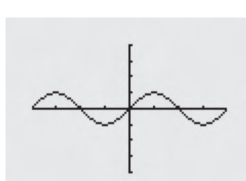
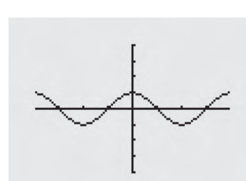
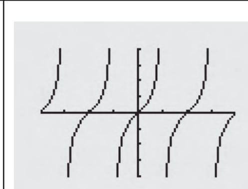
Chapter 5 Trigonometric Functions

5.1 Graphs of Sine, Cosine, and Tangent Functions

KEY CONCEPTS

Characteristics of the Sine, Cosine, and Tangent Functions

	Sine	Cosine	Tangent
Domain	Set of all real numbers	Set of all real numbers	Set of all real numbers, except for integral multiples of $\frac{\pi}{2}$
Range	All real numbers from −1 to 1, inclusive	All real numbers from −1 to 1, inclusive	All real numbers
Period	2π	2π	π
Symmetry	Odd function, that is $\sin(-x) = -\sin(x)$	Even function, that is $\cos(-x) = -\cos(x)$	Odd function, that is $\tan(-x) = -\tan(x)$
***X*-intercept**	$..., -2\pi, -\pi, 0, \pi, 2\pi, 3\pi, ...$	$x = ..., \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, ...$	$x = ..., -2\pi, -\pi, 0, \pi, 2\pi, ...$
***Y*-intercept**	0	1	0
Maximum Value	1, which occurs at $x = ..., \frac{-3\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, ...$	1, which occurs at $x = ..., -2\pi, 0, 2\pi, 4\pi, 6\pi, ...$	
Minimum Value	−1, which occurs at $x = ..., \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}, \frac{11\pi}{2}, ...$	−1, which occurs at $x = ..., -\pi, \pi, 3\pi, 5\pi, ...$	
Asymptotes			Vertical asymptotes occur at $x = ..., \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{5\pi}{2}, \frac{9\pi}{2}, ...$
Sketch of Graph			

A

1. Determine the maximum value, the minimum value, and the values of x where they occur for each function on the interval $x \in [-2\pi, 2\pi]$.

a) $y = \sin x - 5$

b) $y = \cos x + 7$

c) $y = \sin x + 1$

d) $y = \cos x - 3$

2. Sketch a graph of each function in question 1 on the interval $x \in [-2\pi, 2\pi]$.

3. Write an equation for each function.

a) Sine function with an amplitude of $\frac{1}{2}$

b) Cosine function with an amplitude of 3

c) Sine function with an amplitude of 7, reflected in the x-axis

d) Cosine function with an amplitude of $\frac{1}{4}$, reflected in the x-axis

4. Write an equation for each function.

a) Sine function with a phase shift of $\frac{2\pi}{3}$

b) Cosine function with a phase shift of $\frac{4\pi}{5}$

c) Sine function with a phase shift of $\frac{-5\pi}{7}$

d) Cosine function with a phase shift of $\frac{-11\pi}{7}$

5. Write an equation for each function.

a) Sine function with a period of 4π

b) Cosine function with a period of $\frac{\pi}{2}$

c) Sine function with a period of $\frac{3\pi}{2}$

d) Cosine function with a period of $\frac{5\pi}{4}$

B

6. A sine function has an amplitude of 5 and a period of 4π.

a) Write an equation of the function in the form $y = a \sin kx$.

b) Graph the function over two cycles.

7. A cosine function has a maximum value of 1 and a minimum value of -5.

a) Determine the amplitude of the function.

b) Determine the vertical translation.

c) Write an equation of the function in the form $y = a \cos x + c$.

d) Graph the function over two cycles.

8. One cycle of a sine function begins at $x = \frac{-2\pi}{3}$ and ends at $x = \frac{\pi}{3}$.

a) Determine the period of the function.

b) Determine the phase shift of the function.

c) Write the equation of the function in the form $y = \sin k(x - d)$.

d) Graph the function over two cycles.

☆**9.** When seated, an average adult breathes in and out every 4 s. The average minimum amount of air in the lungs is 0.08 L, and the average maximum amount of air in the lungs is 0.82 L. Suppose the lungs have a minimum amount of air at $t = 0$, where t is the time, in seconds.

a) Write a function that models the amount of air in the lungs.

b) Graph the function.

c) Determine the amount of air in the lungs at 5.5 s.

10. A boat is bobbing up and down on the water. The distance between the boat's highest and lowest points is 4 m. The boat moves from its highest point to its lowest point and back to its highest point every 30 s. Write a cosine function that models the movement of the boat in relation to the equilibrium point.

11. The mean average monthly temperature in Toronto, Ontario, is 8.6 ºC. The temperature fluctuates 13.1 degrees above and below the mean temperature. If $t = 1$ represents January, the phase shift of the sine function is 4.

a) Write a model for the average monthly temperature in Toronto.

b) According to your model, what is the average temperature in March?

c) According to your model, what is the average temperature in August?

12. A weight hanging from a spring is set in motion by an upward push. It takes 10 s for the weight to complete one cycle of moving from its equilibrium position to 12 cm above, then dropping to 12 cm below its equilibrium position, and finally returning to that original position.

a) Find a sinusoidal function to represent the moving weight.

b) Sketch the graph of the function you wrote in part a).

c) Use the function determined in part a) to predict the height of the weight after 7 s.

d) In the first 10 s, when will the height of the weight be 9 cm below the equilibrium point?

