Chapter 6 Exponential and Logarithmic Functions

6.1 The Exponential Function and Its Inverse

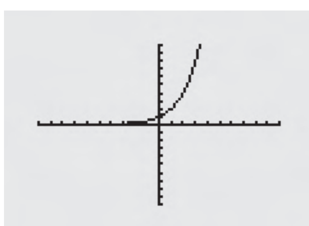
KEY CONCEPTS

An exponential function of the form $y = b^x$, $b > 0$, $b \neq 1$, has

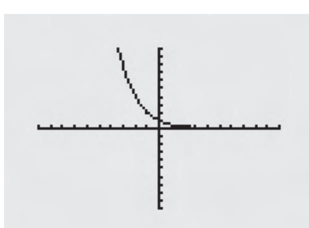
- a repeating pattern of finite differences
- a rate of change that is increasing proportionally to the function for $b > 1$
- a rate of change that is decreasing proportionally to the function for $0 < b < 1$

An exponential function of the form $y = b^x$, $b > 0$, $b \neq 1$,

- has domain $\{x \mid x \in \mathbb{R}\}$
- has range $\{y \in \mathbb{R} \mid y > 0\}$
- has y-intercept of 1
- has horizontal asymptote at $y = 0$
- is increasing on its domain when $b > 1$

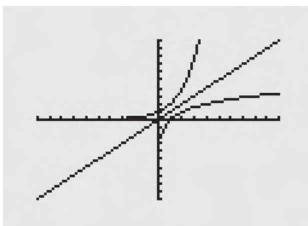


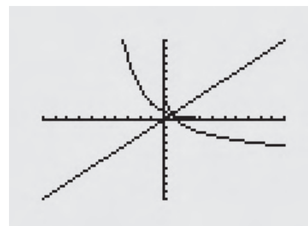
- is decreasing on its domain when $0 < b < 1$



The inverse of $y = b^x$ is a function that satisfies $x = b^y$. This function

- has domain $\{x \in \mathbb{R}, x > 0\}$
- has range $\{y \in \mathbb{R}\}$
- has x-intercept of 1
- has vertical asymptote at $x = 0$
- is a reflection of $y = b^x$ about the line $y = x$
- is increasing on its domain when $b > 1$
- is decreasing on its domain when $0 < b < 1$





A

1. Consider the function $y = 2^x$.

a) Identify the key features of the function

i) domain and range

ii) x-intercept, if it exists

iii) y-intercept, if it exists

iv) intervals for which the function is positive and intervals for which it is negative

v) intervals for which the function is increasing and intervals for which it is decreasing

vi) equation of the asymptote

b) Sketch a graph of the function.

c) On the same set of axes, sketch a graph of the inverse of the function.

d) Identify the key features, as in part a) i) to iv), of the inverse of the function.

2. a) Which of the following is an exponential function? Explain how you know.

i)

x	y
1	2
2	8
3	18
4	32

ii)

x	y
1	5
2	25
3	125
4	625

b) Write an equation for the data that is exponential.

3. Which of the following functions are exponential? For the exponential functions, write an equation to fit the data.

a)

x	y
1	3
2	5
3	7
4	9
5	11

b)

x	y
1	1.5
2	4.5
3	13.5
4	40.5
5	121.5

c)

x	y
1	5
2	11
3	21
4	35
5	53

d)

x	y
1	3.2
2	2.56
3	2.048
4	1.6384
5	1.3107

B

4. **Use Technology** Graph the function $y = 2.5^x$ over the domain $0 \le x \le 10$ using graphing technology.

   a) Determine the average rate of change of y with respect to x for each interval. Round your answers to two decimal places.

      i) $x = 1$ to $x = 2$

      ii) $x = 2$ to $x = 3$

      iii) $x = 3$ to $x = 4$

      iv) $x = 4$ to $x = 5$

   b) Estimate the instantaneous rate of change of y with respect to x at each of the endpoints in part a). Round your answers to two decimal places.

   c) Describe how these rates are changing over the given domains.

5. **Use Technology** Graph the function $y = \left(\frac{3}{4}\right)^x$ over the domain $-10 \le x \le 0$ using graphing technology.

   a) Determine the average rate of change of y with respect to x for each interval. Round your answers to two decimal places.

      i) $x = -5$ to $x = -4$

      ii) $x = -4$ to $x = -3$

      iii) $x = -3$ to $x = -2$

      iv) $x = -2$ to $x = -1$

      v) $x = -1$ to $x = 0$

   b) Estimate the instantaneous rate of change of y with respect to x at each of the endpoints in part a). Round your answers to two decimal places.

   c) Describe how these rates are changing over the given domains.

