Chapter 7 Tools and Strategies for Solving Exponential and Logarithmic Equations

7.1 Equivalent Forms of Exponential Equations

KEY CONCEPTS

- Exponential functions and expressions can be expressed in different ways by changing the base.

Example

Express in terms of a power with a base of 3.

a) 81 **b)** 9^5 **c)** $\sqrt{9} \times \sqrt[7]{27}^{\,5}$ **d)** 17

Solution

a) $81 = 3^4$

b) $9^5 = (3^2)^5 = 3^{10}$

c) $\sqrt{9} \times \sqrt[7]{27}^{\,5} = (3^2)^{\frac{1}{2}} \times (3^3)^{\frac{5}{7}}$

$= 3 \times 3^{\frac{15}{7}}$

$= 3^{\frac{22}{7}}$

d) $3^x = 17$

$\log_3 3^x = \log_3 17$

$x = \log_3 17$

$17 = 3^{\log_3 17}$

- Changing the base of one or more exponential expressions is a useful technique for solving exponential equations.

Example

Solve each equation.

a) $5^{x+3} = 125^x$ **b)** $3^{2x} = 243^{x-5}$

Solution

a)
$$\begin{aligned}
5^{x+3} &= 125^x \\
5^{x+3} &= (5^3)^x \\
5^{x+3} &= 5^{3x} \\
x + 3 &= 3x \\
3 &= 2x \\
x &= \frac{3}{2}
\end{aligned}$$

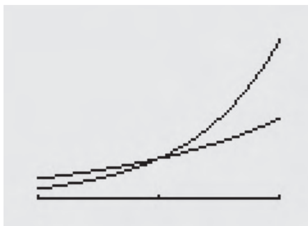
b)
$$\begin{aligned}
3^{2x} &= 243^{x-5} \\
3^{2x} &= (3^5)^{x-5} \\
3^{2x} &= 3^{5x-25} \\
2x &= 5x - 25 \\
-3x &= -25 \\
x &= \frac{25}{3} \\
x &= 8\frac{1}{3}
\end{aligned}$$

- Graphing technology can be used to solve exponential equations.

Example

Solve the equation $4^{x+1} = 2^{x-1}$ using a graphing calculator.

Solution



Solution is $\left(-4, \frac{1}{32}\right)$.

A

1. Write each expression with base 2.

a) 16^4

b) 4^7

c) $\left(\frac{1}{8}\right)^3$

d) 11

2. Write each expression with base 3.

a) 9^4

b) 27^7

c) $\left(\frac{1}{9}\right)^3$

d) $\frac{1}{5}$

3. Write each expression as a power of the base indicated.

a) 64^3, base 4

b) $\left(\frac{1}{4}\right)^5$, base 2

c) 5^3, base 4

4. Write each expression as a single power of 3.

a) $\sqrt[3]{81}$

b) $\frac{\sqrt{27}}{\sqrt[4]{9}}$

5. **Use Technology** Solve. Check your answers by using graphing technology.

a) $7^x = 49^{x+5}$

b) $4^{t-3} = 32^{\frac{t+2}{3}}$

c) $36^{3x-1} = 216^{5-x}$

6. **a)** Write 3^9 as a power of a number other than 3.

b) Write 3^9 as a power with a base that is different from the one you chose in part a).

c) Use a calculator to check that these powers are equivalent.

7. Write each expression as a single power of 4.

a) $(\sqrt{256})^5$

b) $\sqrt[5]{64}$

c) $\sqrt{1024} \times (\sqrt[5]{64})^4$

d) $(\sqrt{2})^9 \times (\sqrt[4]{2})^6$

8. **Use Technology** Solve. Check your answers using graphing technology.

a) $2^{x+3} = 4^{2x-5}$

b) $7^{w-2} = 49$

c) $3^{x+2} = 9^{x-5}$

d) $49^{4w+1} = 7^{2w+7}$

9. **Use Technology** Solve. Check your answers using graphing technology.

a) $9^{3x} = 27^{x-3}$

b) $4^{5x-1} = 8^{8x+7}$

c) $625^{4y+3} = 25^{y-2}$

d) $128^{2k-3} = 64^{k+3}$

B

☆**10.** Consider the equation $36^{4x} = 1296^{3x-7}$.

a) Solve this equation by expressing both sides as powers of a common base.

b) Solve the same equation by taking the common logarithm of both sides.

c) Which technique do you prefer? why?

11. Use Technology Solve. Check your answers using graphing technology.

a) $\sqrt{27} = 3^{x+5}$

b) $8^{k-2} = (\sqrt{128})^{2k}$

12. Consider the equation $10 = 3^x$.

a) Solve the equation for x by taking the common logarithm of both sides.

b) Use the result in part a) to show that $10 = 3^{\frac{1}{\log 3}}$.

c) Apply algebraic reasoning to show that $10 = 4^{\frac{1}{\log 4}}$.

13. Consider the equation $256^{3x} = 16^{x-2}$.

a) Solve this equation by expressing both sides as a power with base 4.

b) Solve the same equation by expressing both sides as a power with base 16.

c) Use Technology Solve the same equation using graphing technology.

d) Reflect on these methods. Which do you prefer? why?

14. Consider the equation $5^{3x-1} = 125^{2x}$.

a) Solve this equation by expressing both sides as powers of a common base.

b) Solve the same equation by taking the logarithm, base 5, of each side.

15. a) Solve. Give exact answers.

i) $5 = 10^x$

ii) $3 = 10^x$

iii) $7 = 10^x$

b) Use your answers to part a) to state a formula that could be used to solve $b = 10^x$ for x.