13. A Ferris wheel completes one revolution every 90 s. The cars reach a maximum of 55 m above the ground and a minimum of 5 m above the ground. The height, *h,* in metres, above the ground can be modelled using a sine function of the form $y = a \sin kt + c$, where t represents time, in seconds.

a) Determine the amplitude of the function.

b) Determine the vertical translation of the function.

c) What is the desired period of the function?

d) Determine the value of k that results in the period desired in part c).

C

14. Use Technology

a) Determine the form of the graph of $y = \tan x - 3$. Verify your answer using graphing technology.

b) Determine the form of the graph of $y = 2 \tan x$. Verify your answer using graphing technology.

c) Determine the form of the graph of $y = \tan\left(x - \frac{\pi}{6}\right)$. Verify your answer using graphing technology.

d) Determine the form of the graph of $y = \tan 4x$. Verify your answer using graphing technology.

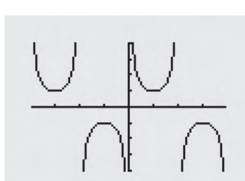
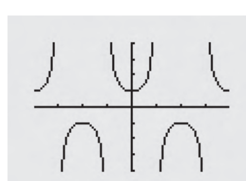
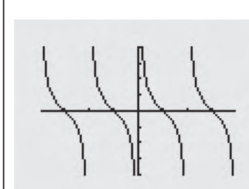
☆**15.** Consider the graph of $y = \frac{1}{4}x$.

a) How many times will this graph intersect the graph of $y = \cos x$ if both are graphed on the same set of axes with no limits on the domain? Justify your answer.

b) Illustrate your answer graphically.

KEY CONCEPTS

Characteristics of the Reciprocal Trigonometric Functions

	Cosecant	**Secant**	**Cotangent**
Domain	Set of all real numbers except πn, where n is an integer	Set of all real numbers except $\frac{\pi}{2}n$, where n is an odd integer	Set of all real numbers, except for multiples of πn, where n is an integer
Range	All real numbers greater than or equal to 1, or less than or equal to -1	All real numbers greater than or equal to 1, or less than or equal to -1	All real numbers
Period	2π	2π	π
***X*-intercept**	None	None	The x-intercepts are located at $\frac{\pi}{2}n$, where n is an odd integer
***Y*-intercept**	None	1	None
Positive Turning Point	$y = 1$ when $x = \frac{\pi}{2} + 2\pi n$, where n is an integer	$y = 1$ when $x = \pi n$, where n is an even integer	
Negative Turning Point	$y = -1$ when $x = \frac{3\pi}{2} + 2\pi n$, where n is an integer	$y = -1$ when $x = \pi n$, where n is an odd integer	
Asymptotes	Vertical asymptotes occur at $x = \pi n$, where n is an integer	Vertical asymptotes occur at $x = \frac{\pi}{2}n$, where n is an odd integer	Vertical asymptotes occur at $x = \pi n$, where n is an integer
Sketch of Graph			

A

1. **Use Technology** Use graphing technology to determine all values of x in the interval $[0, 2\pi]$ such that $\csc x = -7$. Round your answers to two decimal places.

2. **Use Technology** Use graphing technology to determine all values of x in the interval $[0, 2\pi]$ such that $\sec x = 5$. Round your answers to two decimal places.

3. **Use Technology** Use graphing technology to determine all values of x in the interval $[0, 2\pi]$ such that $\cot x = -12$. Round your answers to two decimal places.

B

4. **a)** Describe the graph of $y = \csc x$ in terms of a transformation of the graph of $y = \sec x$.

   b) Is there more than one transformation that will accomplish this? Explain your answer.

5. **Use Technology**

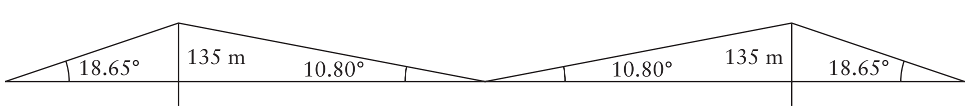
   a) Use graphing technology to show that there is at least one value of x such that $\sec x = \cos^{-1} x$.

   > A reminder: $y = \cos^{-1} x$ is the inverse function of $y = \cos x$.

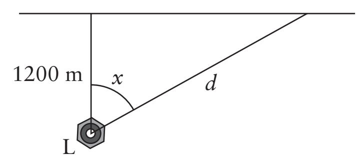
   b) Determine a value of x that makes the equation in part a) true.

   c) Verify your value from part b).

6. One of the world's longest suspension bridges is across the Humber Estuary in England. The towers of this bridge reach about 135 m above the level of the deck. The angles of elevation of the towers seen from the centre of the bridge and from either end are 10.80° and 18.65°, respectively. How long is the Humber Bridge?



☆7. A lighthouse with a rotating beam is located 1200 m south of a coastal cliff that runs west to east.



   a) Determine a relation for the distance from the lighthouse to the point where the light strikes the cliff in terms of the angle of rotation x.

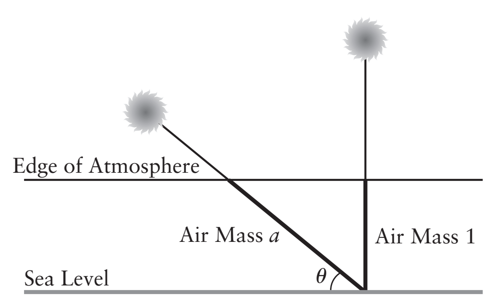
   b) Determine an exact expression for this distance when $x = \frac{7\pi}{12}$.

   c) Sketch a graph of the relation in part a) on the interval $x \in [0, \pi]$.

