velocity using both degree measure and radian measure
- Determine exact values for trigonometric ratios of special angles
- Use technology to determine values for trigonometric ratios
- Use equivalent trigonometric expressions to simplify calculations
- Apply the compound-angle formulas for sine and cosine
- Solve problems involving compound-angle formulas
- Prove trigonometric identities