6. The atmospheric pressure, P, on a balloon or plane decreases with increasing height. This pressure, measured in millimetres of mercury, is related to the distance in kilometres, h, above sea level by the formula $P = 760(2.5)^{-0.145h}$.

   a) What is the pressure

   i) at sea level, $h = 0$?

   ii) at 500 m?

   iii) at 2 km?

   iv) at 5 km?

   b) Graph the function. Does it appear to be exponential? Explain your answer.

   c) Determine the average rate of change from sea level to 500 m.

   d) Estimate the instantaneous rate of change after

   i) 500 m

   ii) 1 km

7. Consider the function $f(x) = 3^x$.

   a) Sketch the graph of $f(x)$.

   b) Graph the line $y = x$ on the same set of Cartesian coordinates.

   c) Sketch the inverse of $f(x)$ on the same set of Cartesian coordinates.

   d) Identify the key features of both the graph and its inverse.

   i) domain and range

   ii) x-intercept, if it exists

   iii) y-intercept, if it exists

   iv) intervals for which the function is positive and intervals for which it is negative

   v) intervals for which the function is increasing and intervals for which it is decreasing

   vi) equation of the asymptote

8. Consider the function $g(x) = \left(\frac{1}{3}\right)^x$.

   a) Sketch the graph of $g(x)$.

   b) Graph the line $y = x$ on the same set of Cartesian coordinates.

   c) Sketch the inverse of $g(x)$ on the same set of Cartesian coordinates.

   d) Identify the key features of both the graph and its inverse.

   i) domain and range

   ii) x-intercept, if it exists

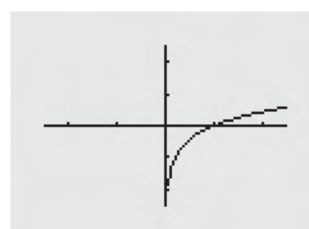
   iii) y-intercept, if it exists

   iv) intervals for which the function is positive and intervals for which it is negative

   v) intervals for which the function is increasing and intervals for which it is decreasing

   vi) equation of the asymptote

9. **a)** Copy the graph.



   b) Graph the line $y = x$ on the same grid.

   c) Graph the inverse of this function by reflecting in the line $y = x$.

   d) Write an equation for the inverse function.

10. The probability, P percent, of having an accident while driving a car is related to the alcohol level of the driver's blood by the formula $P = e^{kt}$, where k is the constant. Accident statistics show that the probability of an accident is 25% when the blood alcohol level is $t = 0.15$.

   a) Find k. Use $P = 25$.

   b) At what blood alcohol level is the probability of having an accident 50%?

11. The number of watts, w, provided by a space satellite's power supply after d days is given by the formula $w = 50(1.75)^{-0.004d}$.

a) How much power will be available after 30 days?

b) How much power will be available after 1 year (365 days)?

c) What is the power supply's average rate of change, in watts, between 30 days and 60 days? between 11 months and 12 months?

d) What is the value of the power supply's instantaneous rate of change, in watts, at 30 days? at 1 year?

12. A failure of o-ring seals caused the Challenger space shuttle disaster in 1986. After the disaster, there was a study of the 23 shuttle launches that had preceded the fatal flight. A mathematical model was developed involving the relationship between the Celsius temperature, x, around the o-rings and the number, y, of eroded or leaky primary o-rings. The model stated that $y = \dfrac{6}{1 + e^{-(5.085 - 0.114\,6x)}}$, where the number 6 indicates the 6 primary o-rings on the spacecraft.

a) What is the predicted number of eroded or leaky primary o-rings at a temperature of 37.5°C?

b) What is the predicted number of eroded or leaky primary o-rings at a temperature of 15.5°C?

c) What is the predicted number of eroded or leaky primary o-rings at a temperature of −7.5°C?

d) Use Technology Use a calculator to graph the equation. At what temperature is the predicted number of eroded or leaky o-rings 1? 2? 3?

e) What is the average rate of erosion of the o-rings from 37.7°C to 15.5°C? from 15.5°C to −7.5°C?

13. The deeper you are under water, the less sunlight reaches you. The percent of sunlight, P, that reaches a depth d, in metres, can be modelled by the function. $P = 100\,(0.85)^d$

a) Sketch a graph of P for the interval $[0, 10]$.

b) Sketch the inverse of P.

c) Use your graph of the inverse of P to determine the depth at which the percentage of sunlight is 50%. Test your value for depth in the original equation for P.

d) Use your graph of P to calculate the average rate at which the sunlight is absorbed over the first 5 m.

e) Is the instantaneous rate at which sunlight is absorbed at a depth of 5 m greater

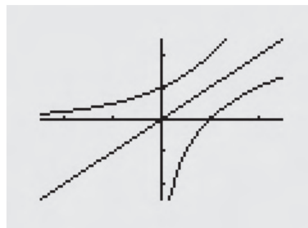
C

14. Complete the table of key features for $f(x)$ and its inverse.

	$f(x) = \left(\frac{1}{2}\right)^x$	**Inverse of f**
Domain		
Range		
x-Intercept		
y-Intercept		
Intervals for which $f(x)$ is positive		
Intervals for which $f(x)$ is increasing		
Equation of asymptote		

KEY CONCEPTS

The logarithmic function is the inverse of the exponential function.



The logarithmic function is denoted as $y = \log_b x$. We say y equals the logarithm of x to the base b.

The function is defined only for $b > 0$, $b \neq 1$.

Exponential equations can be written in logarithmic form, and vice versa.

$y = b^x \quad \Leftrightarrow \quad x = \log_b y$

$y = \log_b x \quad \Leftrightarrow \quad x = b^y$

Exponential and logarithmic functions are defined only for positive values of the base that are not equal to one:

$y = b^x, b > 0, x > 0, b \neq 1$

$y = \log_b x, b > 0, y > 0, b \neq 1$

The logarithm of x to base 1 is only valid when $x = 1$, in which case y has an infinite number of solutions and is not a function.

Common logarithms are logarithms with a base of 10. It is not necessary to write the base for common logarithms: $\log x$ means the same as $\log_{10} x$.

