Chapter 8 Combining Functions

8.1 Sum and Difference of Functions

KEY CONCEPTS

- Some combined functions are formed by adding or subtracting two or more functions.
- The superposition principle states that the sum of two or more functions can be found by adding the ordinates (y-coordinates) of the function at each abscissa (x-coordinate). For example, the sum of $f(x) = 2x$ and $f(x) = x - 1$ is $f(x) = 3x - 1$, which is calculated by adding the y values at every x value.
- The superposition principle also applies to the difference of two functions.
- The domain of the sum or difference of functions is the domain common to the component function.

Example

Determine an equation for the function $h(x) = f(x) + g(x)$ in each case. Then, graph $h(x)$ and state the domain and range of the function.

a) $f(x) = 5x + 2$, $g(x) = 6$

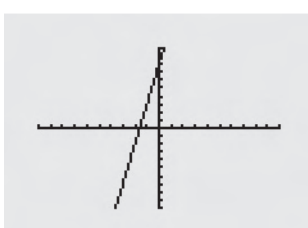
b) $f(x) = x^2 - 1$, $g(x) = x$

Solution

a) $h(x) = f(x) + g(x)$

$h(x) = 5x + 2 + 6$

$h(x) = 5x + 8$

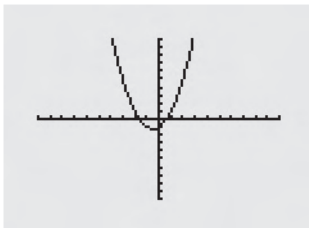


The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}\}$.

b) $h(x) = f(x) + g(x)$

$h(x) = x^2 - 1 + x$

$h(x) = x^2 + x - 1$



The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R} \mid y \geq -1.25\}$.

Example

Student Council is selling prom tickets. The cost to rent the hall is \$1500, plus a meal cost of \$45 per ticket. The Council has decided to sell the tickets for \$60 each.

a) Write an equation to represent

- the total cost, C, as a function of the number, n, of prom tickets sold
- the revenue, R, as a function of the number, n, of prom tickets sold

b) Graph the functions in part a) on the same set of axes. Identify the point of intersection and explain the meaning of its coordinates.

c) Profit, P, is the difference between revenue and expenses. Develop an algebraic and graphical model for the profit function.

d) Under what circumstances will Student Council lose money? make money?

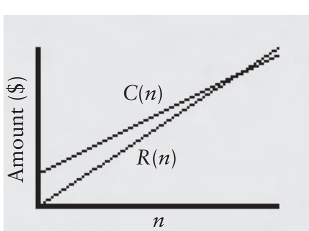
e) Identify the domain and range of the cost, revenue, and profit functions in the context of this problem.

Solution

a) $C(n) = 1500 + 45n$

$R(n) = 60n$

b)

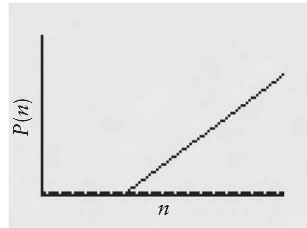


The point of intersection is (100, 6000). This represents the break-even point where the prom would be making money.

c) Profit = Revenue − Costs

$P(n) = 60n - (1500 + 45n)$

$P(n) = 15n - 1500$



d) Student Council will lose money if it sells fewer than 100 tickets. The Council will make money if it sells more than 100 tickets.

e) Cost function: $C(n) = 1500 + 45n$

The domain is $\{n \in \mathbb{R} \mid n \geq 0\}$ and the range is $\{C(n) \in \mathbb{R} \mid C(n) \geq 1500\}$.

Revenue function: $R(n) = 60n$

The domain is $\{n \in \mathbb{R} \mid n \geq 0\}$ and the range is $\{R(n) \in \mathbb{R} \mid R(n) \geq 0\}$.

Profit function: $P(n) = 15n - 1500$

The domain is $\{n \in \mathbb{R} \mid n \geq 0\}$ and the range is $\{P(n) \in \mathbb{R} \mid P(n) \geq -1500\}$.

A

1. For each pair of functions, find

i) $y = f(x) + g(x)$

ii) $y = f(x) - g(x)$

iii) $y = g(x) - f(x)$

a) $f(x) = 6x$ and $g(x) = x - 4$

b) $f(x) = -3x + 7$ and $g(x) = -2x + 3$

c) $f(x) = x^2 - 2$ and $g(x) = 4$

d) $f(x) = 2x^2 + 3x - 7$ and $g(x) = 5x - 2$

2. Let $f(x) = 6x + 7$ and $g(x) = 2x - 5$.

a) Determine an equation for the function $h(x) = f(x) + g(x)$. Then, find $h(-2)$.

b) Determine an equation for the function $h(x) = f(x) - g(x)$. Then, find $h(4)$.

c) Determine an equation for the function $h(x) = g(x) - f(x)$. Then, find $h(-3)$.

3. Let $f(x) = 2x^2 + 3$ and $g(x) = 5x - 7$.

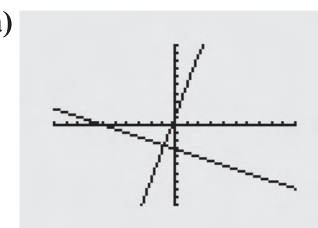
a) Determine an equation for the function $h(x) = f(x) + g(x)$. Then, find $h(0)$.

b) Determine an equation for the function $h(x) = f(x) - g(x)$. Then, find $h(-5)$.

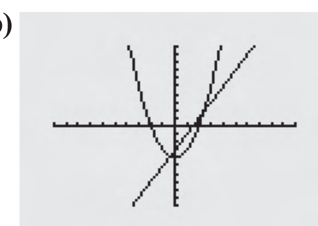
c) Determine an equation for the function $h(x) = g(x) - f(x)$. Then, find $h(6)$.

4. Copy each graph of $f(x)$ and $g(x)$. Then, apply the superposition principle to graph $y = f(x) + g(x)$. Give the domain and range of $y = f(x) + g(x)$.

a)



b)



5. For each graph in question 4, use the superposition principle to graph $y = f(x) - g(x)$.

☆**6.** Let $f(x) = |x|$ and $g(x) = 5$.

a) Graph each of the following:

i) $y = f(x) + g(x)$

ii) $y = f(x) - g(x)$

iii) $y = g(x) - f(x)$

b) Explain how you could also produce each of these combined functions by applying transformations to the graph of $f(x) = |x|$.

c) Give the domain and range of each combined function.

