University Preparation 1: Extending Algebraic Skills

UP 1.1 Factoring Complex Equations

KEY CONCEPTS

- Complex algebraic expressions can be factored by applying a combination of the factoring techniques that you are familiar with: common factoring, difference of squares, and trinomial factoring.

Example 1

Factor $9(x - 1)^2 - 25(2x + 3)^2$.

Solution

In the expression $9(x - 1)^2 - 25(2x + 3)^2$ let $m = x - 1$ and $n = 2x + 3$ to obtain $9m^2 - 25n^2$.

Factor $9m^2 - 25n^2$, using difference of squares.

$= (3m - 5n)(3m + 5n)$

Substitute for m and n.

$= [3(x - 1) - 5(2x + 3)][3(x - 1) + 5(2x + 3)]$

Expand.

$= [3x - 3 - 10x - 15][3x - 3 + 10x + 15]$

Simplify.

$= (-7x - 18)(13x + 12)$

Example 2

Factor $x^2 + 6xy + 9y^2 - a^2 + 12a - 36$.

Solution

$x^2 + 6xy + 9y^2 - a^2 + 12a - 36$

Group the trinomials.

$= (x^2 + 6xy + 9y^2) - (a^2 - 12a + 36)$

Factor each trinomial.

$= (x + 3y)^2 - (a - 6)^2$

Factor this difference of squares.

$= [(x + 3y) - (a - 6)][(x + 3y) + (a - 6)]$

Simplify.

$= (x + 3y - a + 6)(x + 3y + a - 6)$

A

1. Factor.

a) $(x - 2)^2 - 4$

b) $n^2 - (n + 1)^2$

c) $(3x + y)^2 - 1$

d) $(2x + 3)^2 - 49x^2$

e) $3(x + 4)^2 - 75$

f) $2(5 - x)^2 - 72x^2$

g) $(x - 1) - (x - 1)^2$

2. Factor.

a) $(x - 6)^2 - (2x + 3)^2$

b) $(x + 2)^2 - (x - 7)^2$

c) $\frac{1}{9}(x + 2)^2 - (7x - 5)^2$

d) $0.01(1 - 8x)^2 - (1 + 3x)^2$

B

3. Factor.

a) $2(4x - 1)^2 - 8(3 - x)^2$

b) $48(5 - x)^4 - 3(x + 1)^4$

c) $32(a - b + 2c)^2 - 128(a - b - 2c)^2$

d) $64(x + 2y)^4 - 16(2x - y)^4$

4. Factor.

a) $x^2 + 4x + 4 - y^2$

b) $x^2 - 6x + 9 - 4y^2$

c) $a^2 + 10a + 25 - 9b^2$

☆**d)** $4m^2 - 4n^2 - 8np - 4p^2$

e) $2p^2 + 8qr - 8r^2 - 2q^2$

f) $x^4 - y^2 + 2yz - z^2$

g) $x^2 - 2xy + y^2 - 9a^2$

5. Factor.

a) $9a^2 + 18ab + 9b^2 - 4x^2 + 8xy - 4y^2$

b) $16s^2 - 32st + 16t^2 - p^2 - 4pq - 4q^2$

c) $25r^2 - 60rs + 36s^2 - 9g^2 + 12gh - 4h^2$

☆**d)** $s^2 - 14st + 49t^2 - a^2 - 18ab - 81b^2$

C

6. Factor.

a) $(x + 1)^2 + 11(x + 1) + 28$

b) $(2x + 3)^2 + 8(2x + 3) - 9$

☆**c)** $6(x^2 - 1)^2 + 23(x^2 - 1) + 7$

d) $52(3 - 2x)^2 - 57(3 - 2x)(x + 3) - 10(x + 3)^2$

7. Factor.

a) $x^{4n} - y^{6n}$

b) $9x^{6n} - 12x^{3n}y^{2n} + 4y^{4n}$

c) $16x^{4n + 2} + 24x^{2n + 1}y^{4n} + 9y^{8n}$

UP 1.2 Techniques for Solving Complex Equations

KEY CONCEPTS

- You have previously learned to solve quadratic equations by factoring or using the quadratic formula.
- These methods may be applied to solve certain complex equations by using substitution to convert them to quadratic equations.

Example 1

Solve $\frac{1}{x^6} + \frac{9}{x^3} + 8 = 0$.

Solution

Write the equation $\frac{1}{x^6} + \frac{9}{x^3} + 8 = 0$ in quadratic form as follows:

$\left(\frac{1}{x^3}\right)^2 + 9\left(\frac{1}{x^3}\right) + 8 = 0$

Let $w = \frac{1}{x^3}$.

$w^2 + 9w + 8 = 0$

Factor: $(w + 8)(x + 1) = 0$

$w = -8$ or $w = -1$

Revert to x: $\frac{1}{x^3} = -8$ or $\frac{1}{x^3} = -1$

Take the reciprocal of each side: $x^3 = -\frac{1}{8}$ or $x^3 = -1$

$x = -\frac{1}{2}$ or $x = -1$

Example 2

Determine the roots of the equation $x^2 + x + \frac{12}{x^2 + x} = 8$.

Solution

Let $n = x^2 + x$ ①.

Substitute ① in $(x^2 + x) + \frac{12}{(x^2 + x)} = 8$.

$n + \frac{12}{n} = 8$

Multiply each term by n: $n^2 + 12 = 8n$

$n^2 - 8n + 12 = 0$ Factor.