16. a) Solve $16^{3x+2} = 64^{5-3x}$ by expressing both sides of the equation as powers of 4.

b) Solve $16^{3x+2} = 64^{5-3x}$ by expressing both sides of the equation as powers of 2.

c) Use Technology Solve $16^{3x+2} = 64^{5-3x}$ by using graphing technology.

d) Which of the methods is "best"? Explain.

C

☆**17. a)** Solve each inequality.

i) $3^{2x} > 9^{x+3}$

ii) $8^{3x+5} < 32^{x+7}$

b) Use a sketch to help explain how you can use graphing technology to check your answers.

KEY CONCEPTS

- An equation maintains balance when the common logarithm is applied to both sides.
- The power law of logarithms ($\log_b x^n = n \log_b x$, $b > 0$, $x > 0$) is a useful tool for solving a variable that appears as part of an exponent.

Steps:

1. Isolate the term containing the variable on one side of the equation.

2. Take the base 10 logarithm of each side of the equation.

3. Apply the power law of logarithms to rewrite the equation without exponents.

4. Solve for the variable and check the results.

Example

Solve the equation $5^{3x-2} = 3^{x+5}$.

Solution

a)

$$\begin{aligned}
5^{x-2} &= 3^{x+5}\\
\log 5^{3x-2} &= \log 3^{x+5}\\
(3x - 2)\log 5 &= (x + 5)\log 3\\
3x\log 5 - 2\log 5 &= x\log 3 + 5\log 3\\
3x\log 5 - x\log 3 &= 5\log 3 + 2\log 5\\
x(3\log 5 - \log 3) &= 5\log 3 + 2\log 5\\
x &= \frac{5\log 3 + 2\log 5}{3\log 5 - \log 3}\\
x &= 2.33
\end{aligned}$$

- When a quadratic equation is obtained, methods such as factoring and applying the quadratic formula are useful.
- Some algebraic methods of solving exponential equations lead to extraneous roots that are not valid solutions to the original equation.

Example

Solve the equation $5^{2x} - 5^x = 20$.

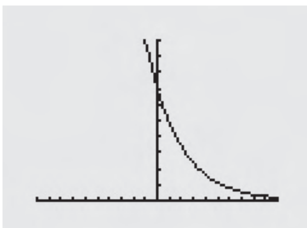
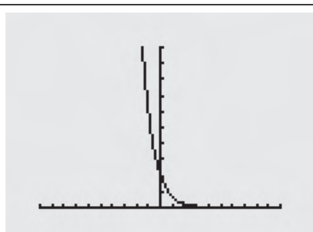
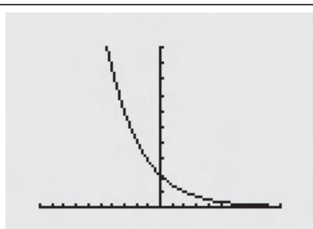
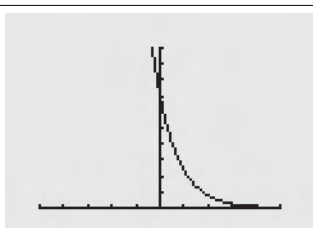
Solution

$$\begin{aligned}
5^{2x} - 5^x &= 20\\
(5^x)^2 - 5^x - 20 &= 0\\
(5^x - 5)(5^x + 4) &= 0
\end{aligned}$$

$$5^x - 5 = 0 \qquad 5^x + 4 = 0$$
$$5^x = 5 \qquad 5^x = -4$$
$$x = 1 \qquad \textit{extraneous}$$

A

1. Match each graph with its equation.

Graph	Equation
1 	$y = 200\left(\frac{1}{3}\right)^x$
2 	$y = 70\left(\frac{1}{3}\right)^x$
3 	$y = 70\left(\frac{1}{3}\right)^{\frac{x}{3}}$
4 	$y = 200\left(\frac{1}{3}\right)^{\frac{x}{3}}$

2. Solve for x. Round the answers to two decimal places.

a) $2^x = 73$

b) $8 = 1.4^x$

c) $49 = 7(2.05)^x$

d) $7 = 1.4^{x-3}$

e) $0.75 = 1.3^{x+3}$

f) $25 = \left(\frac{1}{3}\right)^{2x}$

g) $14 = \left(\frac{1}{3}\right)^{\frac{x}{5}}$

h) $200 = 50\left(\frac{1}{4}\right)^{\frac{5}{x}}$

3. Use the decay equation for polonium-218, $A(t) = A_0\left(\frac{1}{2}\right)^{\frac{y}{3.1}}$, where A is the amount remaining after t minutes and A_0 is the initial amount.

a) How much will remain after 120 s from an initial sample of 120 mg?

b) How long will it take for this sample to decay to 25% of its initial amount of 120 mg?

c) Would your answer to part b) change if the size and initial sample were changed? Explain why or why not.

4. Solve each equation. Leave answers in exact form.

a) $3^x = 4^{x-2}$

b) $7^{x+3} = 5^x$

c) $9^{x-4} = 4^{x+2}$

d) $8^{3x+5} = 5^{x+2}$

5. Find approximate values for your answers to question 4, correct to three decimal places.

6. Consider the equation $3^{2x} + 3^x - 12 = 0$.

a) Write the equation in the form $az^2 + bz + c = 0$, where $z = 3^x$, and then identify the coefficients a, b, and c.

b) Solve the equation using the quadratic formula.

c) Identify any extraneous roots.