7. Let $f(x) = \cos x$ and $g(x) = 2^x$. Work in radians.

a) Sketch these functions on the same set of axes.

b) Sketch a graph of $y = f(x) + g(x)$.

c) Sketch a graph of $y = f(x) - g(x)$.

d) Check your answers using graphing technology.

B

8. A bicycle courier has a fixed delivery cost of \$12.50 per day, plus a variable cost of \$0.25/package. He earns \$4.25 per package he delivers, in revenue. The maximum number of packages that he can deliver in a day is 27.

a) Write an equation to represent

- the total cost, C, as a function of the number, p, of packages delivered
- the revenue, R, as a function of the number, p, of packages delivered

b) Graph $C(p)$ and $R(p)$ on the same set of axes.

c) Identify the break-even point and explain what its coordinates mean.

d) Develop an algebraic and graphical model for the profit function, $P(p) = R(p) - C(p)$.

e) Identify the domain and range in the context of this problem for $C(p)$, $R(p)$, and $P(p)$.

f) What is the maximum daily profit the courier can earn?

9. Refer to question 8. The bicycle courier has found a way to improve the efficiency of his operation that will allow him to either reduce his fixed cost to \$10.25 per day *or* reduce his variable cost to \$0.15/package.

a) Which of these two options has the more favourable effect on

i) the break-even point?

ii) the potential maximum daily profit?

b) What advice would you give the bicycle courier?

☆**10.** Lara is going on a summer camping vacation, the first part to be spent canoeing, the second part to be completed by hiking. The time Lara will spend canoeing is given by the function $T_c(x) = \frac{(x^2 + 4)^{\frac{1}{2}}}{3}$, and the time she will spend hiking is given by the function $T_h(x) = \frac{12 - x}{5}$.

a) Sketch graphs of these functions on the same set of axes.

b) Graph the combined function $T_c(x) + T_k(x)$.

c) Identify the domain and range of $T_c(x) + T_k(x)$.

11. Consider the combined function $T(x) = f(x) + g(x) + h(x)$, where $f(x) = 3x + 1$, $g(x) = 2x^2$, and $h(x) = 3^x$.

a) Graph $f(x)$, $g(x)$, and $h(x)$ on the same set of axes. Use colours or different line styles to easily distinguish the curves.

b) Graph the combined function $T(x)$.

c) The function $T(x)$ appears to converge with one of the three component functions. Which one is it?

d) Explain the result in part c) by considering the rates of change of the component functions.

12. Let $f(x) = x - 4$ and $g(x) = x + 7$.

a) Write an expression for $-g(x)$.

b) Graph $f(x)$ and $-g(x)$ on the same set of axes.

c) Add these functions to produce $f(x) + [-g(x)]$.

d) Graph $f(x)$ and $g(x)$ together on another set of axes.

e) Subtract these functions to produce $f(x) - g(x)$ and compare this result to the one obtained in part c).

13. A computer manufacturing company produces laptop computers for \$175 per unit, plus a fixed operating cost of \$750 000. The company sells the laptops for \$599.00 per unit.

a) Determine a function, $C(x)$, to represent the cost of producing x units.

b) Determine a function, $S(x)$, to represent sales of x units.

c) Determine a function that represents the company's profit.

14. Let $f(x) = mx^2 + 7x + 8$ and $g(x) = 3x^2 - nx + 3$. The functions are combined to form the new function $h(x) = f(x) + g(x)$. Points (1, 18) and (−1, 14) satisfy the new function. Determine $f(x)$ and $g(x)$.

15. Consider two functions, f and g. Explain why the domains of f and g must be the same in order to add or subtract the functions.

16. Describe how to determine the type of function for $(f + g)(x)$ if we know the degree of $f(x)$ and the degree of $g(x)$.

17. a) Sketch graphs of $f(x) = \sin x$ and $g(x) = \cos x$ on the same set of axes. Use the domain $-3\pi \leq x \leq 3\pi$.

b) Use the principle of superposition to sketch a graph of $y = f(x) + g(x)$.

c) Determine the equation of $y = f(x) + g(x)$. Express your answer as a sine function.

d) Sketch a graph of $y = f(x) - g(x)$.

e) Determine the equation of $y = f(x) - g(x)$. Express your answer as a sine function.

C

18. Let $f(x) = \sin x$ and $g(x) = x$.

a) Use graphing technology to graph $f(x)$ and $g(x)$ on the same set of axes. Work in radians.

b) Predict the shape of $h(x) = f(x) + g(x)$. Sketch a graph of your prediction.

c) Use graphing technology to check your prediction.

d) Hide $f(x)$ so that only $g(x)$ and $h(x)$ are visible. How many intersection points do there appear to be?

e) Describe what you notice about where the line $g(x) = x$ intersects the graph of $h(x)$.

f) Consider the curvature (up versus down) of $h(x)$. Explain why this is so.

KEY CONCEPTS

- A combined function of the form $y = f(x)g(x)$ represents the product of two functions, $f(x)$ and $g(x)$.
- A combined function of the form $y = \frac{f(x)}{g(x)}$ represents the quotient of the two functions, $f(x)$ and $g(x)$, for $g(x) \neq 0$.
- The domain of the product and quotient of functions is the domain common to the component functions. The domain of a quotient function, $y = \frac{f(x)}{g(x)}$, is further restricted by excluding any values that make the denominator, $g(x)$, equal to zero.
- Products and quotients of functions can be used to model a variety of situations.