A

1. Rewrite each equation in logarithmic form:

a) $3^5 = 243$

b) $\frac{1}{216} = 6^{-3}$

2. Evaluate without using a calculator.

a) $\log_8 64$

b) $\log_2 \frac{1}{64}$

c) $\log 100$

d) $\log 10^{-2.6}$

3. Rewrite each equation in exponential form:

a) $\log_4 64 = 3$

b) $y = \log 30$

4. Rewrite $125 = 5^x$ in logarithmic form.

5. Evaluate $\log_3\left(\frac{1}{27}\right)$.

6. Rewrite $\log_5 625 = 4$ in exponential form.

7. Rewrite each equation in logarithmic form.
   a) $9 = 3^2$
   b) $16 = 4^2$
   c) $6^{-2} = \frac{1}{36}$
   d) $\left(\frac{1}{3}\right)^3 = \frac{1}{27}$
   e) $7^x = y$
   f) $10^4 = 10\,000$
   g) $\frac{1}{125} = 5^{-3}$
   h) $a = b^x$

8. Evaluate each logarithm.
   a) $\log_2 8$
   b) $\log_4 1024$
   c) $\log_3 \left(\frac{1}{9}\right)$
   d) $\log_5 \left(\frac{1}{625}\right)$
   e) $\log_{\frac{1}{2}} 16$
   f) $\log_{\sqrt{2}} 4$
   g) $\log_5 3125$
   h) $\log_3 2187$
   i) $\log_7 49^4$
   j) $\log_2 8^3$

9. Evaluate each common logarithm.
   a) $\log 100\,000$
   b) $\log \left(\frac{1}{1000}\right)$
   c) $\log 0.001$
   d) $\log 10^{-9}$
   e) $\log 100$
   f) $\log \left(\frac{1}{10^7}\right)$

10. Rewrite in exponential form.
   a) $\log_2 8 = 3$
   b) $\log_3 \left(\frac{1}{9}\right) = 2$
   c) $\log_a 3 = 6$
   d) $\log_3 2 = x$
   e) $\log_2 6 = x$
   f) $\log_2 x = 1.3$
   g) $\log_{\sqrt{2}} \pi = x$
   h) $\log_5 3x = 7y$

11. Sketch a graph of each function. Then, sketch a graph of the inverse of each function. Label each graph with its equation.
   a) $y = 3^x$
   b) $y = 5^x$

12. Estimate the value of each logarithm, correct to one decimal.
   a) $\log 117$
   b) $\log 0.123\,4$
   c) $\log 14$
   d) $\log 2156$
   e) $\log 0.012$
   f) $\log 7500$

B

13. The number of visitors to a popular web site is tripling every month. The time, t, in days, for a number, N, of visitors to see the site is given by the equation $t = \frac{\log N}{\log 3}$. How long will it take until the number of visitors to the web site reaches each number?
   a) 1000
   b) 1 000 000

14. Johannes Kepler, a German mathematician and astronomer, developed an equation to determine how long a planet takes to revolve around the Sun based on the planet's distance from the Sun in kilometres. $T = 1.5 \log d - 0.7$. How long is the orbit around the Sun of

a) Earth, if it is 150 000 000 km from the Sun?

b) Neptune, if it is 4 473 million km from the Sun?

15. The function $D(t) = 93 \log t + 65$ relates the distance, in kilometres, that a particle travels in relation to the time, t, in seconds.

a) If the particle travels for 45 s, estimate its distance travelled.

b) If the particle travels for 798 km, estimate the time it takes to travel that far.

16. There are initially 5000 bacteria in a culture. The number of bacteria, N, doubles every hour, so the number of bacteria after t hours will be $N(t) = 5000(2)^t$.

a) How many bacteria are present after 3 days?

b) How long would it take to have 7 million bacteria?

c) How long would it take to have 150 million bacteria?

17. The pH of a chemical solution is a measure of its acidity and is defined as $\text{pH} = -\log(H^+)$, where H^+ is the concentration of hydrogen ions in moles per litre.

a) What is the pH value of a solution with a hydrogen concentration of 0.000 047 mol/L?

b) Find the hydrogen concentration for a pH value of 11.

18. Our star, the Sun, appears billions of times brighter than the other stars because it is relatively near to us. Hence, astronomers are interested in comparing the brightness of stars if they could all be viewed from the same distance. The luminosity of a star refers to its brightness at a distance of 32.6 light years. By allowing for the Sun's magnitude and its distance from us, astronomers have established the following formula for the luminosity, L, of a star relative to the Sun: $\log L = 0.4(5 \log d - m) - 1.1$, where m is the magnitude of the star, and d is its distance from Earth in light years.

a) How many times as luminous as the Sun are the stars in the table?

Some Stellar Distances

Star	Magnitude	Distance (Light Years)
Sirius	−1.46	8.7
Vega	0.04	26.5
North Star	1.99	680.0
Deneb	1.26	1600.0

b) The distance to the Sun is 1.55×10^{-5} light years. Verify that the luminosity of the Sun, as defined by the given equation, is 1.

19. The age of a bone can be determined from the fraction of carbon-14 that remains in the bone. The age is calculated by using the formula $A = -19\,000(\log R)$, where A is the age in years and R is the fraction of carbon-14 remaining.

a) How old is a bone that has only $\frac{3}{4}$ of its original carbon-14?

b) How old is a bone that has only 10% of its original carbon-14?

c) Express the formula in exponential form.

d) Use your answer to part c) to calculate the percent of carbon-14 remaining in a bone from an animal that died 100 years ago.