7. Consider the equation $7^{2x} = 2(7^x) + 3$.

a) Write the equation in the form $az^2 + bz + c = 0$, where $z = 7^x$, and then identify the coefficients a, b, and c.

b) Solve the equation using the quadratic formula.

c) Identify any extraneous roots.

B

8. A bacteria culture starts with 3000 bacteria. After 3 h, the estimated count is 48 000. What is the doubling period?

☆**9.** A 50-mg sample of radioactive iodine decays to 23 mg after 12 min.

a) Determine the half-life of the radioactive iodine.

b) How long will it take for this sample to decay to 5 mg?

10. After 90 days, a sample of silver-110, Ag^{110}, has decayed to about 80% of its original size.

a) Determine the half-life of Ag^{110}.

b) Graph the amount of Ag^{110} remaining as a function of time.

c) Describe how the graph would change if the half-life were

i) shorter

ii) longer

d) Describe how the graph would change if the initial sample size were

i) greater

ii) less

☆**11.** Does the equation $3^{2x} - 5(3^x) - 6 = 0$ have any real solutions? Explain your answer.

12. Solve and check for any extraneous roots. Where necessary, round answers to two decimal places.

a) $5(2)^{2-3x} = 50$

b) $182 = 2(3)^{4x+3}$

c) $9^{2x} + 4(9^x) = 21$

d) $7^{2x} = 2(7^x) + 3$

e) $6(2^{2x}) + 13(2^x) = 5$

f) $12(4^{2x}) = -5(4^x) + 2$

13. Use Technology Check your solutions to question 12 using graphing technology.

14. Tenzin has purchased a minivan for \$35 000. The value, V, in dollars, of the minivan as a function of time, t, in years, depreciates according to the function $V(t) = 35000\left(\frac{1}{2}\right)^{\frac{t}{3}}$. How long will it take for Tenzin's minivan to depreciate to

a) half of its initial value?

b) 15% of its initial value?

15. Pratheep bought a computer system for \$2500. The value of the system depreciates at a rate of 20% each year.

a) Determine an exponential function to model the value of the system over time.

b) Graph the function you found in part a).

c) Use the graph to determine the value of Pratheep's system three years from now.

d) Verify your result from part c) algebraically.

16. To determine whether a person has a thyroid deficiency, a radioactive iodine with a half-life of 8.2 days is injected into the bloodstream. A healthy thyroid is able to absorb all of the radioactivity. The amount, R, of radioactivity present after t days can be modelled by the relationship $R(t) = R_0\left(\frac{1}{2}\right)^{\frac{t}{8.2}}$, where R_0 is the initial dose. How long will it take for 87.5% of the iodine to be absorbed into a healthy person's body?

17. A sample of radioactive iodine-131 atoms has a half-life of about 8 days. A formula that models the number of iodine-131 atoms that remain is $P = P_0\left(2^{\frac{-t}{8}}\right)$, where P is the number of iodine-131 atoms that remain after time, t, in days, and P_0 is the number of iodine-131 atoms that are initially present. Suppose that 1 000 000 iodine-131 atoms are initially present.

a) How long will it take until there is

i) half of the initial value of iodine-131 atoms?

ii) 10% of the initial value of iodine-131 atoms?

b) Sketch a graph of the decay formula.

c) Suppose that the iodine-131 decays more slowly than the current half-life of 8 days. What effect will this have on the

i) equation?

ii) graph?

☆**18.** One day, a movie trailer to the new James Bond film was posted on the Internet. Suppose that 5000 people see the video on the first day after it is posted and that the number doubles every day after that.

a) Write an expression to describe the number of people who have seen the trailer t days after it is posted.

b) One week later, a second trailer is posted. Suppose that 3500 people see this trailer the first day after it is posted and that this number triples every day after that. Write an expression to describe the number of people who have seen the second trailer t days after it is posted.

c) Set the two expressions from part a) and part b) equal and solve for t using tools and strategies of your choice. What does this solution mean?

19. The population of a colony of bacteria grows according to the formula $P(t) = 4(1.40)^{\frac{t}{24}}$, where P is the population, in thousands, and t is the time, in hours.

a) How long does it take the population to reach 10 000, to the nearest hour?

b) Calculate the time it takes for the population to double, to the nearest hour.

20. The maximum height that a ball reaches after bounce number n is given by the equation $H = 2.0(0.90)^n$, where H is the height, in metres.

a) What is the ball's maximum height after the fifth bounce?

b) What is the first bounce after which the maximum height is less than 10 cm?

21. Rewrite the equation $P(t) = 4(1.40)^{\frac{t}{24}}$ with base 1.40 replaced with 2.

22. The general equation for population growth is $P(t) = P_0\left(1 + \frac{R}{100}\right)^{\frac{t}{t_0}}$, where R is the growth rate, in percent, over time period t_0. Suppose a population grew from 10 000 to 25 000 in 6 years. If time is measured in years, calculate

a) the yearly growth rate

b) the growth rate per decade (10 years)

C

23. A radioactive substance has a half-life of 15 days.

a) Develop an equation for the amount of radioactive substance left as a function of time for the given scenario.

b) Find the domain and the range of this function.

KEY CONCEPTS

- The product law of logarithms states that $\log_b x + \log_b y = \log_b(xy)$ for $b > 0$, $b \neq 1$, $x > 0$, $y > 0$.
- The quotient law of logarithms states that $\log_b x - \log_b y = \log_b\left(\frac{x}{y}\right)$ for $b > 0$, $b \neq 1$, $x > 0$, $y > 0$.
- The laws of logarithms can be used to simplify expressions and solve equations.