Example

Let $f(x) = 2x + 3$ and $g(x) = 2x^2 + x - 3$. Determine an equation for each combined function. Sketch a graph of the combined function and state its domain and range.

a) $y = f(x)g(x)$

b) $y = \frac{f(x)}{g(x)}$

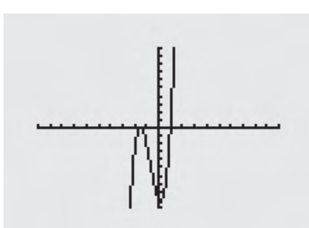
Solution

a) $y = f(x)g(x)$

$y = (2x + 3)(2x^2 + x - 3)$

$y = 4x^3 + 2x^2 - 6x + 6x^2 + 3x - 9$

$y = 4x^3 + 8x^2 - 3x - 9$



The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}\}$.

b) $y = \dfrac{f(x)}{g(x)}$

$y = \dfrac{2x + 3}{2x^2 + x - 3}$

$y = \dfrac{2x + 3}{(2x + 3)(x - 1)}$

$y = \dfrac{1}{x - 1}$

The domain is $\left\{x \in \mathbb{R} \middle| x \neq \frac{-3}{2}, 1\right\}$ and the range is $\{y \in \mathbb{R} | y \neq 0\}$.

Example

The cost, C, in thousands of dollars, for Widgets-R-Us to produce x thousand of its widgets is given by the function $C(x) = 230 + 0.24x + 0.000\,1x^2$. The revenue, R, from the sales of its widgets is given by the function, $R(x) = xD(x)$, where $D(x)$ is the demand function, that is, the price for a widget when x widgets are sold.

a) Write the function that represents the company's profits on sales of x of the widgets.

b) Graph the cost, revenue, and profit functions.

c) What is the company's profit on the sale of 1000 widgets?

Solution

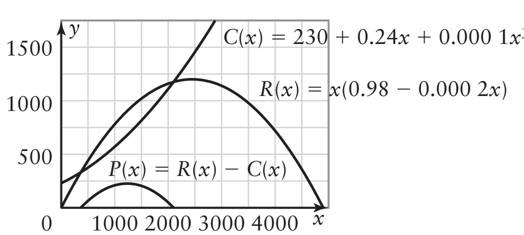
a) Profits = Revenue − Costs

$P(x) = xD(x) - C(x)$

$P(x) = 0.98x - 0.000\,2x^2 - (230 + 0.24x + 0.000\,1x^2)$

$P(x) = -0.000\,3x^2 + 0.74x - 230$

b)



c) $P(x) = -0.000\,3x^2 + 0.74x - 230$

$P(1000) = -0.000\,3(1000)^2 + 0.74(1000) - 230$

$P(1000) = 210$

In this example, the profit on the sale of 1000 widgets is $210.

A

1. Let $f(x) = x + 3$ and $g(x) = x^2 - 9$. Develop an algebraic and graphical model for each combined function. Then, give the domain and range of the combined function. Identify any holes or asymptotes.

   a) $y = f(x)g(x)$

   b) $y = \dfrac{f(x)}{g(x)}$

   c) $y = \dfrac{g(x)}{f(x)}$

2. Let $u(x) = \sin x$ and $v(x) = 0.5^x$. Develop an algebraic and a graphical model for each combined function. Then, give the domain and range of the combined function. Identify any holes or asymptotes.

   a) $y = u(x)v(x)$

   b) $y = \dfrac{u(x)}{v(x)}$

   c) $y = \dfrac{v(x)}{u(x)}$

B

3. A small town initially has a population of 1500 people. The population, P, of people grows as a function of time, t, in years, as $P(t) = 1500(1.5)^t$. The surrounding farms supply the town with food. The amount, F, of food is decreasing as the size of the farms is decreasing because of urban sprawl, according to the function $F(t) = 5000(0.25)^t$.

   a) Graph the functions $P(t)$ and $F(t)$ on the same set of axes. Describe the nature of these functions.

   b) Determine the domain and range of these functions.

   c) Identify the point of intersection of these two curves. Determine the coordinates, to two decimal places, and explain what they mean. Call this point in time the crisis point.

   d) Graph the function $y = F(t) - P(t)$. Explain the significance of this function.

   e) What is the t-intercept of the function $y = F(t) - P(t)$? How does this relate to the crisis point?

   f) Comment on the validity of the mathematical model for $P(t)$ for t values greater than this intercept.

4. Refer to the equations from question 3.

   a) Graph the function $y = \dfrac{F(t)}{P(t)}$ on a different set of axes. What does this function represent? Describe the shape of this function.

   b) What are the coordinates of this function at the crisis point? Explain the meaning of these coordinates.

   c) Describe the living conditions of the people in the town before, at, and after the crisis point.

☆ 5. Let $f(x) = \sqrt{x^2 - 4}$ and $g(x) = \cos x$.

   a) Graph $f(x)$ and describe its shape. Is this function even, odd, or neither?

   b) Graph $g(x)$ on the same set of axes. Is this function even, odd, or neither?

   c) **Use Technology** Predict the shape of $y = f(x)g(x)$. Sketch a graph of your prediction. Then, check your prediction using graphing technology.

   d) Give the domain and range of $y = f(x)g(x)$.

6. Refer to the functions in question 5.

   a) Graph $y = \dfrac{g(x)}{f(x)}$. Is this function even, odd, or neither? Give the domain and range.

   b) Graph $y = \dfrac{f(x)}{g(x)}$. Is this function even, odd, or neither? Give the domain and range.

7. Alex has decided to breed rabbits. The initial population was 12 rabbits and is growing at a rate of 55% per year. The population, P, as a function of time, t, in years, can be modelled by the function $P(t) = 12(1.55)^t$. However, Alex has to feed the rabbits. The amount of food that can sustain the hutch is given by the equation $F(t) = 16 + 0.05t$.

a) Graph $P(t)$ and $F(t)$ on the same set of axes and describe the trends.

b) Graph the function $y = F(t) - P(t)$, and describe the trend. Does the hutch enjoy a food surplus or suffer from a food shortage? Explain. What about in years to come?

c) Identify the coordinates of the maximum of $y = F(t) - P(t)$ and explain what they mean.

8. The food production per capita in question 7 is the amount of food per rabbit.

a) Graph the function $y = \frac{F(t)}{P(t)}$ and describe its trend.

b) At what time is the food production per capita a maximum for the rabbit hutch? Does this point coincide with the maximum in part c) of question 7? Explain the result.

c) What does it mean when $\frac{F(t)}{P(t)} > 1$ and $\frac{F(t)}{P(t)} < 1$? When are these conditions projected to occur in the rabbit hutch?