KEY CONCEPTS

The techniques for applying transformations to logarithmic functions are the same as those used for other functions.

$y = \log x + c$ translates up c units if $c > 0$
translates down c units if $c < 0$

$y = \log(x - d)$ translates right d units if $d > 0$
translates left d units if $d < 0$

$y = a \log x$ stretch vertically by a factor of $|a|$ if $|a| > 1$
compress vertically by a factor of $|a|$ if $|a| < 1$
reflect in the x-axis if $a < 0$

$y = \log kx$ compress horizontally by a factor of $\left|\frac{1}{k}\right|$ if $|k| > 1$
stretch horizontally by a factor of $\left|\frac{1}{k}\right|$ if $|k| < 1, k \neq 0$
reflect in the y-axis if $k < 0$

It is easier to perform multiple transformations in a series of steps:

Step 1: Ensure that the function is in the form $f(x) = a \log[k(x - d)] + c$.

Step 2: Apply any horizontal or vertical stretches or compressions.

Step 3: Apply any reflections.

Step 4: Apply any horizontal or vertical translations.

A

1. **a)** Graph the function
$y = \log(x - 2) + 6$.

b) State the key features of the function.

i) domain and range

ii) x-intercepts, if any exist

iii) y-intercept, if it exists

iv) equations of the asymptotes

2. **a)** Graph the function
$y = -2 \log (3x + 6)$

b) State the key features of the function.

i) domain and range

ii) x-intercepts, if any exist

iii) y-intercept, if it exists

iv) equations of the asymptotes

3. Match each equation with its graph.

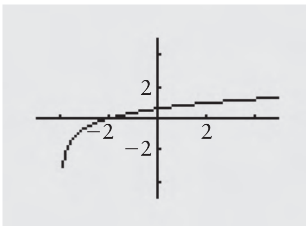
a) $y = \log x + 2$

b) $y = \log x - 2$

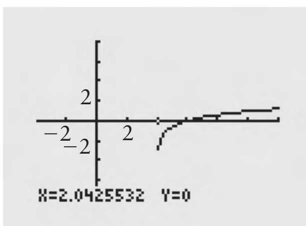
c) $y = \log(x - 2)$

d) $y = \log(x + 2)$

i)



ii)



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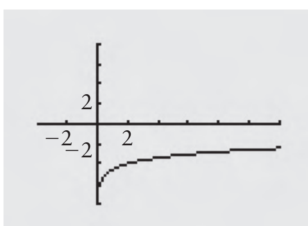

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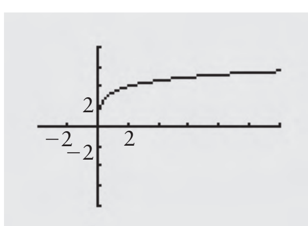
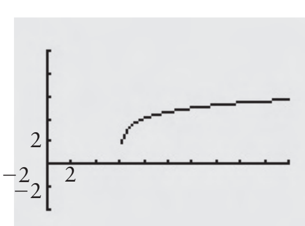


4. Let $f(x) = \log x$. Describe the transformation that would map f onto the following functions in each case.

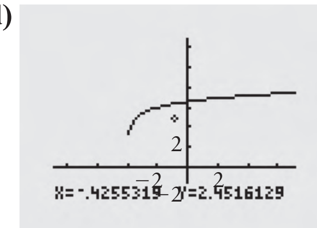
a) $y = \log(x + 3)$

b) $y = \log(x - 2) + 4$

c)



d)



5. Sketch a graph of each function. Identify the key features of each.

a) $y = \log x - 3$

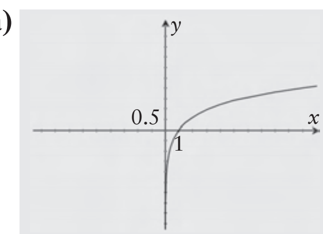
b) $f(x) = \log(x - 4)$

c) $y = \log(x - 2) + 3$

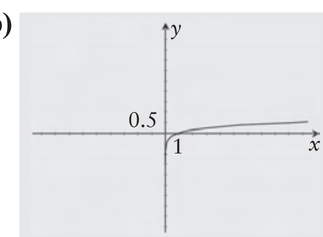
d) $f(x) = \log(x + 4) - 2$

6. Each of the following graphs can be generated by stretching or compressing the graph of $y = \log x$. Write an equation to describe each graph.

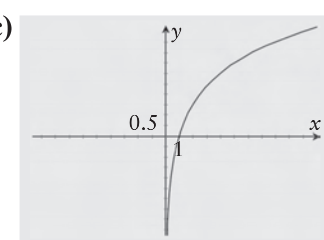
a)



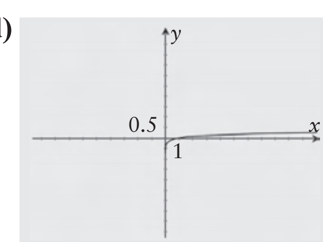
b)



c)



d)