$(n - 6)(n - 2) = 0$

$n = 6$ or $n = 2$

Revert to x: $x^2 + x = 6$ or $x^2 + x = 2$

Now solve each quadratic equation: $x^2 + x = 6$ or $x^2 + x = 2$

$x^2 + x - 6 = 0$ $\quad$ $x^2 + x - 2 = 0$

$(x + 3)(x - 2) = 0$ $\quad$ $(x + 2)(x - 1) = 0$

$x = -3$ or $x = 2$ $\quad$ $x = -2$ or $x = 1$

The roots are $x = -3, -2, 1,$ or 2.

A

1. a) Use the substitution $m = x^2$ to convert the quartic equation $x^4 - 17x^2 + 16 = 0$ into a quadratic equation.
   b) Solve the new equation in part a).

2. Use the method of question 1 to find all the roots of $x^4 - 26x^2 + 25 = 0$.

3. a) Use the substitution $n = 2^x$ to convert the equation $2^{2x} - 6(2^x) + 8 = 0$.
   b) Solve the new equation in part a).

B

4. i) State a suitable substitution that will convert each of the following to a quadratic equation.
   ii) Use your substitution to find all the roots of each equation.
   a) $x^4 - 5x^2 + 4 = 0$
   b) $x^4 - 16 = 0$
   c) $x^4 - 16x^2 + 60 = 0$
   d) $x^4 - 36x^2 + 35 = 0$
   e) $x^4 - 20x^2 + 64 = 0$
   f) $x^4 - 29x^2 + 100 = 0$
   g) $x^4 - 5x^2 + 6 = 0$
   h) $x^4 - 3x^2 - 10 = 0$

5. i) State a suitable substitution that will convert each of the following to a quadratic equation.
   ii) Use your substitution to find all the roots of each equation.
   a) $5^{2x} - 6(5^x) + 5 = 0$
   b) $3^{2x} - 30(3^x) + 81 = 0$
   c) $2^{2x} - 12(2^x) + 32 = 0$
   d) $2^{2x} - 18(2^x) + 32 = 0$
   e) $2^{2x} - 10(2^x) + 16 = 0$
   f) $3^{2x} - 12(3^x) + 27 = 0$
   ☆g) $5^{2x} - 30(5^x) + 125 = 0$

6. Solve.
   a) $4(4^{2x-1} + 1) = 5(4^x)$
   b) $3(3^{2x}) + 1 = 10(3^x) - 2$
   c) $x^2(x^2 - 1) + 2 = 9x^2 - 7$

7. Solve.
   a) $(x - 5)^2 + 7(x - 5) + 10 = 0$
   b) $(3x + 2)^2 = 5(3x + 2)$
   c) $(x^2 + 2x)^2 + 4 = 11(x^2 + 2x) - 20$
   d) $(x^2 - 3x)^2 - 5 = 2(x^2 - 3x) + 3$
   ☆e) $(x^2 - 2x)^2 - 4 = 2(x^2 - 2x) - 1$

8. Solve.
   a) $x^2 = -\frac{36}{x^2} + 13$ b) $\frac{3}{x^2} + \frac{7}{x} = -2$
   c) $\frac{1}{x^2} + \frac{1}{x} = 12$ d) $\frac{1}{x^4} - \frac{9}{x^2} + 20 = 0$
   e) $x^{-4} - 9x^{-2} + 8 = 0$

C

9. Solve and verify.
   a) $\frac{1}{(x-2)^2} - \frac{12}{(x-2)} + 35 = 0$
   b) $\frac{2}{(x+4)^2} + \frac{7}{(x+4)} = -3$
   c) $(x^2 + x) + \frac{24}{(x^2 + x)} = 14$

10. Solve and verify.
   a) $\left(x + \frac{6}{x}\right)^2 - 2\left(x + \frac{6}{x}\right) - 35 = 0$
   b) $\left(x + \frac{4}{x}\right)^2 = 9\left(x + \frac{4}{x}\right) - 20$
   c) $\left(x - \frac{1}{x}\right)^2 = 7\left(x - \frac{1}{x}\right) - 12$

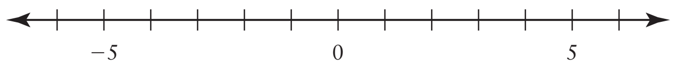
11. a) State a suitable substitution to convert $\left(x^2 + \frac{1}{x^2}\right) = 10 - x - \frac{1}{x}$ into a quadratic equation.
   b) Use your substitution from part a) to determine all the roots of the equation.
   c) Verify your answer(s) in part b).

University Preparation 2: Absolute Value

UP 2.1 Solving Equations Involving Absolute Value

KEY CONCEPTS

On a number line, the numbers −5 and 5 are each located 5 units from 0, one to the left of 0 and the other to the right. Each number is said to have an absolute value of 5. Write them as $|-5| = 5$ and $|5| = 5$.



- $|x| = x, x \geq 0$, and $|x| = -x, x < 0$
- Absolute value equations are solved by considering case i) let $x \geq 0$ and case ii) let $x < 0$.
- In some situations, a solution to an absolute value equation may not satisfy the initial restriction, and so the value found is not valid.
- To check whether a value is a solution, substitute into the left side and right side of the original equation to see if it satisfies the given equation.

Definition of Absolute Value

i) $|x| = x, x \geq 0$

ii) $|x| = -x, x < 0$

Example 1

Evaluate.

a) $|4|$

b) $|-4|$

Solution

a) $|4| = 4$, since the absolute value of a positive number is the number itself.

b) $|-4| = 4$, since the absolute value of a negative number is a positive number.

Example 2

Solve for x.

a) $|x| = 3$

b) $|x - 1| = 3$

Solution

a) $|x| = 3$

Case i) $x \geq 0$; then, for these values of x, $|x| = x$ and, so, $x = 3$.