C

☆**9.** Consider the functions $f(x) = 3^{-x}$ and $g(x) = x^2$. Sketch a graph of the function $y = f(x)g(x)$ and describe its key features. Explain its end behaviour.

10. Let $f(x) = x - 3$ and $g(x) = x^2 - 4x + 3$.

a) Determine each combined function.

i) $y = f(x) - g(x)$

ii) $y = \frac{f(x)}{g(x)}$

b) State the domain and range of $y = \frac{f(x)}{g(x)}$.

11. At the Canadian National Exhibition there is a swing ride that can be modelled by the function $s(t) = 5\cos(4t) \times 0.85^t$, where s is the horizontal displacement from the rest position in metres, as a function of time, t, in seconds.

a) What was the initial horizontal displacement of the swing?

b) Sketch how the shape of the graph would change if

i) the air resistance were reduced

ii) the length of the swing were lengthened.

Justify your answers with mathematical reasoning.

12. The algebraic tests used to decide whether a function is even or odd are as follows.

- A function f is even provided $f(-x) = f(x)$.
- A function f is odd provided $f(-x) = f-(x)$.

a) Suppose f and g are both odd. Prove that $y = f(x)g(x)$ is even.

b) Suppose f is even and g is odd. Prove that $y = f(x)g(x)$ is odd.

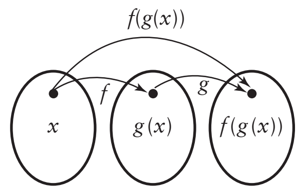
c) Suppose f and g are both even. Prove that is $y = f(x)g(x)$ even.

d) Is the product of functions in any way analogous to the multiplication of numbers when it comes to evenness and oddness? Explain.

13. Let $f(x) = 3x^2 + mx - 1$ and $g(x) = nx^2 + 2x + 4$. The functions are combined to form a new function $h(x) = f(x) \times g(x)$. Points (1, 33) and (−1, −3) satisfy the new function. Determine $f(x)$ and $g(x)$.

KEY CONCEPTS

- $f(g(x))$ denotes a composite function, that is, one in which the function $f(x)$ depends on the function $g(x)$. This can also be written as $(f \circ g)(x)$.



- To determine an equation for a composite function, substitute the second function into the first, as read from left to right. To determine $f(g(x))$, substitute $g(x)$ for x in $f(x)$.

- To evaluate a composite function $f(g(x))$ at a specific value, substitute the value into the equation of the composite function and simplify, or evaluate $g(x)$ at the specific value and then substitute the result into $f(x)$.

Example

Let $f(x) = 3x^2$ and $g(x) = x - 2$. Determine an equation for each composite function, graph the function, and give its domain and range.

a) $y = f(g(x))$

b) $y = g(f(x))$

c) $y = f(f(x))$

d) $y = g(g(x))$

e) $y = g^{-1}(g(x))$

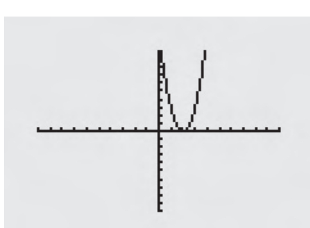
Solution

a) $y = f(g(x))$

$y = 3(x - 2)^2$

$y = 3(x^2 - 4x + 4)$

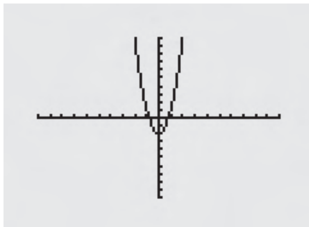
$y = 3x^2 - 12x + 12$



The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R} \mid y \geq 0\}$.

b) $y = g(f(x))$

$y = 3x^2 - 2$



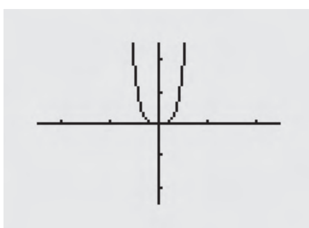
The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R} | y \geq -2\}$.

c) $y = f(f(x))$

$y = 3(3x^2)^2$

$y = 3(9x^4)$

$y = 27x^4$

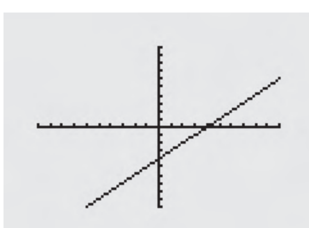


The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R} | y \geq 0\}$.

d) $y = g(g(x))$

$y = (x - 2) - 2$

$y = x - 4$

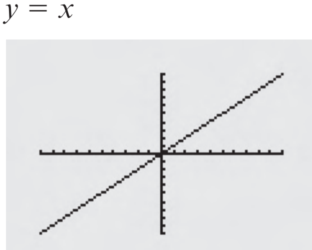


The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}\}$.

e) $y = g^{-1}(g(x))$

$y = (x - 2) + 2$

$y = x$



The domain is $\{x \in \mathbb{R}\}$ and the range is $\{y \in \mathbb{R}\}$.

A

1. An Internet service provider has modelled its projected number of business subscribers as $B(t) = 50t + 2500$, and the projected average number of Internet connections per business subscriber as $C(t) = 0.004t^2 + 3.2$, where t is the time, in weeks, over the next year. The projected total number of business Internet connections is then $N(t) = B(t) \times C(t)$.

   a) Determine the growth rate of the number of business connections after t weeks.

   b) Determine the growth rate of the number of business connections after 12 weeks.

2. Let $f(x) = x - 2$ and $g(x) = (x - 1)^2$. Determine a simplified algebraic model for each composite function.

   a) $y = f(g(x))$ **b)** $y = g(f(x))$

   c) $y = f(f(x))$ **d)** $y = g(g(x))$

   e) $y = f^{-1}(f(x))$

3. Graph each composite function in question 2. Give the domain and range.

4. **Use Technology** Check your answers to question 3 using graphing technology.