7. Sketch a graph of each function. For each, state the domain and the range.

   a) $y = \log x$

   b) $y = \log x - 2$

   c) $y = -2 \log x$

   d) $y = \log(-2x)$

   e) $y = \log\left(\frac{1}{2}x\right)$

   f) $y = \log(x - 2)$

8. Sketch a graph of each function.

   a) $f(x) = -2 \log(x + 3)$

   b) $f(x) = \log[2(x + 3)] - 4$

   c) $f(x) = 3 \log(-(x + 3))$

   d) $f(x) = \frac{1}{2} \log (2x + 6)$

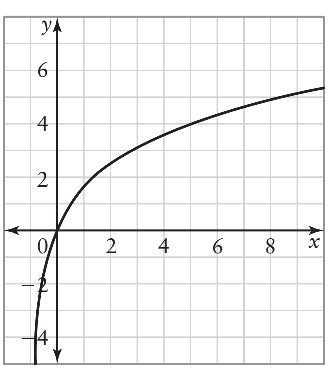
B

9. a) A graph is produced by applying the following transformations, in order, to the graph of $y = \log x$.

   - reflection in the x-axis
   - horizontal stretch by a factor of 2
   - horizontal translation, left 5 units
   - vertical translation, down 3 units

   b) If steps 2) and 3) of the transformations were interchanged, what would the equation of the graph be?

10. Determine an equation for the graph shown.

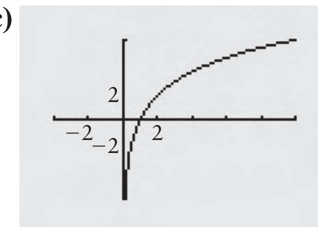


11. Let $f(x) = \log x$. Describe the transformation that would map f onto the following functions in each case.

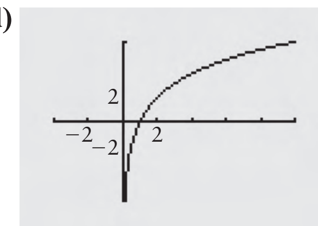
    a) $y = \frac{1}{3} \log(x)$

    b) $y = \log\left(\frac{1}{4}x\right)$

    c)



    d)



☆12. a) Sketch a graph of each function.

   i) $y = -\log(x - 5)$

   ii) $f(x) = \log(-x) + 2$

   iii) $y = 2 \log(x + 4)$

   iv) $f(x) = \log\left(\frac{1}{3}x\right) - 2$

   b) For each function identify the following:

   i) the domain

   ii) the range

   iii) an equation for the asymptote

13. Sketch a graph of each function.

    a) $f(x) = 3 \log[5(x - 2)] + 1$

    b) $y = \log(-x - 3)$

    c) $f(x) = 4 \log(3x + 6) + 5$

    d) $y = -4 \log\left(\frac{1}{2}x - 5\right) - 2$

14. **Use Technology** Sketch the graph of $y = -4 \log(x + 5) - 2$ by hand. Then, check your answer using graphing technology.

15. The generation time for bacteria is the time that it takes for the population to double. The generation time, G, can be found using experimental data and the formula $G = \dfrac{t}{3.3 \log(f/b)}$ where t is the time period, b is the number of bacteria at the beginning of the experiment, and f is the number of bacteria at the end of the experiment. The generation time for mycobacterium tuberculosis is 16 h. How long will it take 4 of these bacteria to multiply into 1024 bacteria?

16. **a)** Graph the function $f(x) = 2 \log x + 3$.

b) Graph the line $y = x$ on the same grid.

c) Graph the inverse function f^{-1} by reflecting f in the line $y = x$.

d) Determine the key features of f^{-1}.

i) domain

ii) range

iii) equation of the asymptote

17. Graph each function.

a) $y = \log_2 (x + 3) - 4$

b) $y = -4 \log_2 (3x + 4) + 2$

☆18. Suppose that the Tidbinbilla Deep Space Tracking Station in Australia received a signal, S, as a function of time that is given by $S = \frac{2}{5} \log(\cos x) - 3$.

a) What are the domain and range of this function? Explain your reasoning.

b) Use Technology Graph this function using graphing technology.

c) Do you think that the graph could be a code signal?

19. The growth of a \$1000 investment at an interest rate of 6% per year compounded annually can be modelled by the function $n(A) = 40 \log A - 120$, where n is the number of years needed to grow to A dollars.

a) Use the formula to calculate the number of years needed for the investment to

i) double to \$2000

ii) triple to \$3000

b) Sketch a graph of n versus A for $0 \le A \le 3000$. Then, use your graph to estimate the value of the investment after 8 years.

c) In real life there must be a restriction on the domain of this function. What is this restriction? Explain.

C

20. Describe how the graph of each function can be obtained using transformations of the graph of $y = \log x$.

a) $y = -\log 2(x + 3)$

b) $y = 3 \log(-x) + 4$

21. Use transformations to explain why $y = -\log(-x)$ and $y = \log x$ are inverses of each other.

22. **Use Technology**

a) Compare the graphs of each pair of functions.

i) $y = \log x + 1$ and $y = \log 10x$

ii) $y = \log x + 2$ and $y = \log 10^2x$

iii) $y = \log x + 3$ and $y = \log 10^3x$

b) Use the pattern from part a) to graph $y = \log 10^4x$, without using technology,

KEY CONCEPTS

The power law of logarithms states that $\log_b x^n = n\log_b x$ for $b > 0$, $b \neq 1$, $x > 0$, and $n \in \Re$. This property can be used to solve equations with unknown exponents.