Example

Express as a single logarithm.

a) $\log_{10} 12 + \log_{10} 7 - \log_{10} 3$

b) $\log_a b + \log_a (7c) + \log_a (3b) - \log_a c$

c) $\log(x^2 + 2x - 3) - \log(2x + 6)$

Solution

a)
$$\begin{aligned}\log_{10} 12 + \log_{10} 7 - \log_{10} 3 &= \log_{10} \frac{(12 \times 7)}{3}\\ &= \log_{10} 28\end{aligned}$$

b)
$$\begin{aligned}\log_a b + \log_a (7c) + \log_a (3b) - \log_a c &= \log_a \frac{21cb^2}{c}\\ &= \log_a 21b^2\end{aligned}$$

c)
$$\begin{aligned}\log(x^2 + 2x - 3) - \log(2x + 6) &= \log \frac{x^2 + 2x - 3}{2x + 6}\\ &= \log \frac{(x + 3)(x - 1)}{2(x + 3)}\\ &= \log \frac{(x - 1)}{2}\end{aligned}$$

Example

Express as a single logarithm. Then, evaluate.

a) $\log_{10} 8 + \log_{10} 1.25$

b) $\log_2 80 - \log_2 5$

c) $\log 40 + \log 5 - \log 2$

d) $\log_7 245 + \log_7 \frac{1}{5}$

Solution

a) $\log_{10} 8 + \log_{10} 1.25 = \log_{10}(8 \times 1.25)$
$= \log_{10} 10$
$= 1$

b) $\log_2 80 - \log_2 5 = \log_2 \frac{80}{5}$
$= \log_2 16$
$= 4$

c) $\log 40 + \log 5 - \log 2 = \log \frac{40 \times 5}{2}$
$= \log 100$
$= 2$

d) $\log_7 245 + \log_7 \frac{1}{5} = \log_7\left(245 \times \frac{1}{5}\right)$
$= \log_7 49$
$= 2$

Example

Write as a sum or difference of logarithms. Simplify if possible.

a) $\log_3 8$

b) $\log_2(x^2 yz^4)$

c) $\log_b\left(\frac{x^2}{yz^3}\right)$

Solution

a) $\log_3 4 + \log_3 2$

b) $2\log_2 x + \log_2 y + 4\log_2 z$

c) $2\log_b x - \log_b y - 3\log_b z$

A

1. Simplify, using the laws of logarithms.

a) $\log 8 + \log 5$

b) $\log 56 - \log 7$

c) $\log_2 8 + \log_2 3$

d) $\log_5 80 - \log_5 16$

2. Use a calculator to evaluate each result in question 1, correct to three decimal places.

3. Simplify each algebraic expression. State any restrictions on the variables.

a) $\log(2x) + \log(3y) + \log z$

b) $\log_2 x + \log_2(3y) - 5 \log z$

c) $3 \log a + 4 \log y - 2 \log z$

d) $2 \log(3x) + 3 \log y - \frac{1}{2} \log(4w)$

4. Evaluate, using the product law of logarithms.

a) $\log_3 9 + \log_3 729$

b) $\log_5 4 + \log_5(31.25)$

c) $\log_4 32 + \log_4 256$

d) $\log 25 + \log 4 + \log 10$

5. Evaluate, using the quotient law of logarithms.

a) $\log_3 13\,122 - \log_3 2$

b) $\log_5 4375 - \log_5 7$

c) $\log_4 36\,864 - \log_4 9$

d) $\log 2800 - \log 4 - \log 7$

6. Evaluate, using the laws of logarithms.

a) $\log_3 12 - \log_3 4$

b) $\log_4 24 + \log_4 \frac{64}{3} - \log_4 32$

c) $\log_5 \sqrt{225} - \log_5 \sqrt{9}$

d) $\frac{1}{2} \log_3 225 - \log_3 5 + 3 \log_3 3$

7. Evaluate, using the laws of logarithms.

a) $\log_4 12 - \log_4 3$

b) $3 \log 6 + 2 \log 5 - \log 54$

8. Write $\log_3 5 + 2 \log_3(x - 2)$ as a single logarithm.

9. Write $\log\left(\frac{ab^4}{\sqrt[4]{c}}\right)$ as sums and differences of logarithms. Simplify, if possible.

10. Simplify and state restrictions necessary on the variable.

a) $\log(x^2 - 4x - 12) - \log(3x - 18)$

b) $\log(x^3 - 27) - \log(x - 3)$

11. Write as a sum or difference of logarithms. Simplify, if possible.

a) $\log_5(xy)$

b) $\log_8\left(\frac{a}{b}\right)$

c) $\log_2 40$

d) $\log_6 120$

B

12. Simplify. State any restrictions on the variables.

a) $\log\left(\frac{\sqrt{x}}{x^2}\right)$

b) $\log\left(\frac{x^5}{\sqrt[3]{x}}\right) + \log(x^2)^4$

c) $\log \sqrt[3]{x^4} + \log \sqrt{x} + \log(\sqrt{x})^6$

d) $4 \log x - 5 \log \sqrt{x} + \frac{1}{2} \log x^8$

☆ **13.** Simplify. State any restrictions on the variable.

a) $\log(x^2 - 9) - \log(x + 3)$

b) $\log(x^2 - 11x + 28) - \log(x - 7)$

c) $\log(x^2 + 5x - 36) - \log(3x - 12)$

d) $\log(2x^2 + 7x + 6) - \log(x^2 - 4)$

14. Simplify. State any restrictions on the variables.

a) $\log(m^5) - \log(m^2) + \log m$

b) $\log(\sqrt[3]{p}) + \log(\sqrt{p}) + \log(\sqrt[6]{p})$

c) $\log(x^2 - 5x - 6) - \log(x - 3)$

d) $\log(6x^2 + 5x - 6) + \log(2x - 3) - \log(4x^2 - 9)$

15. Use Technology

a) Graph the function $f(x) = \log x$.

b) Graph the function $g(x) = 2f(x)$.

c) Graph the function $h(x) = f(x^2)$.

d) How are the functions $g(x)$ and $h(x)$ related? What law of logarithms does this illustrate?