☆5. Let $f(x) = x^2 - 3x + 5$ and $g(x) = \frac{1}{x-2}$. Evaluate:

   a) $g(f(1))$ **b)** $f(g(4))$

B

6. In the Student Council election, the popularity, P, as a percent of voters, of Candidate A can be modelled by a function of time, t, in days throughout the campaign, as $P(t) = 50 - 0.4t$. The popularity, $R(t)$ of the opposing Candidate B can be modelled by a composite function of $P(t)$, $R(P(t)) = 40 + 0.65(50 - P(t))$.

   a) Graph $P(t)$ and describe the trend.

   i) What is the popularity of Candidate A at the beginning of the campaign?

   ii) What is the rate of change of this function, and what does it mean?

   iii) Can you tell if these are the only two candidates in the election? If you can, explain how. If you cannot, describe what additional information is required.

   b) Graph $R(t)$and describe the trend.

   i) What is the popularity of Candidate B at the beginning of the campaign?

   ii) What is the rate of change of this graph? what does it mean?

   iii) If it can be assumed that these are the only two candidates running for election, which candidate do you think will win? Does your answer depend on something? Explain.

   c) Assume that there are at least three candidates running in the election.

   i) Graph the composite function $V(t) = 100 - [P(t) + R(t)]$. What does this graph represent? Describe the trend.

   ii) Assuming that this is a three-candidate election, and that there is no undecided vote, can you tell who will win this election? Explain.

   iii) Repeat part c) assuming that there are four candidates.

7. Let $f(x) = x^4$.

   a) Determine $f^{-1}(x)$

   b) Determine $f\left(f^{-1}(x)\right)$

   c) Determine $f^{-1}\left(f(f^{-1}(x))\right)$

   d) Compare your answers to parts b) and c).

   e) Determine $f(f^{-1}(2))$, $f(f^{-1}(4))$, and $f(f^{-1}(-2))$. What do you notice?

8. Let $f(x) = x^3$, $g(x) = x^5$, and $h(x) = \cos x$. Work in radians.

a) Predict what the graph of $y = f(h(x))$ will look like, and sketch your prediction.

b) Check your prediction using graphing technology.

c) Is the function in part a) periodic? Explain.

d) Identify the domain and range.

9. **a)** Repeat question 8 for $y = g(h(x))$.

b) Compare $y = f(h(x))$ and $y = g(h(x))$. How are these functions similar? different?

10. In an electric circuit, the current through a resistor, in amperes, is given by $I = 4.85 - 0.001t^2$, and the resistance, in ohms, is given by $R = 15.00 + 0.11t$, where t is the time, in seconds. The voltage, V, in volts, across the resistor is the product of the current and the resistance.

a) Determine an equation for the voltage as a function of time.

b) Sketch a graph of this function.

c) How long will it take for the voltage to reach 77 V, to the nearest second?

☆11. A manufacturing company models its weekly production of alarm clocks since 2007 by the function $N(t) = 300 + 75t$, where t is the time, in years, since 2007, and N is the number of alarm clocks. The size of the company's workforce can be modelled by the composite function $W(N) = 2\sqrt{N}$.

a) Write the size of the workforce as a function of time.

b) State the domain and range of the new function and sketch its graph.

C

12. To rent a sailboat for an evening cruise costs $C(h) = 2500 + 50h$, where h is the number of hours of the cruise. During the month of June, when there are a number of proms, the sailboat's rental company is offering a 5% discount.

a) Write the new cost equation for an evening cruise including the discount.

b) If a prom committee has raised \$2600, how long can they cruise at the new price?

13. Let $f(x) = 2x^3$, $g(x) = 4x - 7$, and $h(x) = \frac{-1}{x}$.

a) Determine a simplified algebraic model for each composite function.

i) $f(g(x))$

ii) $h(g(x))$

iii) $g^{-1}(h(x))$

b) Evaluate $f(h(-1))$.

14. Let $f(x) = |x|$, $g(x) = \sin x$, and $h(x) = x^3$. Work in radians.

a) What is the domain of $f(x)$?

b) Use this information to predict the shape of the graph of the composite function $y = f(g(x))$. Sketch your prediction.

c) Check your prediction in part b) using graphing technology. Give the domain and range of $y = f(g(x))$.

d) Use your result in part c) to predict the shape of the graph of the composite function $y = f(h(x))$. Sketch your prediction.

e) Check your prediction in part d) using graphing technology. Give the domain and range of $y = f(h(x))$.

15. Let $f(x) = 25 - x^2$ and $g(x) = \frac{1}{x - 9}$.

a) Determine the domain and range of $y = f(g(x))$.

b) Determine the domain and range of $y = g(f(x))$.

c) Use graphing technology to verify your answers in parts a) and b).

KEY CONCEPTS

- Solutions to problems involving combined functions can sometimes lead to a range of acceptable answers. When this happens, techniques for solving inequalities are applied.
- There are a number of ways to graphically illustrate an inequality involving a combined function.
- Algebraic and graphical representations of inequalities can be useful for solving problems involving combined functions.

A

1. Let $f(x) = 2x$ and $g(x) = x^2 - 1$.

a) Graph the functions on the same set of axes. Identify the points of intersection.

b) Illustrate the regions for which

i) $f(x) > g(x)$

ii) $g(x) > f(x)$

2. Solve $\cos x > 2^x$.

3. Talk-Us's cost, C, for shipping and storing n cell phones can be modelled by the function $C(n) = n + \frac{100}{n}$, where n represents years.

a) Graph this function and explain its shape. What is the domain of interest for this problem?

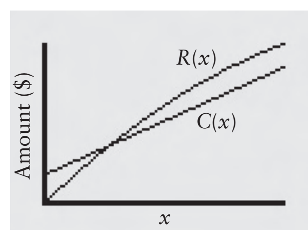
b) Determine the maximum and minimum number of cell phones that can be ordered at any one time to keep costs below \$55, assuming that inventory has fallen to zero.

c) What is the optimum order size that will minimize storage costs?

d) Why might this be the best number to order?

Use the following information to answer questions 4 and 5.

Revenue and cost functions for the Howard's Hamburgers restaurant chain are shown on the graph.



4. a) Suppose that Howard's Hamburgers found a way to reduce its variable cost to \$0.35. How would this affect

i) the minimum and maximum number of burgers the chain could sell?

ii) the maximum potential profit?

b) Sketch a graph to illustrate your explanation.