Any logarithm can be expressed in terms of common logarithms using the change of base formula:

$$\log_b m = \frac{\log m}{\log b},\ b > 0,\ b \neq 1,\ m > 0$$

This formula can be used to evaluate logarithms or graph logarithmic functions with any base.

A

1. Evaluate $\log_4 64^3$.

2. Solve for x: $5 = 2.5^x$.

3. Evaluate $\log_6 23$.

4. Graph the function $y = \log_3 x$.

5. An investment earns 12% interest, compounded annually. The amount, A, that the investment is worth as a function of time, t, in years, is given by $A = 1500(1.12)^t$.

a) Use the equation to determine the value of the investment.

i) initially, when $t = 0$

ii) after 2 years

iii) after 4 years

b) How long will it take for the investment to

i) double in value?

ii) quadruple in value?

6. Evaluate.

a) $\log_3 27^5$

b) $\log_5 15^2$

c) $\log 1000^{-2}$

d) $\log 0.001^{\frac{1}{3}}$

7. Evaluate.

a) $\log_8 \sqrt{16}$

b) $\log_{125} 5^{12}$

c) $\log_3\left(\sqrt[3]{729}\right)^9$

d) $\log_5\left(\sqrt[4]{\frac{1}{125}}\right)^{12}$

8. Solve x to two decimal places.

a) $5^x = 8$

b) $6^{3x} = 10$

c) $6(3)^{2x-3} = 18$

d) $2(7)^{4x-5} = 30$

B

9. A guaranteed investment certificate (GIC) earns 5% interest, compounded annually. The amount, A, that the investment is worth as a function of time, t, in years, is given by $A = 1500(1.05)^t$.

a) Use the equation to determine the value of the GIC.

i) initially, when $t = 0$

ii) after 2.5 years

iii) after 5 years

b) How long will it take for the investment to

i) double in value?

ii) quadruple in value?

10. Evaluate, correct to three decimal places.

a) $\log_5 67$

b) $\log_3 34$

c) $\log_9 5$

d) $-\log_{15} 5$

e) $\log_{\frac{1}{3}} 20$

f) $\log_{\frac{5}{7}} 9$

11. Write as a single logarithm.

a) $\dfrac{\log 9}{\log 4}$

b) $\dfrac{\log 12}{\log 5}$

c) $\dfrac{\log\left(\frac{1}{3}\right)}{\log\left(\frac{5}{7}\right)}$

d) $\dfrac{\log (x + 3)}{\log (x - 5)}$

12. Solve for x, correct to three decimal places.

a) $3 = \log 5^x$

b) $10\ 000 = 100 \log 10^x$

c) $5 = \log_2 16^x$

d) $24 = 4 \log_3 5^x$

☆13. An investment earns 12% compounded semi-annually. The amount, A, that the investment is worth as a function of time, t, in years, is given by $A(t) = 1200(1.06)^{2t}$.

a) What was the initial value of the investment? Explain.

b) How long will it take the investment to triple in value?

14. a) Evaluate $\log_3 27^7$ without using the power law of logarithms.

b) Evaluate the same expression by applying the power law of logarithms.

c) Which method do you prefer? Why?

☆15. Suppose that the Ares 1 crew-launch vehicle is approaching the International Space Station (ISS) that is orbiting Earth. When Ares 1 is 250 km from the ISS, it must be slowed down, by controlled braking. The time, t, in hours, required to reach a distance, d, in kilometres, from the ISS during controlled braking can be modeled by $t = \log_{0.5}\left(\frac{d}{250}\right)$. The docking sequence can be initialized when the craft is within 1 km of the station's docking bay.

a) How long after the controlled braking starts should the docking procedures begin?

b) What are the domain and range of this function? What do these features represent?

16. An investment pays 4% interest, compounded annually.

a) Write an equation that expresses the amount, A, of an investment of a function of time, t, in years.

b) Determine how long it will take for this investment to

i) triple in value

ii) quadruple in value

c) Determine the percent increase in value of the account after

i) 6 years

ii) 12 years

C

17. Use $\log_b x = \dfrac{\log_k x}{\log_k b}$ for any $x > 0$, $k > 0$, $b > 0$ to express each of the following terms of base 3 logarithms.

a) $\log_4 1024$

b) $\log 37$

18. An investment pays 5.5% interest, compounded semi-annually.

a) Write an equation to express the amount, A, of the investment as a function of time, t, in years.

b) Determine how long it will take for this investment to

i) double in value

ii) triple in value

6.5 Making Connections: Logarithmic Scales in the Physical Sciences

KEY CONCEPTS

Logarithmic scales provide a convenient method of comparing values that typically have a very large range.

Several phenomena in the physical sciences can be described using logarithmic scales.

The pH scale is defined as $\text{pH} = -\log[H^+]$ $\text{pH} = -\log(H^+)$ where H^+ is the hydronium ion concentration in the substance.

A

☆**1.** The pH scale is used to measure the acidity or alkalinity of a chemical solution. It is defined as $\text{pH} = -\log(H^+)$, where H^+ is the concentration of hydronium ions, measured in moles per litre.

a) Hydrochloric acid has a hydronium ion concentration of approximately 100 mol/L. What is its pH?

b) Battery acid has a hydronium ion concentration of 5×10^{-9} mol/L. What is its pH?

c) A soft drink has a pH of approximately 3. What is the concentration of hydronium ions in a soft drink?

d) What has a greater concentration of hydronium ions, soft drinking water with a pH of 5 or tomato juice with a pH of 4? by how much?