16. Use the laws of logarithms to write y as a function of x for each of the following. Then, state the domain of the function.

a) $\log(xy) = 2 \log(x - 3)$

b) $\log(y) + 3 = \log(y + 1) + \log(x)$

c) $\log\left(\frac{x^2}{y}\right) = 2 \log(x + 5)$

17. Prove that $\frac{1}{\log_x 10} + \frac{1}{\log_y 10} = \frac{1}{\log_{xy} 10}$.

$\left(\text{Hint: recall that } \log_n m = \frac{\log m}{\log n}\right)$

18. a) Graph the functions $f(x) = 7 \log x$ and $g(x) = 5 \log x$.

b) Graph the difference of these two functions $r(x) = f(x) - g(x)$.

c) Graph the function $s(x) = \log x^2$.

d) How are the functions $r(x)$ and $s(x)$ related? What law of logarithms does this illustrate?

19. Ifra, a recent business school graduate, has been offered entry-level positions with two firms. Firm A offers a starting salary of \$45 000 per year, with a \$2500 per year increase guaranteed each subsequent year. Firm B offers a starting salary of \$37 000, with a 7% increase every year after that.

a) After how many years will Ifra earn the same amount at either firm?

b) At which firm will she earn more after

i) 10 years?

ii) 20 years?

c) What other factors might affect Ifra's choice? Explain how these factors may influence her decision.

20. Energy is needed to transport a substance from outside a living cell to inside the cell. This energy, E, is measured in kilocalories per gram molecule. It is given by the relationship

$E = 1.4 \log \frac{C_1}{C_2}$, where C_1 represents the concentration of the substance outside the cell, and C_2 represents the concentration inside the cell.

a) Find the energy needed to transport the exterior substance into the cell if the concentration of the substance inside the cell is

i) double the concentration outside the cell

ii) triple the concentration outside the cell

b) What is the sign of E if $C_1 < C_2$? Explain what this means in terms of the cell.

21. The formula for the gain in voltage of an electronic device is

$A_v = 20(\log V_0 - \log V_i)$, where V_0 in the output voltage and V_i is the input voltage.

a) Write a simplified form of this formula, expressing the right side as a single logarithm.

b) Verify the gain in voltage for $V_0 = 22.8$ and $V_i = 14$ using both versions of the formula.

C

☆ 22. Use the product law of logarithms and the power law of logarithms to prove the quotient law of logarithms,

$\log_a \frac{p}{q} = \log_a p - \log_a q$, where $a > 0$, $a \neq 1, p > 0, q > 0$.

KEY CONCEPTS

- It is possible to solve an equation involving logarithms by expressing both sides of a logarithm of the same base: if $a = b$, then $\log a = \log b$, and if $\log a = \log b$, then $a = b$.
- When a quadratic equation is obtained, methods such as factoring or applying the quadratic formula may be useful.
- Some algebraic methods of solving logarithmic equations lead to extraneous roots that are not valid solutions to the original equations.

Example

Find the roots of each equation.

a) $\log(5 - x) = 3$

b) $\log_4(2x + 6) = 3$

Solution

a)
$$\begin{aligned} \log(5 - x) &= 3 \\ 5 - x &= 10^3 \\ 5 - x &= 1000 \\ x &= -995 \end{aligned}$$

b)
$$\begin{aligned} \log_4(2x + 6) &= 3 \\ 2x + 6 &= 4^3 \\ 2x + 6 &= 64 \\ 2x &= 58 \\ x &= 29 \end{aligned}$$

Example

Solve. Identify and reject any extraneous roots.

a) $\log(x + 2) - 1 = -\log(x - 1)$

b) $\log \sqrt[5]{x^2 - 21x} = \dfrac{2}{5}$

c) $\log_9(x - 5) + \log_9(x + 3) = 1$

Solution

a) $\log(x + 2) - 1 = -\log(x - 1)$

$\log(x + 2) + \log(x - 1) = 1$

$$(x + 2)(x - 1) = 10$$
$$x^2 - x + 2x - 2 = 10$$
$$x^2 + x - 12 = 0$$
$$(x + 4)(x - 3) = 0$$
$$x + 4 = 0 \qquad x - 3 = 0$$
$$x = -4 \qquad x = 3$$
extraneous

b) $$\log \sqrt[5]{x^2 - 21x} = \frac{2}{5}$$
$$(x^2 - 21x)^{\frac{1}{5}} = 10^{\frac{2}{5}}$$
$$x^2 - 21x = 100$$
$$x^2 - 21x - 100 = 0$$
$$(x - 25)(x + 4) = 0$$
$$x - 25 = 0 \qquad x + 4 = 0$$
$$x = 25 \qquad x = -4$$
extraneous

c) $\log_9(x - 5) + \log_9(x + 3) = 1$
$$\log_9(x - 5)(x + 3) = 1$$
$$(x - 5)(x + 3) = 9$$
$$x^2 - 2x - 15 = 9$$
$$x^2 - 2x - 24 = 0$$
$$(x - 6)(x + 4) = 0$$
$$x - 6 = 0 \qquad x + 4 = 0$$
$$x = 6 \qquad x = -4$$
extraneous