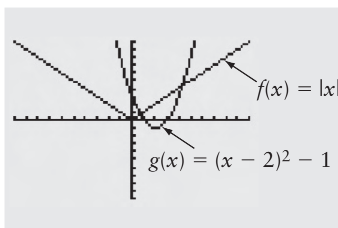
5. a) Suppose that, instead of the variable cost being reduced, the restaurant chain's fixed costs increase by \$225 000. How would this affect

   i) the minimum and maximum number of burgers the chain could sell?

   ii) the maximum potential profit?

   b) Sketch a graph to illustrate your explanation.

6. The graph of two functions is shown.



   a) For what values of x is

   i) $f(x) > g(x)$?

   ii) $f(x) < g(x)$?

   b) Sketch the graph of $y = f(x) - g(x)$ on the interval (0, 5).

   c) For what region is $f(x) - g(x) > 0$? Explain how this corresponds to your answer to part a).

7. Let $u(x) = -2x + 3$ and $v(x) = \left(\frac{1}{2}\right)^x$.

   a) Graph these functions on the same set of axes.

   b) Graph the combined function $y = \dfrac{u(x)}{v(x)}$.

   c) Explain how $y = \dfrac{u(x)}{v(x)}$ can be used to identify the regions where

   i) $u(x) > v(x)$

   ii) $u(x) < v(x)$

☆8. Let $f(x) = -x^3 - x^2 + 17x - 15$ and $g(x) = (x - 5)^2 - 18$.

   a) Graph these functions on the same set of axes.

   b) Identify, by inspecting the graphs, the intervals for which

   i) $f(x) > g(x)$

   ii) $g(x) > f(x)$

9. Solve question 8b) using two other methods.

10. Solve $\cos x > x^2$.

11. Solve $x^3 > \left(\frac{1}{4}\right)^x$.

Use the following information to answer questions 12 to 14. The owner of a local amusement park needs to identify the optimum price for admission tickets to maximize his profits. The number, N, of people who attend the amusement park is a function of the price, p, in dollars. $N(p) = -(p + 5)(p - 17)$, assuming the minimum ticket price is \$12.00.

B

12. a) Graph $N(p)$.

   b) Identify the region for which $N(p) > 0$. What does this suggest about the maximum realistic ticket price? Explain your answer.

   c) Identify the domain and range for which $y = N(p)$ has meaning.

13. The revenue generated, R, in dollars, is $R(p) = N(p) \times p$, where p is the number of tickets sold.

   a) Graph the function $R(p)$ on a graphing calculator.

   b) For what region is $R(p) > 0$? Does this result agree with the result found in question 12b)?

   c) Do the maxima of $N(p)$ and $R(p)$ occur at the same value of p? Explain why or why not.

   d) What does $R(p)$ suggest that the optimum ticket price is? Explain.

14. The cost, C, of running the amusement park can be modelled by a composite function of $N(p)$, $C(p) = 75 + 12N(p)$.

a) Graph the function $C(p)$ on a graphing calculator.

b) Graph the combined function $y = R(p) - C(p)$. What does this function represent?

c) Identify the region for which $R(p) - C(p) > 0$. What is the significance of this region?

d) Do the maxima of $y = R(p)$ and $y = R(p) - C(p)$ occur for the same value of p? Explain why or why not.

e) Identify the optimum ticket price for the amusement park and determine the maximum profit per ticket.

C

15. The Parkdalian Pen Company estimates that the cost of manufacturing x pens is $C(x) = 6000 + 0.8x$ and the revenue is $R(x) = \frac{1}{10\,000}(30\,000x - x^2)$.

a) Graph $R(x)$ and $C(x)$ on the same set of axes.

b) How many points of intersection does this system have? Explain their significance.

c) Identify the region where $R(x) > C(x)$. Why is this region important?

d) Maximum profit occurs when $R(x)$ exceeds $C(x)$ by the greatest amount. Use the superposition principle to graph the function $P(x) = R(x) - C(x)$.

e) Use this function to determine

i) the optimum number of units sold

ii) the maximum profit per unit sold

iii) the total profit, if the optimum number of units are sold

f) Reflect on the shapes of the revenue and cost curves. Suggest some reasons why they are shaped like this.

☆**16.** Stephanie makes and sells jewellery for the gift shop at the museum. Stephanie makes n necklaces in a given week and sells them for $25 - n$ dollars per necklace. Her costs include a fixed cost of \$55 plus \$3.50 per necklace made. Assume that Stephanie sells all of the necklaces that she makes.

a) Write an equation to represent her total weekly cost.

b) Write an equation to represent her total weekly revenue.

c) Write an inequality to express the conditions with which Stephanie will make a profit.

d) How many necklaces should Stephanie make each week in order to make a profit?

e) What is the optimum number of necklaces Stephanie should make in order to earn maximum profit? How much will she earn if she does this?

17. The projected population, P, of a town can be modelled by the function $P(t) = 1500(1.025)^t$, where t is the time, in years, from now. The expected number, N, of people who can be supplied by the local water services can be modelled by the function $N(t) = 4200 + 45.2t$.

a) Determine $y = N(t) - P(t)$ and sketch the graph.

b) Explain what the function in part a) represents.

c) When is $N(t) - P(t) < 0$? Explain what this means.

d) Determine $y = \frac{N(t)}{P(t)}$ and sketch its graph.

e) Explain what the function in part d) represents.

KEY CONCEPTS

- A variety of real world situations can be modelled using combined functions
- To develop a model consisting of a combined function, consider
  - the component functions that could be combined to form the model
  - the nature of the rate of change of the component function
  - the other key features of the graph or equation that fit the given scenario

Example

The following table lists one octave of the frequencies of commonly used notes in North American music, rounded to the nearest hertz (Hz). This is called the Chromatic scale.