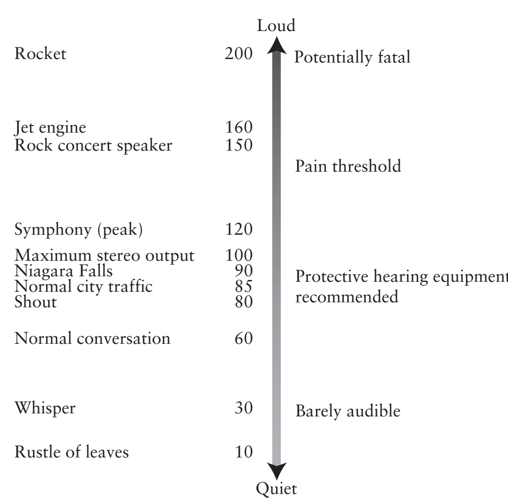
2. The pH of water in a small lake in northern Québec has dropped from 5.4 to 4.8 in the last three years. How many times more acidic is the lake now than it was three years ago?

3. When the pH of the water in a lake falls below 4.7, nearly all the species of fish in the lake are deformed or killed. How many times more acidic than clean rainwater, which has a pH of 5.6, is such a lake?

☆**4.** The difference in sound levels, in decibels, can be found using the equation $\beta_2 - \beta_1 = 10 \log\left(\frac{I_2}{I_1}\right)$, where I_1 and I_2 are the intensities of the two sounds, in watts per square metre.

a) How many times more intense than a shout is the sound of maximum stereo output?

b) The sound level of a normal conversation is approximately 60 dB. The sound level at a rock concert is about 10 500 times more intense. What is the sound level at the rock concert, in decibels?



☆ **5.** The magnitude, M, of an earthquake is measured using the Richter scale, which is defined as $M = \log\left(\frac{I}{I_0}\right)$, where I is the intensity of the earthquake being measured and I_0 is the intensity of a standard, low-level earthquake.

a) How many times more intense than a standard earthquake is an earthquake measuring 4.2 on the Richter scale?

b) What is the magnitude of an earthquake 50 000 times as intense as a standard earthquake?

6. Determine the pH of a solution with each hydronium ion concentration.

a) 0.001 **c)** 10^{-5}

b) 0.000 75 **d)** 4.3×10^{-8}

7. Determine the hydronium ion concentration, in moles per litre, of a solution with each pH.

a) 15 **c)** 11.3

b) 4 **d)** 6.2

Use the following information to complete questions 8 and 9.

The minimum intensity detectable by the human ear is $I_0 = 10^{-12}$ W/m^2 (watts per square metre), and is used as the reference point. The sound level corresponding to an intensity of I watts per square metre is $L = \frac{10 \log I}{I^0}$ or $L = \frac{10 \log I}{10^{-12}}$. This formula is used to determine the decibel rating of a sound.

☆ **8.** Find the decibel ratings, in dB, of a whisper with an intensity in watts per square metre of $1.15 \times 10^{-10}\,\frac{\text{W}}{\text{m}^2}$.

9. Find the decibel ratings, in dB, of a teacher's voice with an intensity of $9 \times 10^{-9}\,\frac{\text{W}}{\text{m}^2}$.

10. A hair dryer has a sound intensity level of 70 dB and an air conditioner has a sound intensity level of 50 dB. How many times more intense is the sound from the hair dryer than the sound from the air conditioner?

11. How many times more intense is the sound of a jet engine than the sound of

a) Niagara Falls?

b) a rustle of leaves?

12 A power mower makes a noise that is measured at 106 dB. Ordinary traffic registers about 70 dB. How many times louder is the mower than the traffic?

B

13. An earthquake occurred that was 1 000 000 times more intense than I_0. What was the measure of this earthquake on the Richter scale?

14. In the South Pacific, there was an earthquake that measured 6.7 on the Richter scale.

a) How many times more intense was this earthquake than a standard low-level earthquake?

b) On July 22, 2001, an earthquake in St. Catharines, Ontario, measured 1.1 on the Richter scale. How many times less intense was the St. Catharines earthquake than the South Pacific earthquake?

15. The 1906 earthquake in San Francisco had a magnitude of 8.3 on the Richter scale. At the same time in Japan there was an earthquake that had a of magnitude of 4.8 that caused only minor damage. How many times more intense was the San Francisco earthquake than the Japan earthquake?

☆**16.** In 1987, Canadian astronomer Ian Sheldon became the first person to observe a supernova in our galaxy since the invention of the telescope. Supernova 1987A increased in brightness by at least 200 times in the first day, and almost 1000 times in the first two days. What change occurred in the magnitude

a) in the first day?

b) in the first two days?

C1. The formula $D = 5e^{-0.4h}$ can be used to find the number of milligrams, D, of a certain drug that is in a patient's blood stream h hours after the drug has been administered. How many milligrams will be present after 1 h? after 6 h?

☆**C2.** A model for the number of people, N, in a college community who have heard a certain rumour is $N = P(1 - e^{-0.15d})$ where P is the total population of the community and d is the number of days that have elapsed since the rumour began. In a community of 1000 students, how many students will have heard the rumour after 3 days?