A

1. **Use Technology** Find the roots of each equation. Check your solutions using graphing technology.
   a) $\log(x - 5) = 1$
   b) $\log(x - 35) = 2$
   c) $15 = 5\log(x + 73)$
   d) $1 - \log(x + 4) = 0$
   e) $\log(x - 6) = 2$
   f) $8 - 4\log 3n = 0$

2. Solve.
   a) $\log_4(x + 9) = 2$
   b) $6 = \log_2(3x - 12)$
   c) $3 - \log_3(x - 17) = 0$
   d) $12 = \log_5(x + 150) + 9$
   e) $\log_8(t + 2) - 1 = 0$
   f) $\log_3(n^2 - 2n - 6) = 2$

3. **Use Technology** Solve. Identify and reject any extraneous roots. Check your solutions using graphing technology.

a) $\log_4(x - 3) + \log_4(x + 3) = 2$

b) $\log x^4 - \log 3 = \log(3x^2)$

c) $\log(v - 2) = 1 + \log(v + 2)$

d) $2 + \log x = \log(x - 9)$

e) $\log_2(x + 3) + \log_2(x - 3) = 4$

f) $\log_7(x + 1) + \log_7(x - 5) = 1$

4. **Use Technology** Solve $\log_5(2x - 1) = 2$. Verify the solution to the equation using graphing technology. Explain your method.

B

5. **Use Technology** Solve. Check the extraneous roots. Check your results using graphing technology.

a) $\log_9 \sqrt{x^2 - 6x} = \frac{1}{2}$

b) $\log \sqrt{x^2 + 21x} = 2$

☆ 6. Solve. Identify any extraneous roots.

a) $\log_{11} x + \log_{11}(x + 1) = \log_{11} 6$

b) $\log(3x + 6) = 1 + \log x$

c) $\log_2(x + 7) - \log_2(2x) = 3$

7. **Use Technology** Find the roots of each equation, correct to two decimal places, using graphing technology. Sketch the graphical solutions.

a) $\log(x + 3) = 1 - \log x$

b) $2\log(x + 3) = \log(4x) - 5$

☆ 8. An airplane altimeter is a gauge that indicates the height of the plane above the ground. The gauge works based on air pressure, according to the formula $h = 18\,400 \log \frac{P_0}{P}$, where h is the height of the airplane above the ground in metres, P is the air pressure at height h, and P_0 is the air pressure at ground level. Air pressure is measured in kilopascals (kPa).

a) Air pressure at the ground is 102 kPa. If the air pressure outside the airplane is 32.5 kPa, what is the height of the airplane?

b) How high would the airplane have to be flying for the outside air pressure at that height to be half of the air pressure at ground level?

c) If the weather changes, then the air pressure at ground level may change. How do pilots take this into account?

9. Another formula to calculate altitude or atmospheric pressure takes temperature into account. If x is the atmospheric pressure (measured in millimetres of mercury), then the formula for the altitude $h(x)$, measured in metres above sea level, is $h(x) = (30T + 8000)\log\left(\frac{P_0}{x}\right)$, where T is the temperature, in degree Celsius at altitude $h(x)$, and P_0 is the atmospheric (barometric) pressure at sea level which is approximately 760 mm of mercury.

a) At what height is a small airplane whose instruments record an outside temperature of 0^0 C and a barometric pressure of 300 mm of mercury?

b) What is the barometric pressure outside a passenger aircraft flying at an altitude of 10 000 metres if the outside air temperature is -100^0 C?

c) What is the barometric pressure at the summit of Mount Everest, which has an elevation of approximately 8900 metres, if the air temperature is 5^0 C?

C

10. Solve the equation $3\log(x - 15) = \left(\frac{1}{4}\right)^x$.

11. Solve the equation $\frac{\log(28 - x^3)}{\log(4 - x)} = 7$.

KEY CONCEPTS

- Different technology tools and strategies can be used to construct mathematical models that describe real situations.
- A good mathematical model
  - is useful for both interpolating and extrapolating from given data in order to make predictions
  - can be used, in conjunction with other considerations, to aid in decision making
- Exponential and logarithmic equations often appear in contexts that involve continuous growth or decay.

Example

A bacterial culture has been growing steadily for several hours. The table below gives the growth of the culture at 5-h intervals, beginning at 9:00 a.m.

Time (hours)	Amount of Bacteria
0	140
5	1415
10	2000
15	2828
20	4000
25	5656
30	8000

a) Create a scatter plot to illustrate this growth trend.

b) Construct a quadratic model to fit the data.

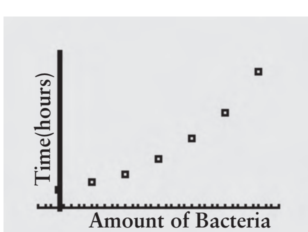
c) Construct an exponential model to fit the data.

d) Which model is better? why?

e) When will the amount of bacteria reach 5000?

Solution

a)



b) $y = 7.47x^2 + 0.76x + 1117.74$

$R^2 = 0.997\ 19$

c) $y = 1000(1.07)^x$

$R^2 = 0.999\ 99$

d) The exponential model is better. It fits the data better and its regression coefficient is closer to 1.

e)
$$5000 = 1000(1.07)^x$$
$$5 = 1.07^x$$
$$\ln 5 = x \ln 1.07$$
$$x = \frac{\ln 5}{\ln 1.07}$$
$$x = 23.78$$

There would be 23.78 growth intervals of 5 h each. Therefore, the amount of bacteria will reach 5000 after 118.9 h.