Note	Frequency (Hz)
C	262
C#	277
D	294
D#	311
E	330
F	349
F#	370
G	392
G#	415
A	440
A#	466
B	494
High C	524
High D	588
High F#	740
High G	784
High A	880

The graph of a pure note can be modelled by the function $I(t) = \sin(2\pi ft)$, where I is the sound intensity; f is the frequency of the note, in hertz; and t is the time, in seconds.

a) Graph the intensity functions for D and high D. Then, graph the combined function of these two notes struck together on domain $[-0.001, 0.005]$. Describe the resultant waveform.

b) The G-major triad is formed by striking the following notes simultaneously:

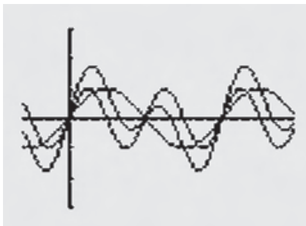
G B D

Graph the combined function for these notes struck together. Explain why these notes sound good together.

c) Graph the intensity functions for D and G# and the combined function for these two notes struck together. Explain why these notes are discordant (i.e. do not sound good together).

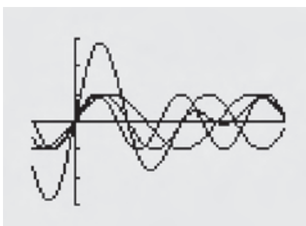
Solution

a)



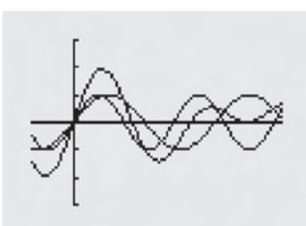
The resultant waveform is a smooth curve that is similar to the initial waveforms.

b)



The notes of the G-major triad sound good together because the resultant curve of the combined function is similar to the waveforms graphed for the individual notes.

c)



The notes D and G# do not sound good together because the resultant curve of the combined function does not follow the waveforms graphed for the individual notes.

1. A D-major triad can be expanded into other D-major chords by adding additional notes of the triad from the next octave.

a) Graph the combined function formed by the following notes being struck simultaneously:

D F# A High F# High G

b) Compare this waveform to the one for the D-major triad.

2. A power chord is formed by dropping the major third (or F# note) from the D-major triad.

a) Graph each D power chord.

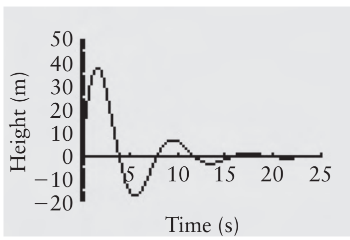
i) D A

ii) D A High A

b) Compare these waveforms to those of the D-major triad and the D-major chord.

3. A skier is skiing down a 200 m hill at a constant speed of 2 m/s through a series of moguls, or small hills. The constant slope of the hill was -2. Assuming that the moguls measure 0.80 m from crest to trough and are roughly 6 m apart, develop an algebraic and a graphical model of the height of the skier versus time.

Use this graph of the path of the bungee jumper to answer questions 4 to 6.

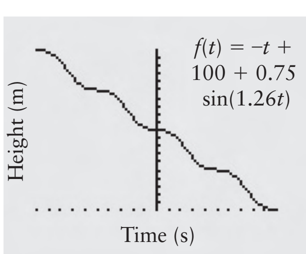


4. Copy the graph of this curve. Sketch how the path of the jumper would change if she dropped from
   a) a greater height
   b) a lower height

5. Sketch how the path of the jumper would change is she were attached to
   a) a longer cord
   b) a shorter cord

6. Sketch how the path of the jumper would change if she were attached to
   a) a springier cord
   b) a stiffer cord

7. A skier is skiing down a hill at a constant speed. His height versus time graph is shown.



Sketch how the graph would change if the moguls were
   a) shorter
   b) farther apart

☆8. A skier is going up and down the same hill at regular intervals. On each run, she skis at an average speed of 1.5 m/s from the top of the hill to the bottom, waits in the chairlift line for about 3 min, and then travels up the chairlift at a speed of 0.75 m/s.
Assume that this hill has no moguls.
   a) Develop a graphical model of the skier's height versus time over the course of several ski runs.
   b) Explain what is happening during each region of the graph for one cycle.
   c) How might you develop an algebraic model to describe this motion?

9. Refer to question 8. Adjust your graph from part a) to illustrate the effect on the skier's height function in each scenario.
   a) After her first run, the lift breaks and she spends an extra 5 min waiting in the lift line.
   b) On her second trip up the chairlift, the lift is stopped for 2 min so that the lift operator can help some new skiers onto the lift.
   c) On her third trip down the hill, the skier tripled her speed.

B

10. The data shown models the growth of a field mouse population where the mice have no predators.

Year	Mice Population
1973	630
1976	2 097
1978	5 275
1979	8 185
1980	12 320
1983	35 487
1986	66 265
1993	91 780
1998	92 875
2004	92 752
2008	93 205

a) Using 1973 as year 0, create a scatter plot of the number of mice, N, versus time, t, in years.

b) Determine a line or curve of best fit, using a method of your choice (e.g., regression analysis, sliders, systematic trial). Justify the type of function that you chose.

c) Write an equation for $N(t)$.

d) There is an outlier in 1993 that does not appear to fit the trend very well. What effect does removing the outlier have on the model for $N(t)$? Does this effect appear to be significant?

☆**11.** While testing the speed of a car in kilometres per hour, $v(t)$ was measured.

Time, t	Kilometres Per Hour, $v(t)$
0	50
1	55
2	62
3	71
4	82
5	95
6	110

The rate of gas consumption for this car, c litres per kilometre, at a speed of $v(t)$ kilometres per hour, is represented by $c(v) = \left(\frac{v(t)}{450} - 0.09\right)^2 + 0.18$.

a) Create a scatter plot of kilometres, $v(t)$, versus time, t, in hours.

b) Determine a line or curve of best fit, using a method of your choice. Justify the type of function that you chose.

c) Write an equation for $v(t)$.

d) Determine algebraically $c(v(t))$.

12. Relative motion uses the superposition principle. Consider the position of a person relative to the ground as she runs back and forth in a train while the train itself is also moving.

a) Suppose the runner's motion relative to the train is $x_1(t) = -\cos(0.2\pi t)$, where x_1 is in metres and t is in seconds. If the train is moving at a constant speed of 1 m/s, its position is $x_2(t) = y$.