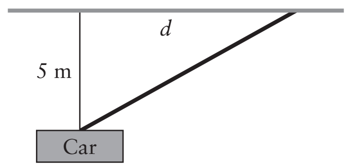
8. One factor that affects daily temperature is the distance sunlight must pass through the atmosphere before it reaches Earth. To express this distance, the term air mass has been coined. Air mass 1 is the distance sunlight must pass through the atmosphere to reach sea level when the Sun is directly overhead. Air mass 2 is two times air mass 1; air mass 3 is three times air mass 1; and so on.

   a) Write an equation that expresses the air mass, a, as a function of the angle of inclination of the Sun, θ.

   b) Graph the function in part a).



☆**9.** A police cruiser is parked such that the beacon on its roof is 5 m from a brick wall. As the beacon rotates, a spotlight moves along the wall. Assuming that the beacon makes one complete rotation in 3 s, find an equation that expresses the distance, d, in metres, as a function of time.



10. a) Explain the difference between $\cot \frac{1}{\sqrt{3}}$ and $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$.

b) Determine a value for each expression in part a).

11. Use Technology Use graphing technology to determine whether it is reasonable to conjecture that $\sec^2 x = \csc^2 x$. If so, justify your conclusion. If not, use the graphs to determine a similar equation that is an identity.

12. A boat is in the water 150 m from a straight shoreline. There is a rotating beam on the boat.

a) Determine a reciprocal trigonometric relation for the distance, d, from the boat to where the light hits the shoreline in terms of the angle of rotation, x.

b) Determine an exact expression for the distance in part a) when $x = \frac{\pi}{6}$.

c) Determine an approximate value, to the nearest tenth of a metre, for the distance from part b).

d) Sketch a graph of the relation in part a) in the interval $x \in \left[0, \frac{\pi}{2}\right]$.

13. a) Sketch the function $y = \sec x$.

b) Determine the shape of each function, and then check by graphing.

i) $y = 2 \sec x$

ii) $y = \sec 3x$

iii) $y = \sec x - 4$

iv) $y = \sec(x + 2)$

14. a) Sketch the function $y = \cot x$.

b) Determine the shape of each function, and then check by graphing.

i) $y = 4 \cot x$

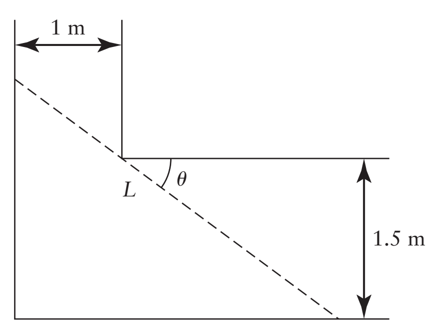
ii) $y = \cot 3x$

iii) $y = \cot x + 5$

iv) $y = \cot(x - 2)$

C

15. Suppose that you are carrying a fully-extended extension ladder horizontally and have to turn from a corridor 1 m wide to a corridor 1.5 m wide. To navigate the corner, you must shorten the ladder by reducing the extension as you make the turn. See illustration.



a) Show that the length, L, in metres, of the ladder as a function of the angle θ is $L = \sec \theta + 1.5 \csc \theta$.

b) Graph L, $0 < \theta < \frac{\pi}{2}$.

c) At what value of θ is L the least?

d) If the least value of L represents the maximum length that the ladder can be to get around the corner, how long is the ladder?

KEY CONCEPTS

The transformation of a sine or cosine function $f(x)$ has the general form $g(x) = af[k(x - d)] + c$.

$\lvert a\rvert > 1$	is a vertical stretch
$0 > \lvert a\rvert > 1$	is a vertical compression
$a < 0$	is a reflection in the x-axis
$\lvert k\rvert > 1$	is a horizontal compression
$0 < \lvert k\rvert < 1$	is a horizontal expansion
$k < 0$	is a reflection in the y-axis
$d > 0$	is a phase shift to the right $\lvert d\rvert$ units
$d < 0$	is a phase shift to the left $\lvert d\rvert$ units
$c > 0$	is a vertical translation up $\lvert c\rvert$ units
$c < 0$	is a vertical translation down $\lvert c\rvert$ units

The period of the transformed function is given by $\frac{2\pi}{k}$.

A

1. Determine the amplitude and period of each sinusoidal function. Then, transform the graph of $y = \sin x$ to sketch a graph of each function.

a) $y = 4 \sin 9x$

b) $y = -2 \cos \frac{1}{2}x$

c) $y = 7 \sin \frac{1}{4}x$

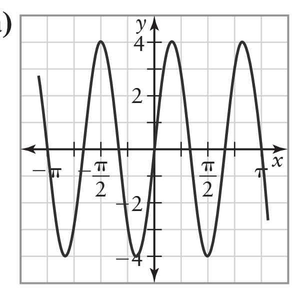
d) $y = \frac{1}{2} \cos 6x$

e) $y = -3 \sin \frac{3}{4}x$

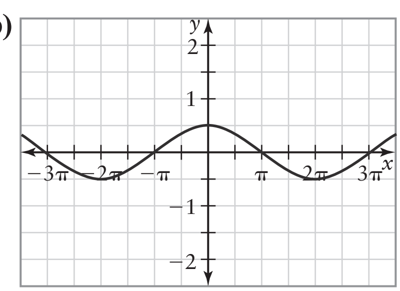
f) $y = \frac{3}{4} \cos \frac{4}{5}x$

2. Determine the equation in the form $y = a \sin kx$ or $y = a \cos kx$ for each graph.

a)



b)