Example

The compound interest formula modelling the future amount, A, of an investment with initial principal, P, is $A = P(1 + i)^n$, where I is the interest rate per compounding period, in decimal form, and n is the number of compounding periods. Sasha wishes to invest \$1500 in a Registered Education Savings Plan that earns 12% interest, compounded quarterly.

a) Write an equation for the value of the investment as a function of time, in years.

b) Determine the value of the investment after 5 years.

c) How long will it take for the investment to double in value?

Solution

a) $A = 1500(1 + 0.03)^{4x}$

b)
$$A = 1500(1 + 0.03)^{4x}$$
$$A = 1500(1.03)^{4\times 5}$$
$$A = 1500(1.03)^{20}$$
$$A = \$2709.17$$

c)
$$A = 1500(1 + 0.03)^{4x}$$
$$3000 = 1500(1 + 0.03)^{4x}$$
$$2 = (1.03)^{4x}$$
$$\ln 2 = 4x \ln 1.03$$
$$x = \frac{\ln 2}{4 \times \ln 1.03}$$
$$x = 5.86$$

It will take about 5.86 years, or 5 years and 10 months, for the investment to double in value.

1. Sasha has a 100-g sample of a radioactive material. He records the amount of radioactive material every week for 6 weeks and obtains the following data.

Week	Weight (g)
0	100
1	88.3
2	75.9
3	69.4
4	59.1
5	51.8
6	45.5

a) Draw a scatter diagram with the week as the independent variable.

b) Find the exponential function of best fit. Express the function of best fit in the form $A = A_0e^{kt}$.

c) Use the solution to part b) to estimate the time it takes until l50 g of material are left.

d) Use the solution to part b) to predict how much radioactive material will be left after 50 weeks.

e) Use Technology Use a calculator to verify the exponential function of best fit.

2. The number of students who were home-schooled in Canada in selected years is shown in the table below.

School Year	Number of Children
1983	925
1985	1830
1988	2250
1990	3010
1992	4700
1993	5880
1994	7350
1995	8000
1996	9200
1997	11 000

a) Draw a scatter diagram with the school year as the independent variable.

b) Find the exponential function of best fit. Express the function of best fit in the form $A = A_0 e^{kt}$.

c) Use the solution to part b) to estimate the time it takes until there are 15 000 children home-schooled in a year.

d) Use the solution to part b) to predict how many students will be home-schooled in 2020.

e) Use Technology Use a calculator to verify the exponential function of best fit.

B

☆ **3.** A $2500 investment earns 5% interest compounded semi-annually.

a) Write an equation for the value of the investment as a function of time, in years.

b) Determine the value of the investment after 5 years.

c) How long will it take for the investment to double in value?

4. Refer to question 3. Suppose that a penalty for early withdrawal of 3% of the initial investment is applied if the withdrawal occurs within the first 5 years.

a) Write an equation for the adjusted value of the investment as a function of time.

b) Describe the effect this adjustment would have on the graph of the original function.

5. The table below shows the population growth of rabbits living in a warren.

Time (years)	Number of Rabbits
1	12
2	15
3	19
4	23
5	29
6	37
7	46

a) Create a scatter plot of the rabbit population over time.

b) Perform the following types of regression to model the data:

i) linear

ii) quadratic

iii) exponential

c) Record the equation for the line or curve of best fit in each case.

d) Assuming that the rabbit population had been steadily growing for several months before the collection of data, which model best fits the situation? why?

e) Use the model to predict when the population will reach 100.

f) Do you think this trend will continue indefinitely? Explain why or why not.

6. The population of a species of animal in a nature reserve grows by 12.2% each year. Initially, there are 200 of that species.

a) Write an equation for the population of the species as a function of time, in years.

b) After 20 years, an epidemic kills all but 200 of the species. After the epidemic, the population grows as it did before. What will be the equation modelling the population after the epidemic?

c) Sketch the graph of the population for $0 \leq t \leq 30$.

C

☆ 7. Newton's Law of Cooling states that the temperature of a heated object will decrease exponentially over time towards the temperature of the surrounding medium. That is, the temperature, u, of a heated object at a given time, t, obeys the law $u(t) = T + (u_0 - T)e^{kt}$, where T is the constant temperature of the surrounding medium, u_0 is the initial temperature of the heated object, and k is a negative number.

a) An object is heated to 100°C and is then allowed to cool in a room that has an air temperature of 21°C. If after 5 min the temperature of the object is 80°C, when will its temperature be 50°C?

b) A thermometer reading −5°C is brought into a room with a constant temperature of 21°C. If the thermometer reads 15°C after 3 min, what will it read after being in the room for 5 min? for 10 min?

8. The difference in two sound levels, β_1 and β_2, in decibels (dB), is given by the logarithmic equation

$\beta_2 - \beta_1 = 10 \log\left(\frac{I_2}{I_1}\right)$, where $\frac{I_2}{I_1}$ is the ratio of their intensities.

a) The sound level of a jet at take off is 140 dB, while the level of a normal conversation is 50 dB. What is the ratio of the intensities of the sound level of the jet versus the level of normal conversation?

b) What is the loudness of a jackhammer (in use) if it is known that this sound has an intensity 10 times that of sound due to heavy city traffic (90 dB)?

Chapter 7: Challenge Questions

C1. Write each as a power of 6.

a) 216

b) $\frac{1}{36}$

c) $\sqrt[5]{6^7}$

C2. Solve each equation. Check your answers using graphing technology.

a) $4^{5x} = 32^{x+2}$

b) $9^{2x+1} = 27^{x-7}$

C3. The half-life of a substance is 80 days. Initially, there are 500 mg of this substance.

a) How much of the substance will remain after 60 days?

b) When will 100 mg of the substance remain?