Case ii) $x < 0$; then, for these values of x, $|x| = -x$ and so $-x = 3$ or $x = -3$.

The solution is $x = -3$ or $x = 3$.

b) $|x - 1| = 3$

Case i) $x - 1 \geq 0$; i.e., $x \geq 1$.
The solution below is valid only for $x \geq 1$.

Then, $|x - 1| = x - 1$.

And so, $x - 1 = 3$

$x = 4$, which satisfies the restriction $x \geq 1$.

Case ii) $x - 1 < 0$; i.e., $x < 1$.
The solution below is valid only for $x < 1$.

Then, $|x - 1| = -(x - 1)$

So, $-(x - 1) = 3$

$-x + 1 = 3$

$-x = 2$

$x = -2$, which satisfies the restriction $x < 1$.
The solution is $x = 4$ and $x = -2$.

Example 3

Solve and check.
$|x - 2| = 3x$

Solution

Case i) $x - 2 \geq 0$, i.e., $x \geq 2$

The solution below is valid only for $x \geq 2$.
$|x - 2| = x - 2$
So, $x - 2 = 3x$
$-2 = 2x$
$x = -1$, which is not a valid solution because it does not satisfy the restriction $x \geq 2$.

Case ii) $x - 2 < 0$, i.e., $x < 2$
The solution below is valid only for $x < 2$.
$|x - 2| = -(x - 2)$
So, $-(x - 2) = 3x$
$-x + 2 = 3x$
$2 = 4x$
$x = 0.5$, which satisfies the restriction $x < 2$.
The solution is $x = 0.5$.

Check:
Substitute $x = 0.5$ in the left side and right side of the given equation, $|x - 2| = 3x$.

L.S. $= |0.5 - 2|$
$= |-1.5|$
$= 1.5$

R.S. $= 3(0.5)$
$= 1.5$
$=$ L.S.

Therefore, $x = 0.5$ is, indeed, a solution.

A

1. Evaluate.
   a) $|6|$
   b) $|-8|$
   c) $|2 - 9|$
   d) $|3(2 - 6) + 1|$
   e) $\left|\frac{6 - 14}{-5 + 3}\right|$

2. Solve for x, if possible.
   a) $|x| = 2$
   b) $|x| = 9$
   c) $|4x| = 36$
   d) $|-3x| = 99$
   e) $2|8x - 12x| = 100$
   f) $|x| - 7 = 32$
   g) $9 - |x| = -26$
   h) $\frac{3}{4}|x| = 6$
   i) $-\frac{3}{4}|-8x| = 24$

B

3. Solve for x, if possible.
   a) $|x + 4| = 3$
   b) $|x - 3| = 6$
   c) $|x - 2| = 7$
   d) $|x + 1| = 5$
   e) $|x - 4| = -2$
   f) $|1 - x| = 8$

4. Solve for x, if possible.
   a) $|2x - 1| = 9$
   b) $|4x - 1| = 3$
   c) $4|x + 3| = 12$
   d) $|2 - 3x| = 11$
   e) $|3 - 2x| = -5$
   f) $-|4x + 2| = -3$
   g) $-|2x - 5| = 3$

5. Solve and check.
   a) $|x - 4| = 5x$
   b) $|x + 5| = -2x$
   c) $3x = |2 - x|$
   d) $|4 + 3x| = 7x$
   e) $4|4x - 3| = 32x$
   f) $\frac{1}{4}|3 - 2x| = \frac{3}{2}$

6. Solve for x, if possible.
   a) $|3x - 4| + |9x - 12| = 12x$
   b) $|3x - 4| + |7 - 2x| = 0$
   c) $|2x - 8| + |12 - 3x| = 0$
   d) $|x - 3| + |3 - x| = 0$
   ☆e) $16x - 3|2x - 1| = |10x - 5|$
   f) $\left|\frac{x - 6}{3}\right| + 4x = 0$

C

7. Solve for x.
   a) $|x + 1| + 2 = 3|x + 1|$
   b) $7|x + 2| = 2|x + 2| + 15$
   c) $|2x| = |x + 5|$
   d) $|3x - 2| = |2x|$
   e) $6|x + 1| - 14 = 4|x + 1| + 6$

8. Solve for x.
   a) $|5x - 3| = |x + 1|$
   b) $|2x - 5| = |x - 1|$
   c) $|2 - 3x| = |5 + x|$
   d) $|x - 3| = |4x - 7|$
   ☆e) $|3x - 7| = |2 - x|$

9. Solve for x, if possible.
   a) $|x + 2| + |2 - x| = 8$
   b) $|2x - 1| - |1 - 2x| = 4$

KEY CONCEPTS

- An inequality is an algebraic expression with one of the following symbols: $\le$, $\ge$, $>$, or $<$.
- To solve absolute value inequalities, apply the definition of absolute value and use the principles of case i) $x \ge 0$ and case ii) $x < 0$.
- In general, for any constant, a, and any positive constant, c, an inequality of the form
  i) $|x + a| \le c$ has the solution $-c - a \le x \le c - a$, and
  ii) $|x + a| \ge c$ has the solution $x \le -c - a$ or $x \ge c - a$.

Example 1

Solve $|x - 6| \le 3$.

Solution

Method 1: Using Cases

Case i) $x - 6 \ge 0$, i.e., $x \ge 6$ ①

Then, $|x - 6| = x - 6$

Solve $x - 6 \le 3$

$x \le 9$ ②

The solution must satisfy ① and ②, so $6 \le x \le 9$.