C3. The number of watts, w, provided by a space satellite's power supply after d days is given by the formula $w = 50(1.75)^{-0.004d}$.

a) How much power will be available after 120 days?

b) How much power will be available after 5 years?

c) What is the power supply's average output, in watts, between 120 days and 150 days? between 24 months and 36 months?

d) What is the value of the satellite's instantaneous watts at 120 days? at 5 years?

C4. If an earthquake is 390 times as intense as an earthquake with a magnitude of 4.2 on the Richter scale, what is the magnitude of the more intense earthquake?

C5. The absolute magnitude of star A is −4.5 and that of star B is 0.2.

a) How many times as bright is star A than star B, to the nearest unit?

b) If the apparent magnitudes of two stars are −2.5 and 1.3, respectively, which star is closer to Earth? Justify your answer.

☆**C6.** The approximate distance above sea level, d, in kilometres, is given by $h = \dfrac{-\ln\left(\frac{P}{760}\right)}{0.145}$ where P is the atmospheric pressure, in mm of mercury.

a) Among the highest inhabited buildings in the world are those in the Indian Tibetan border fort of Basisi. If the atmospheric pressure at Basisi is 271.5 mm of mercury, how far above sea level is the fort?

b) Sir Edmund Hillary was the first person to reach the top of Mount Everest, 8850 m above sea level. Calculate the pressure loss that the mountaineer experienced from sea level to the top of the mountain.

c) Suppose that an aircraft is pressurized to sea level (about 760 mm of mercury) for the duration of a flight. What pressure difference must the jet withstand at 2.1 km above sea level?

C7. Another formula to calculate altitude takes temperature into account. If x is the atmospheric, or barometric, pressure measured in mm of mercury, then the formula for the altitude $h(x)$, measured in m above sea level,

is $h(x) = (30T + 8000)\log\frac{P_0}{x}$. T is the temperature, in degrees Celsius, and P_0 is the barometric pressure at sea level, approximately 760 mm of mercury.

a) At what altitude is an aircraft if its instruments record an outside temperature of −30 °C and a barometric pressure of 241 mm of mercury?

b) Mount McKinley, in Alaska, is the highest peak in North America. What is the atmospheric pressure at the summit, altitude approximately 6970 m, if the air temperature is −25 °C?

C8. The number of years, n, for a piece of machinery to depreciate to a known salvage value can be found using the formula $n = \dfrac{\log s - \log i}{\log(1-d)}$ where s is the salvage value of the machinery, i is its initial value, and d is the annual rate of depreciation.

a) How many years will it take for a piece of machinery to decline in value from \$90 000 to \$10 000 if the annual rate of depreciation is 0.20 (20%)?

b) How many years will it take for a piece of machinery to loose half of its value if the annual rate of depreciation is 15%?

C9. Graph each function.

a) $f(x) = \log_3(x + 1) - 5$

b) $y = -2\log_2(3x - 4) + 2$

C10. Given the exponential function $f(x) = a^x$ and $f^{-1}(x) = \log_a x$, where $a > 0$,

a) for what values of a do the graphs of $f(x) = a^x$ and $f^{-1}(x) = \log_a x$ intersect?

b) Describe as much as you can about the point of intersection of the graphs in part a).

C11. An investment pays 12% interest, compounded weekly.

a) Write an equation that expresses the amount, A, of the investment as a function of time, t, in years.

b) Determine how long it will take for this investment to

i) triple in value

ii) quadruple in value

c) Determine the present increase in value of the account after

i) 15 years

ii) 20 years

C12. Tensing invests \$2500 in savings bonds at an interest rate of 8.5% per year, compounded semi-annually. How l ong will it take for this sum to increase to \$3500?

C13. The sum of \$1000 was invested for 4 years and the interest was compounded annually. If this investment amounts to \$1463.33 after 4 years, what was the interest rate?

C14. A great earthquake in India had a Richter reading of 8.7. A slight tremor occurring in California had a magnitude of 2.5. How many times greater was the earthquake in India?

C15. An Alaskan earthquake was 4 times more intense than a San Francisco earthquake that had a magnitude of 3.4 on the Richter scale. What was the magnitude of the Alaskan earthquake on the same scale?

C16. Find the pH of a 1-L container of water with 0.000 000 1 mol of hydrogen ion.

C17. Find the hydrogen ion concentration of a mildly acidic solution with a pH of 4.2.

C18. Between 1997 and 2007, the annual average pH of precipitation in a northern Ontario town dropped from 5.6 to 4.3. How many times more acidic was the precipitation in 2007 than the precipitation in 1997?

C19. In chemistry, the equation $t = c\log_2 \dfrac{b(a-x)}{a(b-x)}$ is used where x is the concentration of a substance at time t, and a, b, c are constants. Solve this equation to express x as a function of t.

Chapter 6: Checklist

By the end of this chapter, I will be able to:

- Identify the nature of the rate of change and key features of an exponential function
- Identify the shape of the graph and key features of the inverse of an exponential function
- Identify the logarithmic function as the inverse of a corresponding exponential function
- Write an exponential equation in logarithmic form and vice versa
- Estimate and evaluate a logarithm
- Apply transformations to logarithmic functions
- Apply the power law of logarithms
- Solve for an unknown exponent of an exponential equation by applying the power law of logarithms
- Apply the change of base formula to evaluate a logarithm having any base
- Apply the change of base formula to graph a logarithmic function having any base
- Understand the nature of a logarithmic scale
- Solve a variety of problems involving logarithmic scales used in the physical sciences