C4. Solve exactly.

a) $4^{x-2} = 7^x$

b) $2^{x-3} = 3^{x+7}$

C5. The equation $y = a(2)^{bt}$ models an exponential relation.

a) Explain what the restriction is on b. Why does t not have the same restriction?

b) For what values of b will the equation model an exponential growth? Explain why.

c) For what value of b will the equation model an exponential decay? Explain why.

C6. Solve each equation. Check for extraneous roots.

a) $2^{2x} + 7(2^x) + 12 = 0$

b) $2(5^{2x}) - 7(5^x) = 15$

c) $4^{2x} + 9(4^x) + 14 = 0$

d) $7^{2x} + 3(7^x) = 10$

C7. Scientists who study Atlantic salmon have found that the oxygen consumption, O, of a yearling salmon is given by the function $O = 100\left(3^{\frac{3s}{5}}\right)$, where s is the speed that the fish is travelling in metres per second.

a) What is the oxygen consumption of a fish that is travelling at 3 m/s?

b) If a fish has travelled 1.2 km in an hour, what is its oxygen consumption?

C8. Evaluate, using the laws of logarithms.

a) $\log_7 49 + \log_7 343$

b) $\log_5 1125 - \log_5 9$

c) $2 \log 5 + \log 4$

d) $3 \log 4 + 2 \log\left(\frac{5}{4}\right)$

C9. Write as a single logarithm.

a) $\log_8 7 + 2 \log_8 3 - \log_8 3$

b) $\log_a b + c \log_a d - r \log_a s$

C10. Write as a sum or difference of logarithms. Simplify, if possible.

a) $\log_4 \frac{a^2 b}{\sqrt{c}}$

b) $\log_9\left(\sqrt[3]{y^2 + y}\right)$

C11. Simplify and state any restrictions on the variables.

a) $\log(2x + 10) - \log(x^2 - 25)$

b) $\log(2x^2 + 7x + 6) - \log(2x^2 - 7x - 15)$

C12. Solve.

a) $\log_3(5 - x) = 3$

b) $1 - \log_3(4x - 15) = 0$

C13. Solve $\log_6(x + 3) + \log_6(x - 2) = 1$. Check for extraneous roots.

C14. The percent, P, of caffeine remaining in your bloodstream is related to the elapsed time, t, in hours, by $t = 5\left(\frac{\log P}{\log 0.5}\right)$.

a) How long will it take for the amount of caffeine to drop to 25% of the amount consumed?

b) Suppose you drank a cup of coffee after dinner at 5:00 p.m. What percent of the caffeine will remain in your body when you go to bed at 10:30 p.m.?

C15. Use the product and quotient laws of logarithms to prove the power law of logarithms $\log_a(p^c) = c \log_a p$, where $a > 0$, $a \neq 1$, $p > 0$, $q > 0$, and $p, q \in (0, \infty)$, and $c \in \Re$.

C16. Show that if $\log_b a = c$ and $\log_y b = c$, then, $\log_a y = c^{-2}$.

C17. Solve for x and check your solution.

a) $\log_2 x + \log_4 x + \log_8 x + \log_{16} x = 25$

b) $2(5^{6x}) - 9(5^{4x}) + 13(5^{2x}) - 6 = 0$

C18. The following data represent the amount of money an investor has in an investment account each year for 10 years. She wishes to determine the effective rate of return on her investment.

Year	Value of Account
1999	\$10 000
2000	\$10 573
2001	\$11 260
2002	\$11 733
2003	\$12 424
2004	\$13 269
2005	\$13 968
2006	\$14 823
2007	\$15 297
2008	\$16 539

a) Draw a scatter diagram with the year as the independent variable.

b) Find the exponential function of best fit. Express the function of best fit in the form $A = A_0e^{kt}$.

c) Use the solution to part b) to predict the value of the account in the year 2020.

d) Use the solution to part b) to predict how long it will take to have \$25 000 in the account.

e) **Use Technology** Use a calculator to verify the exponential function of best fit.

C19. A savings bond offers interest at a rate of 8.8% compounded semi-annually. Suppose that a \$1500 bond is purchased.

a) Write an equation for the value of the investment as a function of time, in years.

b) Determine the value of the investment after 12 years.

c) How long will it take for the investment to triple in value?

d) Describe how the shape of the graph of this function would change if

i) a bonus of 5% of the principal was added after 5 years had passed

ii) the size of the initial investment were doubled

C20. On average, the number of items, N, per day, on an assembly line, that a quality assurance trainee can inspect is $N = 45 - 26(0.74)^t$, where t is the number of days worked.

a) After how many days of training will the employee be able to inspect 43 items?

b) The company expects an experienced quality assurance employee to inspect 50 items per day. After the training period of 20 days is complete, how close will the trainee be to the experienced employee's quota?

Chapter 7 Checklist

By the end of this chapter, I will be able to:

- Write a power using different bases
- Solve equations involving exponential expressions using a variety of methods (change of base, algebraic manipulation, use of graphing technology)
- Identify and reject extraneous roots to equations involving exponential expressions
- Solve problems involving exponential equations
- Apply the product and quotient laws of logarithms
- State restrictions on variables for expressions involving logarithms
- Solve equations involving logarithmic expressions using a variety of methods (rewrite numbers as logarithms, apply laws of logarithms, use of graphing technology)
- Solve problems involving logarithmic equations
- Construct and evaluate mathematical models involving exponential and logarithmic relationships
- Apply exponential and logarithmic models to solve contextual problems