Case ii) $x - 6 < 0$, i.e., $x < 6$ ①

Then, $|x - 6| = -(x - 6)$

$= -x + 6$

Solve $-x + 6 \le 3$

$-x \le -3$

Divide by -1; change $\le$ to $\ge$.

$x \ge 3$ ②

The solution must satisfy ① and ②, so $3 \le x < 6$.

From case i) and case ii), $6 \le x \le 9$ and $3 \le x < 6$; the solution is $3 \le x \le 9$.

Method 2: Without Cases

For $|x - 6| \le 3$, the equivalent inequality is $-3 \le x - 6 \le 3$.

$-3 \le x - 6$ and $x - 6 \le 3$

$3 \le x$ and $x \le 9$

$3 \le x \le 9$

Strategy Tip: In general, for any constant, a, and any positive constant, c, inequalities of the form

i) $|x + a| \le c$ may be solved by solving the equivalent inequality $-c \le x + a \le$, which results in $-c - a \le x \le c - a$.

ii) $|x + a| \ge c$ may be solved by solving the equivalent inequality $x + a \le -c$ or $x + a \ge c$, which results in $x \le -c - a$ or $x \ge c - a$.

Example 2

Solve $|3x - 4| \geq 2$; then, graph the solution on a number line.

Solution

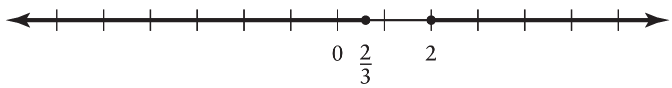
The inequality $|3x - 4| \geq 2$ is of the form $|x + a| \geq c$.

Use strategy tip ii):

$3x - 4 \leq -2$ or $3x - 4 \geq 2$

$3x \leq 2$ or $3x \geq 6$

$x \leq \frac{2}{3}$ or $x \geq 2$



The dots indicate the end points that are included in the solution.

A

1. Solve.

a) $|x| > 3$ **b)** $|x| < 2$

c) $|x| \leq 5$ **d)** $|x| \geq -1$

e) $|4x| \leq 7$ **f)** $|-3x| > 12$

g) $|5x| + 7 < 21$ **h)** $3 - |x| > -4$

i) $21 \geq |-4x| + 5$

2. Solve and graph the solution.

a) $2|x| - 5 > 14$ **b)** $-\frac{3}{4}|x| \geq 6$

c) $\left|-\frac{2}{3}x\right| + 4 < 13$ **d)** $|x| - 6 \leq 4$

e) $1 - |-3x| \geq -20$ **f)** $6 + \left|\frac{5x}{-9}\right| > 3$

g) $9|x| - |-5x| < 28$

B

3. Solve and graph the solution.

a) $|x - 3| > 3$ **b)** $|x + 4| \leq 1$

c) $|x - 1| < 7$ **d)** $|5 - x| \geq 0$

e) $|x + 3| > 10$ **f)** $|8 - x| \leq 15$

g) $|6 + x| > 9$ **h)** $|2 - x| \geq 2$

i) $|x + 10| < 6$

4. Solve.

a) $|2x + 1| > 3$ **b)** $|3 - 5x| \geq 8$

c) $|7x - 2| < 1$ **d)** $|2x - 5| \leq 3$

e) $|3x + 1| < 9$ **f)** $|7 - 4x| + 2 \geq 15$

g) $8 - |2x + 3| < -3$ **h)** $5 + \frac{2}{3}|4 - 6x| \geq 7$

i) $\left|\frac{3x + 1}{6}\right| - 2 > 9$ **j)** $\frac{2}{|x + 4|} \leq 5$

5. Solve.

a) $|3x + 2| < 5x + 1$

b) $|2x + 1| < 3x$

c) $|6 - 3x| \leq x - 2$

☆**d)** $|2 - 3x| > 3x - 6$

e) $|x + 1| \geq x - 4$

f) $\left|\frac{x - 2}{3}\right| > x + 1$

g) $5 - |x + 2| \leq 6x$

C

6. Solve and check.

a) $|3x - 1| \leq |2x + 18|$

b) $|6x + 5| < |5x + 6|$

☆**c)** $|x + 2| > 4 + |x|$

7. Solve and graph your solution.

a) $|x| + |x - 1| < 5$

b) $|x - 1| \geq |x + 2|$

c) $2 - |3x + 6| < |-5x - 10| - 8$

8. Solve.

a) $\left|\frac{x - 1}{x + 3}\right| < 1$

b) $\left|\frac{2x - 3}{4x + 1}\right| \geq 2$

c) $|2x - 4| + 3|x + 5| \leq 2|6 - 3x| - |-5x - 25|$

University Preparation 3: Matrices

UP 3.1 Introduction to Matrices

KEY CONCEPTS

Numerical data is often displayed or organized in the form of a matrix. For instance, teachers often keep a record of class marks in the form of a matrix, with **rows** indicating students' names and marks and **columns** indicating the test title and marks.

	Test 1	Test 2	Test 3
P. Adams	98	88	93
R. Butler	75	81	85
G. Casten	86	89	79
K. Dustop	62	74	83

- A **matrix** is a rectangular array of numbers arranged in rows (horizontal) and columns (vertical). Each number in the array is called an **element** or **entry** of the matrix.
- The **dimensions** or **order** of a matrix are the number of rows by the number of columns in the matrix. In general, an $\boldsymbol{m \times n}$ (read m by n) matrix is a matrix with m rows and n columns.
- In mathematics, it is conventional to use capital letters to denote matrices using square brackets around the elements.
- Matrix $\boldsymbol{A}$ represents the test information shown above.

$$\boldsymbol{A} = \begin{bmatrix} 98 & 88 & 93 \\ 75 & 81 & 85 \\ 86 & 89 & 79 \\ 62 & 74 & 83 \end{bmatrix}$$

The dimensions of this matrix are 4×3. The entry in row 3, column 2 is 89. This entry, or element, can also be identified as a_{32}.