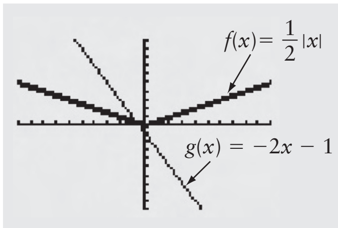
i) Predict what the motion of the runner would look like from the vantage point of a person standing on the ground beside the train as it went by. Sketch a graph of this motion.

ii) Graph $y = x_1(t) + x_2(t)$ to check your answer to part a). Work in radians, and use technology.

b) Repeat part a) but with the train accelerating from rest, so that $x_2(t) = 0.15t^2$.

Chapter 8: Challenge Questions

C1. Copy the graph.



Use the superposition principle to generate a graph of each function.

a) $y = f(x) + g(x)$
b) $y = f(x) - g(x)$
c) $y = g(x) - f(x)$

C2. Let $f(x) = x + 5$, $g(x) = x^2 - 21 + x$, and $h(x) = \left(\frac{1}{2}\right)^x$.

Determine an algebraic and graphical model for each combined function. Identify the domain and range in each case.

a) $y = f(x) + g(x)$

b) $y = f(x) + g(x) + h(x)$

c) $y = f(x) - h(x)$

C3. Use Technology Use graphing technology to check your answers to question C2.

C4. Alexis can earn \$7.50/h as a server, plus an additional \$12.25/h in tips.

a) Graph Alexis's earnings from wages as a function of hours worked.

b) Graph Alexis's earnings from tips as a function of hours worked.

c) Develop an algebraic and a graphical model for Alexis's total earnings.

d) How much can Alexis earn if she works 32 h in one week?

C5. Let $u(x) = x^4$ and $v(x) = \sin x$. Work in radians.

a) What type of symmetry do you predict the combined function $y = u(x)v(x)$ will have? Explain your reasoning.

b) Use Technology Use graphing technology to check your prediction.

C6. Let $f(x) = x - 5$ and $g(x) = x^2 - 9x + 20$.

a) Determine an algebraic and a graphical model for $y = \frac{f(x)}{g(x)}$ and identify its domain and range.

b) Determine an algebraic and a graphical model for $y = \frac{g(x)}{f(x)}$ and identify its domain and range.

C7. Let $f(x) = x^2 + 5x + 1$ and $g(x) = 7x - 4$. Determine an equation for each composite function, graph the function, and give its domain and range.

a) $y = f(g(x))$

b) $y = g(f(x))$

c) $y = g(g(x))$

d) $y = g^{-1}(g(x))$

C8. Let $f(x) = 7\sin\left(\frac{1}{2}x\right)$ and $g(x) = 2^x$.

a) Identify the region for which

i) $f(x) > g(x)$

ii) $g(x) > f(x)$

b) Illustrate this inequality graphically in two different ways.

C9. Let $f(x) = x^3 - 5x^2 - 35x$ and $g(x) = 9 - x^2$.

a) Graph these functions on the same set of axes.

b) Identify, by inspecting the graphs, the interval(s) for which $f(x) > g(x)$.

c) Check your answer to part b) using another method.

C10. A pendulum is released and allowed to swing back and forth according to the equation $x(t) = 8 \sin(3t) \times 0.80^t$, where x is the horizontal displacement from the rest position, in centimetres, as a function of time, t, in seconds.

a) Graph the function. What type of motion is this? Identify the domain and range in the context of this problem.

b) This combined function is the product of two component functions. Identify the component that is responsible for:

i) the periodic nature of the motion

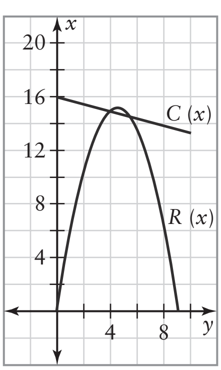
ii) the exponential decay of the amplitude

c) At what horizontal distance from the rest position was the bob of the pendulum released?

d) At what point(s) is the rate of change zero? When does this occur with respect to the motion of the pendulum bob?

e) After what elapsed time will the pendulum's amplitude diminish to 30% of its initial value?

C11. The cost, C, and revenue, R, as functions of the number of burgers sold by Harry's Hamburgers, are shown on the graph.



a) Identify the region(s) for which

i) $C > R$

ii) $R > C$

b) What can you conclude about this business venture?

c) What suggestions would you give to the restaurant in order to help it improve this situation?

C12. A company that produces sunscreen estimates that the cost of manufacturing x bottles of the product is given by $C(x) = 480 - 0.32x + 0.000\,5x^2$ and the revenue is given by $R(x) = 0.78x + 0.000\,3x^2$.

a) Graph $C(x)$ and $R(x)$ on the same set of axes.

b) Identify the break-even point and explain what its coordinates mean.

c) Develop an algebraic and a graphical model for the profit function, $P(x)$.

d) What is the maximum daily profit that the sunscreen company can earn?

C13. A skier is skiing down a 350-m hill at a constant speed of 2 m/s, through a series of moguls, or small hills. The constant slope of the hill is -3. Assuming that the moguls measure 3 m from crest to trough and are roughly 2 m apart, develop an algebraic and a graphical model of the height of the skier versus time.

C14. A skier's height, h, in metres, as a function of time, t, in seconds, can be modelled by the combined function $h(t) = -2t + 150 + 1.95 \sin(1.5t)$.

a) Graph this function.

b) Assuming that the skier stops the first time that her height reaches zero, find the domain and range relevant to the problem.

Chapter 8: Checklist

By the end of this chapter, I will be able to:

- Add or subtract two or more functions graphically using the superposition principal
- Add or subtract two or more functions algebraically
- Combine and simplify a product of functions algebraically
- Combine and simplify a quotient of functions algebraically, and identify any restrictions on the vairable
- Graph the sum, difference, product, or quotient of functions with and without graphing technology and identify the key characteristics of the graph
- Algebraically determine the composition of two or more functions
- Understand the effect of operating on a variable by a function followed by its inverse
- Solve inequalities of combined functions algebraically
- Solve inequalities of combined functions graphically, with and without technologty, using a variety of strategies
- Model contextual situations using combinations of functions
- Solve problems involving various combinations of functions