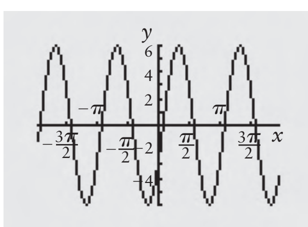


3. Consider the function $y = -\frac{1}{2}\sin\left[3\left(x + \frac{\pi}{6}\right)\right] - 5$.
   a) What is the amplitude?
   b) What is the period?
   c) Describe the phase shift.
   d) Describe the vertical translation.
   e) Sketch a graph of the function for two cycles.

4. Consider the function $y = 7\cos\left[\frac{1}{2}\left(x - \frac{\pi}{4}\right)\right] + 3$.
   a) What is the amplitude?
   b) What is the period?
   c) Describe the phase shift.
   d) Describe the vertical translation.
   e) Sketch a graph of the function for two cycles.

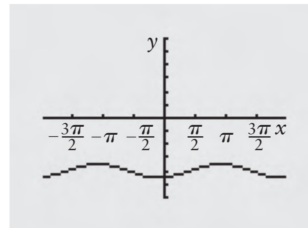
B

5. Model the graph shown using a sine function.



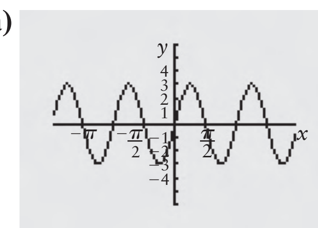
   a) From the graph, determine the amplitude, the period, the phase shift, and the vertical translation.
   b) Write an equation for the function.
   c) Graph the function from part b).
   d) Compare the graph in part c) to the given graph, and verify that the graphs match.

6. Model the graph shown using a cosine function.

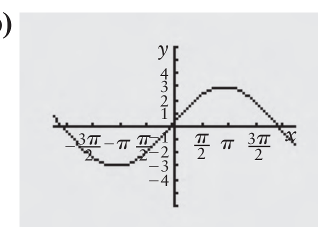


   a) From the graph, determine the amplitude, the period, the phase shift, and the vertical translation.
   b) Write an equation for the function.
   c) Graph the function from part b).
   d) Compare the graph in part c) to the given graph, and verify that the graphs match.

7. Determine an equation for each sine function. Check by graphing.
   a)



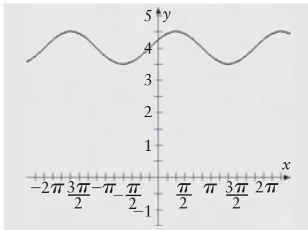
   b)



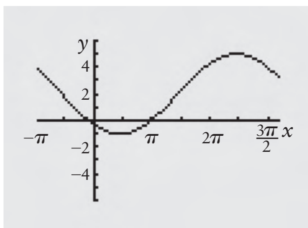
8. A cosine function has a maximum value of 4, a minimum value of −6, a period of $\frac{\pi}{2}$, and a phase shift of $\frac{\pi}{6}$ rad to the right.
   a) Write an equation for the function.
   b) Graph the function and verify that it has the properties given.

9. Determine an equation for each cosine function. Check by graphing.

a)



b)



10. Write the equation of a sine function that has the given characteristics.

a) Amplitude 2
Period π
Phase shift $\frac{\pi}{2}$ to the left

b) Amplitude 3
Period $\frac{\pi}{2}$
Phase shift $\frac{\pi}{3}$ to the right

11. Write the equation of a cosine function that has the given characteristics.

a) Amplitude $\frac{1}{2}$
Period 4π
Phase shift $\frac{\pi}{6}$ rad to the left

b) Amplitude 4
Period 4π
Phase shift π rad to the right
Vertical translation three units down

12. A sine function has a maximum value of 11, a minimum value of -1, a phase shift of $\frac{2\pi}{3}$ rad to the left, and a period of π.

a) Write an equation for the function.

b) Graph the function and verify that it has the properties given.

13. A cosine function has a maximum value of -3, a minimum value of -5, a phase shift of $\frac{\pi}{4}$ rad to the right, and a period of 4π.

a) Write an equation for the function.

b) Graph the function and verify that it has the properties given.

14. The water depth in a harbour is 21 m at high tide and 11 m at low tide. One cycle is completed approximately every 12 h.

a) Find an equation for the water depth as a function of the time, t hours, after low tide.

b) Draw a graph of the function for 48 h after low tide, which occurred at 2:00 p.m.

☆15. A Ferris wheel with a radius of 14 m makes one complete revolution every 30 s. The bottom of the wheel is 2.5 m above the ground.

a) Draw a graph to show how a person's height above the ground varies with time during three revolutions, starting when the person gets on the Ferris wheel at its lowest point.

b) Find an equation for the graph.

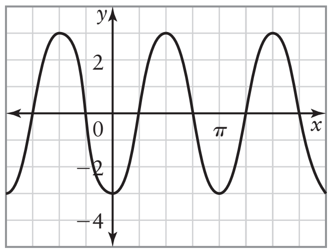
c) Predict how the graph and the equation will change if the Ferris wheel turns more quickly.