- In general, if A is an $m \times n$ matrix, we write $\boldsymbol{A} = [a_{ij}]_{m \times n}$, where i represents the row number and j the column number of element a in a matrix of m rows and n columns. Thus,

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & a_{ij} & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- Two matrices have the **same size** if they each have the same dimensions, that is, the same number of rows and the same number of columns. Two matrices are **equal** if and only if they have the same dimensions and all their corresponding entries are equal.

Matrix Addition and Subtraction

- Matrices that have the same dimensions can be added or subtracted by adding or subtracting corresponding entries or elements.
- The resulting matrix is the same size as the matrices that were added or subtracted.

Scalar Multiplication

- A matrix can be multiplied by a scalar k by multiplying each entry or element by the scalar k. The resulting matrix is the same size as the original matrix.

Properties of Matrix Addition and Scalar Multiplication

For matrices $\boldsymbol{A}$, $\boldsymbol{B}$, and $\boldsymbol{C}$, all of which have the same dimensions, and scalars k and s, the following properties hold true:

1. $\boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ (Commutative property)
2. $\boldsymbol{A} + (\boldsymbol{B} + \boldsymbol{C}) = (\boldsymbol{A} + \boldsymbol{B}) + \boldsymbol{C}$ (Associative property)
3. $s(k\boldsymbol{A}) = (sk)\boldsymbol{A}$ (Associative property)
4. $k(\boldsymbol{A} + \boldsymbol{B}) = k\boldsymbol{A} + k\boldsymbol{B}$ (Distributive property)
5. $(k + s)\boldsymbol{A} = k\boldsymbol{A} + s\boldsymbol{A}$ (Distributive property)

Example 1

Given $\boldsymbol{A} = \begin{bmatrix} -2 & 5 \\ 7 & 3 \end{bmatrix}$, $\boldsymbol{B} = \begin{bmatrix} 1 & -4 \\ -6 & 11 \end{bmatrix}$, and $\boldsymbol{C} = \begin{bmatrix} 0 & -3 \\ -1 & 10 \\ 8 & 4 \end{bmatrix}$, determine the following, if possible.

a) $\boldsymbol{A} + 2\boldsymbol{B}$

b) $-3(4\boldsymbol{C})$

c) $\boldsymbol{C} - \boldsymbol{A}$

Solution

a)
$$\begin{aligned}\boldsymbol{A} + 2\boldsymbol{B} &= \begin{bmatrix} -2 & 5 \\ 7 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & -4 \\ -6 & 11 \end{bmatrix} \\ &= \begin{bmatrix} -2 & 5 \\ 7 & 3 \end{bmatrix} + \begin{bmatrix} 2 & -8 \\ -12 & 22 \end{bmatrix} \\ &= \begin{bmatrix} -2 + 2 & 5 - 8 \\ 7 - 12 & 3 + 22 \end{bmatrix} \\ &= \begin{bmatrix} 0 & -3 \\ -5 & 25 \end{bmatrix}\end{aligned}$$

b)
$$\begin{aligned}-3(4\boldsymbol{C}) &= (-3 \times 4)\boldsymbol{C} = -12\boldsymbol{C} \\ &= -12\begin{bmatrix} 0 & -3 \\ -1 & 10 \\ 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 36 \\ 12 & -120 \\ -96 & -48 \end{bmatrix}\end{aligned}$$

c) $\boldsymbol{C} - \boldsymbol{A}$

It is not possible to subtract $\boldsymbol{A}$ from $\boldsymbol{C}$ since matrix $\boldsymbol{C}$ is a 3×2 matrix and matrix $\boldsymbol{A}$ is a 2×2 matrix.

Matrix Multiplication

- Matrix multiplication—that is, the multiplication of one matrix by another—is based on column-by-row multiplication. If you have two matrices, A and B, you can multiply B by A only if the number of columns in A equals the number of rows in B. Therefore, if A is a 2×2 matrix and B is a 2×3 matrix, you can multiply B by A (AB) because B has two rows and A has two columns. However, you *cannot* multiply A by B (BA) because A has two rows and B has *three* columns. The new matrix will have dimensions 2×3, the same dimensions as the matrix being multiplied. This rule may be written as $A_{m \times n} \times B_{n \times p} = (AB)_{m \times p}$.

Example 2

Given $\boldsymbol{D} = \begin{bmatrix} -2 & 5 \\ 7 & 3 \end{bmatrix}$, $\boldsymbol{E} = \begin{bmatrix} 6 & -3 & 1 \\ 2 & -1 & -2 \end{bmatrix}$, and $\boldsymbol{F} = \begin{bmatrix} 0 & -3 \\ -1 & 10 \\ 8 & 4 \end{bmatrix}$ determine the following, if possible.

a) $\boldsymbol{DE}$

b) $\boldsymbol{EF}$

c) $\boldsymbol{DF}$

d) $\boldsymbol{E}^2$

Solution

a) $\boldsymbol{D}$ is a **2** × **2** matrix and $\boldsymbol{E}$ is a **2** × **3** matrix. The number of rows of $\boldsymbol{D}$ equals the number of columns of $\boldsymbol{E}$, so the product $\boldsymbol{DE}$ exists. The resulting matrix will be a **2** × **3** matrix.

$$\begin{aligned} \boldsymbol{DE} &= \begin{bmatrix} -2 & 5 \\ 7 & 3 \end{bmatrix}\begin{bmatrix} 6 & -3 & 1 \\ 2 & -1 & -2 \end{bmatrix} \\ &= \begin{bmatrix} (-2)(6) + (5)(2) & (-2)(-3) + (5)(-1) & (-2)(1) + (5)(-2) \\ (7)(6) + (3)(2) & (7)(-3) + (3)(-1) & (7)(1) + (3)(-2) \end{bmatrix} \\ &= \begin{bmatrix} -2 & 1 & -12 \\ 48 & -24 & 1 \end{bmatrix} \end{aligned}$$

b) $\boldsymbol{E}$ is a **2** × **3** matrix and $\boldsymbol{F}$ is a **3** × **2** matrix. The number of rows of $\boldsymbol{E}$ equals the number of columns of $\boldsymbol{F}$, so the product $\boldsymbol{EF}$ exists. The resulting matrix will be a **2** × **2** matrix.

$$\begin{aligned} \boldsymbol{EF} &= \begin{bmatrix} 6 & -3 & 1 \\ 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} 0 & -3 \\ -1 & 10 \\ 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} (6)(0) + (-3)(-1) + (1)(8) & (6)(-3) + (-3)(10) + (1)(4) \\ (2)(0) + (-1)(-1) + (-2)(8) & (2)(-3) + (-1)(10) + (-2)(4) \end{bmatrix} \\ &= \begin{bmatrix} 11 & -44 \\ -15 & -24 \end{bmatrix} \end{aligned}$$

c) $\boldsymbol{D}$ is a **2** × **2** matrix and $\boldsymbol{F}$ is a **3** × **2** matrix. The number of rows of $\boldsymbol{D}$ is not equal to the number of columns of $\boldsymbol{F}$, so the product $\boldsymbol{DF}$ is not possible.

d) $\boldsymbol{E}^2$ means $\boldsymbol{E} \times \boldsymbol{E}$. Since $\boldsymbol{E}$ is a **2** × **3** matrix, the number of rows of $\boldsymbol{E}$ is not equal to the number of columns of $\boldsymbol{E}$, so the product $\boldsymbol{E}^2$ is not possible.

A

1. Given $A = \begin{bmatrix} -4 & 3 & 8 \\ 5 & 1 & -2 \end{bmatrix}$, determine:

   a) the dimensions of A

   b) the value of the entries a_{12} and a_{23}

   c) $-A$ **d)** $2(3A)$

   e) $5A - 2A$

☆2. Given $B = \begin{bmatrix} 9 & -6 & 10 \\ -1 & 2 & 7 \end{bmatrix}$ and $C = \begin{bmatrix} -3 & 1 \\ 0 & 4 \end{bmatrix}$, state whether each of the products below exists. If it does exist, state the dimensions of the resulting matrix. If it does not exist, explain why.

   a) BC **b)** CB

   c) B^2 **d)** C^2

B

3. Calculate each product in question 2 that does exist.

   Use the following matrices for questions 4 and 5.

   $P = \begin{bmatrix} 1 & 3 & -1 \\ 2 & -9 & 4 \\ 0 & 5 & -6 \end{bmatrix}$ $Q = \begin{bmatrix} 12 & 3 \\ 0 & -1 \\ -3 & 7 \end{bmatrix}$

   $R = \begin{bmatrix} -6 & 1 & 3 \\ 2 & 4 & -2 \end{bmatrix}$ $S = \begin{bmatrix} -8 & 2 & 0 \\ -5 & 7 & 1 \\ 3 & -1 & 11 \end{bmatrix}$

4. Perform the indicated operation, if it exists. If it does not exist, explain why.

   a) $P + S$ **b)** $3(S + P)$

   c) $R - Q$ **d)** $-\frac{2}{3}Q$

5. Perform the indicated operation, if it exists. If it does not exist, explain why.

   a) QR **b)** $QR - S$

   c) QP **d)** PQ

   e) $(PQ)R$ **f)** P^2

   g) $RS - P^2$ **h)** $PS - P^2$

   i) $SP - 3S$

6. Given that A, B, and C are 2×2 matrices, and k is a scalar, determine whether or not each of the following is true. Support your answer with an example.

   a) $AB = BA$

   b) $(AB)C = A(BC)$

   c) $k(AB) = (kA)B = A(kB)$

7. Determine the unknown values in the following:

   $$\begin{bmatrix} a+2 & b-3 \\ 2c-1 & 3-4d \end{bmatrix} = \begin{bmatrix} 2a-3 & -b+1 \\ 5-c & -2d+5 \end{bmatrix}$$

8. Evaluate.

   a) $\begin{bmatrix} 6 & 12 \\ -11 & 5 \end{bmatrix}\left(\begin{bmatrix} -1 & 3 \\ 7 & 16 \end{bmatrix} - \begin{bmatrix} 4 & -5 \\ 20 & -2 \end{bmatrix}\right)$

   ☆**b)** $\begin{bmatrix} 3 & -4 \end{bmatrix}\begin{bmatrix} -1 & 1 & 2 \\ 8 & 3 & -5 \end{bmatrix} + 4\begin{bmatrix} 2 & -1 & -2 \end{bmatrix}\begin{bmatrix} 3 & 2 & 4 \\ -1 & 0 & 1 \\ 5 & 6 & -2 \end{bmatrix}$

C

9. Given $H = \begin{bmatrix} 2 & 5 \\ -3 & 7 \end{bmatrix}$, determine a matrix $I = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, such that $HI = IH$.