16. An object suspended from a spring is oscillating up and down. The distance from the high point to the low point is 45 cm, and the object takes 4 s to complete 5 cycles. For the first few cycles, the motion is modelled by a sine function, where the distance from the mean position, $d(t)$, is measured in centimetres, with respect to time, in seconds.

a) Sketch a graph of this function for two cycles.

b) Write an equation that describes the distance of the object from its mean position as a function of time.

17. Model the graph shown using a sine function.



a) From the graph, determine the amplitude, the period, the phase shift, and the vertical translation.

b) Write an equation for the function.

c) Graph the function you found in part b) and compare it to the given graph. Verify that the two graphs match.

d) Determine a model for the above graph using a cosine function.

e) Verify that the graph of your equation in part d) matches the given graph.

C

18. In the theory of biorhythms, a sine function of the form $P = 50 \sin \omega t + 50$ is used to measure the percent, P, of a person's potential at a time, t, where t is measured in days and $t = 0$ is the person's birthday. Three characteristics are commonly measured:

Physical potential: period of 23 days
Emotional potential: period of 28 days
Intellectual potential: period of 33 days

a) Find ω for each characteristic.

b) Graph all three functions.

c) Is there a time, t, when all three characteristics have 100% potential? When is it?

d) Suppose that you are 20 years old today ($t = 7305$ days). Describe your physical, emotional, and intellectual potential for the next 30 days.

19. Suppose that the length of time between consecutive high tides is approximately 12.5 h. On June 28, high tide occurred at 3:38 a.m. (3.633 3 h) and low tide occurred at 10:08 a.m. (10.133 3 h). Water heights are measured as the amounts above or below the mean low water mark. The height of the water at high tide was 8.2 m and the height of the water at low tide was −0.6 m.

a) Approximately when will the next high tide occur?

b) Find the sinusoidal function of the form $y = a \sin k(x - d) + c$ that fits the data.

c) Draw a graph of the function found in part b).

d) Use the function found in part b) to predict the height of the water at the next high tide.

☆**20.** Sales of Ocean King boogie boards fluctuate sinusoidally from a low of 50 units per week each February 1($t = 1$) to a high of 350 units per week each August 1($t = 7$). Model the weekly sales, $s(t)$, of Ocean King boogie boards, where t is time in months.

a) Find an equation for the boogie board sales as a function of the time, t months.

b) Draw a graph of the function for a two-year period.

21. A Ferris wheel at an amusement park completes one revolution every 40 s. The wheel has a diameter of 16 m and its centre is 12 m above the ground. A rider boards on the lowest point on the wheel.

a) Model the height, h, in metres, above the ground of a rider using a sine function in the form $h = a \sin[k(t - d)] + c$, where t represents the time, in seconds.

b) Sketch a graph of the model over two cycles.

KEY CONCEPTS

Trigonometric equations can be solved

- algebraically by hand or
- graphically with technology

To solve a trigonometric equation that is not an identity, find all values of the variable that make the equation true.

There are often multiple solutions. Ensure that you find all solutions that lie in the domain of interest.

Factoring can often solve quadratic trigonometric equations.

A

1. Determine approximate solutions for each equation in the interval $x \in [0, 2\pi]$ to the nearest hundredth radian.

a) $\sin x - \frac{1}{2} = 0$

b) $\cos x + 0.9 = 0$

c) $\tan x - 1 = 0$

d) $\sec x + 5 = 0$

e) $3 \cot x + \sqrt{3} = 0$

f) $\csc x + 3 = 0$

2. Determine exact solutions for each equation in the interval $x \in [0, 2\pi]$.

a) $\sin x + \frac{\sqrt{2}}{2} = 0$

b) $\cos x + \frac{\sqrt{3}}{2} = 0$

c) $\tan x - \sqrt{3} = 0$

d) $\cot x + 5 = 0$

3. Determine approximate solutions for each equation in the interval $x \in [0, 2\pi]$ to the nearest hundredth of a radian.