10. Determine the unknown values.

    a) $\begin{bmatrix} 2 & -3 \\ 5 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -7 \\ 4 \end{bmatrix}$

    b) $\begin{bmatrix} x & 0 & 1 \\ 2 & -3 & 5 \\ 7 & y & 0 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 0 \\ -3 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 3 & -2 \\ 1 & 4 \end{bmatrix} = \begin{bmatrix} y & z \\ -11 & 3 \\ x & 4 \end{bmatrix}$

11. Given $M = \begin{bmatrix} 2 & -1 \\ -6 & 3 \end{bmatrix}$, verify that $M^2 = 5M$.

12. Is the following true for all 2×2 matrices A and B? Justify your answer.
    $(A + B)(A - B) = A^2 - B^2$

KEY CONCEPTS

- One very useful application of matrices is to determine the solution to a system of linear equations. For instance, consider the following linear system of two lines in the Cartesian plane.

  $2x - 3y = 7$
  $5x + y = 9$

  Solve this system of linear equations by using the method of elimination or the method of substitution. Another way to solve the system is to use **determinants**.

  The **determinant** (det) of a 2 × 2 matrix $\boldsymbol{A} = \begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is

  $\det \boldsymbol{A} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - b_1a_2$, where $a_1b_2 - b_1a_2$ is a real number.

  The determinant of a 3 × 3 matrix $\boldsymbol{B} = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is

  $$\det \boldsymbol{B} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

  Example 1

  Find the determinant of each matrix below.

  a) $\boldsymbol{A} = \begin{bmatrix} 7 & -2 \\ 5 & 12 \end{bmatrix}$

  b) $\boldsymbol{B} = \begin{bmatrix} -3 & 1 & 6 \\ 4 & -5 & 2 \\ 3 & 8 & 7 \end{bmatrix}$

  Solution

  a) $\boldsymbol{A} = \begin{bmatrix} 7 & -2 \\ 5 & 12 \end{bmatrix}$

  $$\begin{aligned} \det \boldsymbol{A} &= \begin{vmatrix} 7 & -2 \\ 5 & 12 \end{vmatrix} \\ &= 7(12) - (-2)(5) \\ &= 94 \end{aligned}$$

  b) $\boldsymbol{B} = \begin{bmatrix} -3 & 1 & 6 \\ 4 & -5 & 2 \\ 3 & 8 & 7 \end{bmatrix}$

  $$\begin{aligned} \det \boldsymbol{B} &= \begin{vmatrix} -3 & 1 & 6 \\ 4 & -5 & 2 \\ 3 & 8 & 7 \end{vmatrix} \\ &= (-3)\begin{vmatrix} -5 & 2 \\ 8 & 7 \end{vmatrix} - (1)\begin{vmatrix} 4 & 2 \\ 3 & 7 \end{vmatrix} + (6)\begin{vmatrix} 4 & -5 \\ 3 & 8 \end{vmatrix} \\ &= -3[(-5)(7) - 2(8)] - [4(7) - 2(3)] + \\ &\quad 6[4(8) - (-5)(3)] \\ &= -3(-51) - (22) + 6(47) \\ &= 413 \end{aligned}$$

- When two lines **intersect in a point,** the corresponding linear system is said to have a **unique solution.** Determinants are useful because not only can they be used to determine if a linear system has a unique solution, but they can also be used to find the point of intersection.

Cramer's Rule

The linear system $a_1x + b_1y = c_1$
$a_2x + b_2y = c_2$
has a unique solution if and only if the determinant $\boldsymbol{D} = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \neq 0$.

If $\boldsymbol{D} \neq 0$ the unique solution is $x = \dfrac{D_1}{D}$, $y = \dfrac{D_2}{D}$, where $D_1 = \begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}$ and $D_2 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$.

Example 2

Determine whether each linear system below has a unique solution. Use Cramer's rule to find the unique solution, if it exists.

a) $4x - 5y = 2$
$3x + 2y = 6$

b) $2x - 3y = 5$
$-4x + 6y = 7$

Solution

a) $4x - 5y = 2$
$3x + 2y = 6$

$$\begin{aligned} \boldsymbol{D} &= \begin{vmatrix} 4 & -5 \\ 3 & 2 \end{vmatrix} \\ &= 8 + 15 \\ &= 23 \end{aligned}$$

Since $\boldsymbol{D} \neq 0$, the linear system has a unique solution.

The solution is

$$x = \frac{D_1}{D} = \frac{\begin{vmatrix} 2 & -5 \\ 6 & 2 \end{vmatrix}}{23} = \frac{34}{23}$$

$$y = \frac{D_2}{D} = \frac{\begin{vmatrix} 4 & 2 \\ 3 & 6 \end{vmatrix}}{23} = \frac{18}{23}$$

The point of intersection is $(x, y) = \left(\frac{34}{23}, \frac{18}{23}\right)$.

b) $2x - 3y = 5$
$-4x + 6y = 7$

$$\begin{aligned} \boldsymbol{D} &= \begin{vmatrix} 2 & -3 \\ -4 & 6 \end{vmatrix} \\ &= 12 - 12 \\ &= 0 \end{aligned}$$

Since $\boldsymbol{D} = 0$, the linear system does not have a unique solution.