a) $\sin^2 x - 0.49 = 0$

b) $\cos^2 x - \frac{64}{144} = 0$

c) $\tan^2 x - 121 = 0$

d) $\sec^2 x - 4.9 = 0$

e) $\cot^2 x - 3.6 = 0$

4. Determine exact solutions for each equation in the interval $x \in [0, 2\pi]$.

a) $\sin^2 x - \frac{3}{4} = 0$

b) $\cos^2 x - \frac{1}{4} = 0$

c) $2 \tan^2 x - 12 = 0$

d) $2 \csc^2 x - 8 = 0$

5. Use Technology Solve the following equations on the interval $x \in [0, 2\pi]$ using graphing technology.

a) $x + 5 \cos x = 0$

b) $x - 4 \sin x = 0$

c) $\sin x + \cos x = x$

d) $x^2 - 2 \cos x = 0$

6. Solve the following equations on the interval $x \in [0, 2\pi]$.

a) $\sin x - \sin x \tan x = 0$

b) $2 \cos^2 x + \cos x = 0$

c) $2 \sin^2 x - \sin x - 1 = 0$

d) $6 \cos^2 x + \cos x - 1 = 0$

e) $8 \cos^2 x + 14 \cos x = -3$

f) $\csc x - 3 \csc x \sec x = 0$

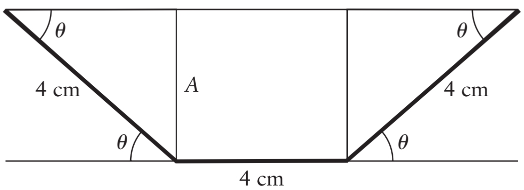
g) $\sec^2 x + 2 \sec x - 8 = 0$

h) $3 \sec^2 x - 5 \sec x - 2 = 0$

i) $\csc^2 x + 5 \csc x - 6 = 0$

B

☆ **7.** A rain gutter is to be constructed of aluminum sheets 12 cm wide. Lengths 4 cm from each edge are marked off, then each length is bent up at an angle, θ. See the illustration. The area, A, of the opening as a function of θ is given by $A = 16 \sin \theta(\cos \theta + 1)$, $0° < \theta < 90°$.



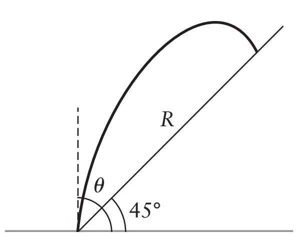
a) In Calculus, you would be asked to find the angle, θ, that maximizes A by solving the equation $2 \cos^2 \theta + \cos \theta - 1 = 0$, $0° < \theta < 90°$. Solve this equation for θ.

b) What is the maximum area, A, of the opening?

c) Graph A, $0° < \theta < 90°$, and find the angle, θ, that maximizes the area, A. Also, find the maximum area. Compare the results with the answers found earlier.

8. An object is propelled upward at an angle, θ, $45° < \theta < 90°$, to the horizontal with an initial velocity of v_0 metres per second from the base of a plane that makes an angle of 45° with the horizontal. See the illustration. If air resistance is ignored, the distance, R, that the object travels up the inclined plane is given by

$$R = \frac{{v_0}^2\sqrt{2}}{32}(2 \sin \theta \cos \theta - 2 \cos^2 \theta).$$



a) In Calculus, you would be asked to find the angle, θ, that maximizes R by solving the equation $2 \sin \theta \cos \theta + 1 - 2 \sin^2 \theta = 0$. Solve this equation for θ.

b) What is the maximum distance, R, if $v_0 = 32$ m/s?

c) Graph R, $45° < \theta < 90°$, and find the angle, θ, that maximizes the distance, R. Also, find the maximum distance. Use $v_0 = 32$ m/s. Compare the results with the answers found earlier.

9. The horizontal distance that a projectile will travel in the air is given by the equation $r = \frac{v^2}{g} \sin 2\theta$, where r is the range, in metres; v is the launch speed, in metres per second; g is the acceleration due to gravity, 9.8 m/s^2; and θ is the angle above the horizontal that the projectile is aimed at.

a) If you can throw a baseball with an initial speed of 34.8 m/s, at what angle of elevation, θ, should you direct the throw so that the ball travels a distance of 107 m before striking the ground?

b) Determine the maximum distance that you can throw the ball.

C

10. Determine the solutions for $2 \sec^2 x - \tan^4 x = -1$ in the interval $x \in [-2\pi, 2\pi]$.

☆ **11.** Determine the solutions for $\frac{1 + \sec x}{\sec x} = \frac{\sin^2 x}{1 - \cos x}$ in the interval $x \in [-2\pi, 2\pi]$.

12. In the study of heat transfer, the equation $x + \tan x = 0$ occurs.

a) Graph $y = -x$ and $y = \tan x$ for $x \geq 0$.

b) Find the first two positive solutions of $x + \tan x = 0$, correct to two decimal places.

KEY CONCEPTS

- The instantaneous rates of change of a sinusoidal function follow a sinusoidal pattern.
- Many real-world processes can be modelled with a sinusoidal function, even if they do not involve angles.
- Modelling real-world processes usually requires transformations of the basic sinusoidal functions.

A

☆ **1.** The position of a particle as it moves horizontally is described by the given equation $s(t) = 12 \sin \frac{\pi t}{90} + 15$. If s is the displacement, in metres, and t is the time, in seconds, do the following:

a) Determine the average Rate of Change of $s(t)$ in the following time intervals, rounded to three decimal places.

i) 5 s to 10 s

ii) 9 s to 10 s

iii) 9.9 s to 10 s

iv) 9.99 s to 10 s

b) Estimate a value for the instantaneous rate of change of $s(t)$ at $t = 10$ s.

c) What physical quantity does this instantaneous rate of change represent?

d) When you expect the instantaneous rate of change of $s(t)$ to be the same as at $t = 14$ s?

B

2. The water depth in a harbour is 8 m at low tide and 20 m at high tide. One cycle is completed approximately every 12 h.

a) Find an equation for the water depth, $d(t)$, in metres, as a function of the time, t, after high tide, which occurs at 3:00 a.m.

b) Draw a graph of the function for 48 h after high tide.

c) Determine the average rate of change of $d(t)$ in the following time intervals, rounded to three decimal places.

i) 8:00 a.m. to 9:00 a.m.

ii) 8:30 a.m. to 9:00 a.m.

iii) 8:40 a.m. to 9:00 a.m.

iv) 8:50 a.m. to 9:00 a.m.

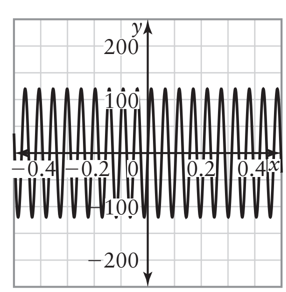
v) 8:55 a.m. to 9:00 a.m.

vi) 8:59 a.m. to 9:00 a.m.

d) Estimate a value for the instantaneous rate of change of $d(t)$ at t = 9:00 a.m.

e) What physical quantity does this instantaneous rate of change represent?

3. An electromotive force, E, in volts in a certain ac circuit can be modelled by $E = a \sin kt$. A graph of electromotive force versus time is shown.



a) Use the graph to determine values for a and k.

b) Determine an equation for the electromotive force.

4. The average monthly temperatures in Brockton, Ontario, are represented in the following table. Since the data represent average monthly temperatures collected over many years, the data will not vary much from year to year and so will essentially repeat each year. In other words, the data are periodic.

Month	Average Monthly Temperature (°C)
January	−1.3
February	0.8
March	3.9
April	8.9
May	14.0
June	19.4
July	23.1
August	21.9
September	16.8
October	10.8
November	3.9
December	−0.6

a) Write a sine function to model the data.

b) Make a scatter plot of the data. Then, graph your model on the same set of axes. How well does it fit the data?

c) Check your model using a sinusoidal regression. How does the regression equation compare with the model?

d) Select a point on the graph where the instantaneous rate of change of the average monthly temperature appears to be a maximum.

e) Estimate the instantaneous rate of change of the average monthly temperature at this point.

f) What does this instantaneous rate of change of the average monthly temperature represent?

5. The following table shows, by year, the number, in millions, of unemployed people in a country's labour force.

Year	Unemployed (millions)
1996	9.539
1997	9.312
1998	9.237
1999	8.237
2000	7.701
2001	7.528
2002	8.047
2003	9.628
2004	10.613
2005	9.940
2006	8.996
2007	9.404

a) Make a scatter plot of the data.

b) Use the table and the graph to write a sinusoidal function to model the data.

c) Graph your model on the same set of axes as the scatter plot. Comment on the fit.

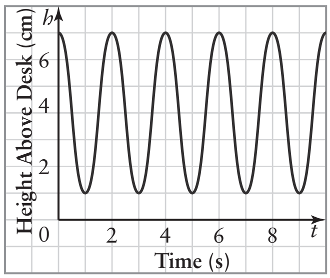
d) Check your model using a sinusoidal regression. How does the regression equation compare with the model?

e) Refer to the graph and select a point on the graph where the instantaneous rate of change of the population appears to be a minimum.

f) Estimate the instantaneous rate of change of the population at this point.

g) What does the instantaneous rate of change of the population represent?

6. A weight is suspended on a spring and set in motion such that it bobs up and down vertically. The graph shows the height, h, in centimetres, of the weight above a desk after time, t, in seconds. Use the graph to determine a model of the height versus time using a sine function.



a) Select a point on the graph where the instantaneous rate of change of the height appears to be a maximum.

b) Estimate the instantaneous rate of change of the height at this point.

c) What does this instantaneous rate of change of the height represent?

7. The height, h, in metres, of a seat above the ground as a Ferris wheel turns can be modelled using the function $h = 18 \sin\left(\frac{\pi t}{60}\right) + 20$, where t is the time, in seconds.

a) Determine the average rate of change of h in the following time intervals, rounded to three decimal places.

i) 6 s to 12 s

ii) 11 s to 12 s

iii) 11.9 s to 12 s

iv) 11.99 s to 12 s

b) Estimate a value for the instantaneous rate of change of h at $t = 12$ s.

c) Would you expect the instantaneous rate of change to be the same at $t = 3$ s? Justify your answer.

C

8. Sketch a graph of the inverse trigonometric relation $y = \cos^{-1} x$ such that the range covers the interval $[0, 2\pi]$.

a) Is this relation a function in this range? If so, explain how you know. If not, show how it can be made into a function by restricting the range.

b) Determine a value of x where the instantaneous rate of change appears to be a maximum.

c) Estimate the instantaneous rate of change for the value of x in part b).

9. Sketch the graph of the secant function on the interval $x \in [0, 2\pi]$.

a) For what values of x does the instantaneous rate of change appear to be equal to 0? reach a maximum value? reach a minimum value?

b) Plot the instantaneous rates of change on the same set of axes as the secant function.

c) Describe the pattern formed by the instantaneous rate of change of the secant function.

10. Consider the function $f(x) = \sin \theta$.

a) Find the instantaneous rate of change for the points $\theta = -\pi, \frac{-\pi}{2}, 0, \frac{\pi}{2}, \pi$

b) On a new graph, plot your answers to (θ, instantaneous rate of change). What function have you graphed?

c) Based on your result, what statement can be made about the instantaneous rate of change of $f(x) = \sin \theta$?

Chapter 5: Challenge Questions

C1. A security camera is scanning a long, straight fence along one side of a military base. The camera is located 10 m from the centre of the fence. If d represents the distance along the fence from the centre and t is time, in seconds, then $d = 10 \tan \frac{\pi}{40}t$ models the point being scanned.

a) Graph the equation for $-20 \le t \le 20$.

b) Find the location that the camera is scanning at 3 s.

c) Find the location that the camera is scanning at 15 s.

C2. In Victoria, British Columbia, the first high tide was 3.99 m at 12:03 a.m. The first low tide of 0.55 m occurred at 6:24 a.m. The second high tide occurred at 12:19 p.m.

a) Find the amplitude of a sinusoidal function that models the tides.

b) Find the vertical shift of the sinusoidal function that models the tides.

c) What is the period of the sinusoidal function that models the tides?

d) Write a sinusoidal function to model the tides, using t to represent the number of hours, in decimals, since midnight.

e) According to your model, determine the height of the water at noon.

C3. Consider the graph of $y = \frac{x}{6}$.

a) How many times will this graph intersect the graph of $y = \cos x$ if both are graphed on the same set of axes with no limits on the domain? Justify your answer.

b) Illustrate your answers graphically.

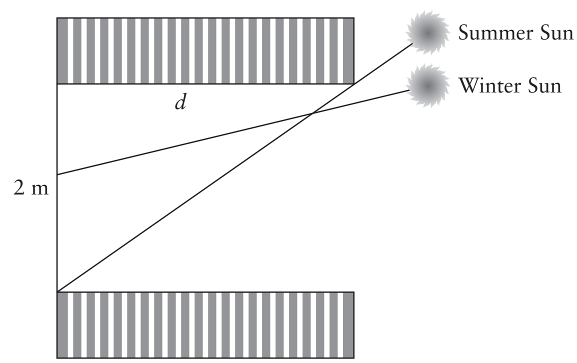
C4. Consider the graphs $y = \sin^{-1} x$ and $y = \cos^{-1} x$. Name the y coordinate of the points of intersection of the two graphs.

☆ **C5.** One energy-saving idea is to design houses with recessed windows on the side facing the sun. Sunlight will enter the window in winter when the sun's angle is low but it will be blocked in the summer when the angle is much higher. The highest angle of elevation of the Sun occurs at noon in mid-June when the angle of elevation, in degrees, is $23.5 + l$, where l is the latitude of the location.

a) Assume that the window shown is designed to block the sun's rays when the sun is at its highest elevation in the summer. Express the depth, d, in metres, of the recess as a function of latitude, l.

b) Draw the graph of d versus l.

c) Determine the value of d for your latitude.



C6. Use Technology The position of a particle as it moves horizontally is described by the equation $s = 2 \sin t - \cos t$, $t \in [0, 2\pi]$, where s is the displacement, in metres, and t is the time, in seconds.

a) Using technology, graph the position of the particle relative to time.

b) At what time or times during the cycle is the instantaneous rate of change of the position equal to 0? At what times is it a minimum? Justify your answers.

C7. A wave travelling on a guitar string can be modelled by the equation $D = 0.5\sin(6.5x)\sin(2500t)$, where D is the displacement, in millimetres, at the position x metres from the left end of the string at time t seconds. Find the first positive time when the point 0.5 m from the left end has a displacement of 0.01 mm.

C8. The equation $T(t) = 12.5 + 15.8\sin\left(\frac{\pi}{6}t - \frac{2\pi}{3}\right)$ models the average monthly temperature of Toronto, Ontario. In this equation, t denotes the number of months with January represented by 1. During which two months is the average temperature 12.5°C?

C9. Determine solutions for $2\tan^2 x - 3\sec x = 0$ in the interval $x \in [-2\pi, 2\pi]$.

C10. Determine solutions for $\frac{1}{\sin x - \cos x} = \cos x + \sin x$ in the interval $x \in [-2\pi, 2\pi]$.

C11. How many solutions in the interval $x \in [0, 2\pi]$ should you expect for the equation $a\sin(bx + d) + c = c + \frac{a}{2}$, if $a \neq 0$ and b is a positive integer?

C12. Determine an exact value for $\cos\left(\cos^{-1}\left(\frac{\sqrt{2}}{2}\right) - \frac{\pi}{2}\right)$.

C13. Engineers use the equation $\tan x = \frac{v^2}{224\,000}$ to calculate the angle at which a curved section of the highway should be banked. In the equation, x is the angle of the bank and v is the speed limit on the curve, in kilometres per hour.

a) Calculate the angle of the bank, to the nearest tenth of a degree, if the speed limit is 100 km/h.

b) The four turns at the Indianapolis Motor Speedway are banked at an angle of 9.2°. What is the maximum speed through these turns, to the nearest kilometre per hour?

C14. The following table gives the average population, in thousands, of a northern university town for each month throughout the year. The population is greater in the winter and smaller in the summer, and it repeats this pattern from year to year.

Month	Population (thousands)
January	10.5
February	9.3
March	7.8
April	6.0
May	4.9
June	4.5
July	4.7
August	5.8
September	7.6
October	9.4
November	10.6
December	10.9

a) Write a sine function to model the data.

b) Make a scatter plot of the data. Then, graph your model on the same set of axes. How well does it fit the data?

c) Check your model using a sinusoidal regression. How does the regression equation compare with the model?

d) At what time or times during the cycle is the instantaneous rate of change of the population equal to 0? At what times is it a maximum? Justify your answers.

Chapter 5: Checklist

By the end of this chapter, I will be able to:

- Determine amplitude, vertical translation, phase shift, and period of a sinusoidal function given its equation
- Determine the equation of a sinusoidal function given its amplitude, vertical translation, phase shift, and period
- Determine angles that have a given sine, cosine, tangent, cosecant, secant, or cotangent
- Solve linear and quadratic trigonometric equations
- Model real-world situations using reciprocal trigonometric equations
- Given a data set, model the data using a sinusoidal function
- Determine values for inverse trigonometric relations