A

1. The values of the determinants $\boldsymbol{D}$, $\boldsymbol{D_1}$, and $\boldsymbol{D_2}$ of a linear system are given. State if the system has a unique solution. If it does, state the solution.

   a) $\boldsymbol{D} = 4$, $\boldsymbol{D_1} = 9$, $\boldsymbol{D_2} = -17$

   b) $\boldsymbol{D} = 0$, $\boldsymbol{D_1} = -23$, $\boldsymbol{D_2} = 15$

   c) $\boldsymbol{D} = -9$, $\boldsymbol{D_1} = 22$, $\boldsymbol{D_2} = 36$

2. Describe the similarities and differences between a matrix and a determinant. Use examples to support your answer.

B

3. Evaluate the determinant for each of the following matrices.

   a) $\begin{vmatrix} 5 & -3 \\ 1 & 9 \end{vmatrix}$

   b) $\begin{vmatrix} 7 & -1 \\ -6 & 4 \end{vmatrix}$

   c) $\begin{vmatrix} 3 & 1 & 6 \\ -4 & 2 & -1 \\ 5 & 8 & -3 \end{vmatrix}$

   d) $\begin{vmatrix} -\frac{1}{2} & 4 & 5 \\ 1 & -3 & 2 \\ 0 & 8 & -4 \end{vmatrix}$

   ☆**e)** $5\begin{vmatrix} 9 & 10 \\ -7 & 4 \end{vmatrix} - \frac{2}{3}\begin{vmatrix} 6 & 27 \\ 18 & -15 \end{vmatrix}$

4. Determine whether each linear system below has a unique solution. If it does, use Cramer's rule to find the solution.

   a) $5x - 2y = 3$
   $4x + 3y = 8$

   b) $-x + 3y = 1$
   $3x - 9y = 4$

   c) $8x - 5y = -4$
   $2x - 3y = 2$

   d) $-4x - 2y = 6$
   $10x + 5y = -15$

   e) $1.5x + 3y = -8$
   $4.5x - 2y = 7$

☆5. Determine the value(s) of x such that

   $\begin{vmatrix} 1 & 1 - x \\ x - 4 & 2 \end{vmatrix} = 0.$

6. Use determinants to find the values of a and b, such that the point $(4, -3)$ is the unique solution to the linear system below.
   $ax + 2y = -2$
   $3x + by = -9$

C

7. A system of three equations in three variables x, y, and z has a unique solution if, and only if, the 3×3 determinant $\boldsymbol{D} \neq 0$. In this case, the unique solution is $x = \frac{D_1}{D}$, $y = \frac{D_2}{D}$, and $z = \frac{D_3}{D}$, where D_i is the determinant obtained from D by replacing the ith column with the constants on the right-hand side of the linear system. Determine whether each linear system that follows has a unique solution.

   a) $4x - y + 2z = 3$
   $-x + 3y + z = 1$
   $5x + 2y - z = -7$

   b) $x + y + 2z = -2$
   $3x - y + 14z = 6$
   $x + 2y = -5$

   c) $2x + y + z = -4$
   $3y - 2z = 2$
   $3x + y + 2z = -7$

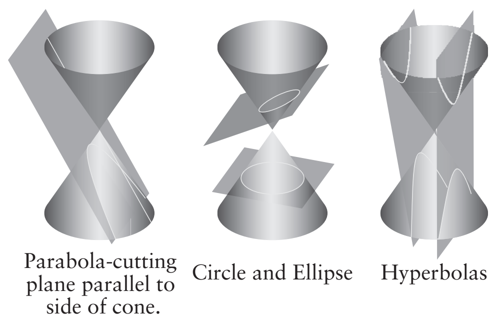
8. Use Cramer's rule to find a quadratic function, $f(x)$, such that $f(1) = 5$, $f(2) = 6$, and $f(-2) = 26$.

University Preparation 4: CONICS

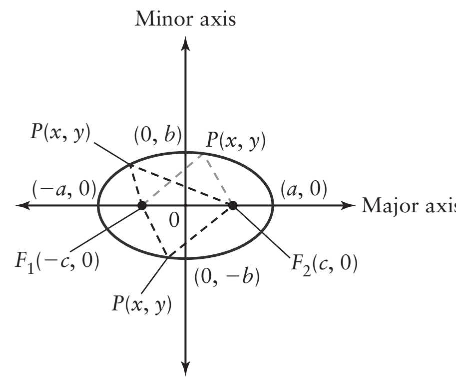
UP 4.1 Introduction

KEY CONCEPTS

- The ancient Greeks identified the **conic sections** as a special group of curves created when a plane slices a cone, similar to the one shown below, at different angles. The four curves in this group are the **parabola**, the **circle**, the **ellipse**, and the **hyperbola**.
- The ancient Greeks defined a cone as the surface generated when a line is rotated through a fixed point.



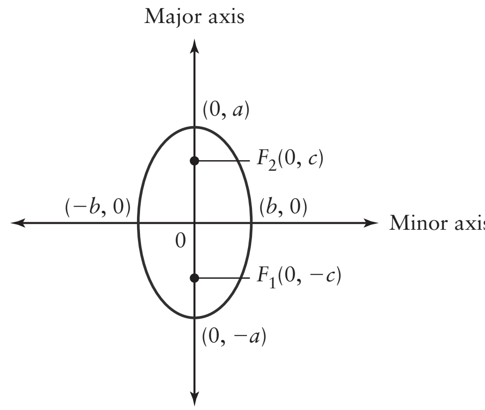
KEY CONCEPTS



- An ellipse is defined as the set of all points for which the **sum** of the distances to two fixed points, F_1 and F_2, (called **foci**, plural of focus) is constant, that is, $PF_1 + PF_2 = k$.
- The general equation of an ellipse with centre (0, 0) and foci $F_1(-c, 0)$ and $F_2(c, 0)$ on the x-axis may be developed from the basic definition. Note that $\mathbf{c^2 = a^2 - b^2}$.

  The distance from point $P(x, y)$ to $F_1(-c, 0)$ is
  $PF_1 = \sqrt{(x + c)^2 + (y - 0)^2}$.
  The distance from point $P(x, y)$ to $F_2(c, 0)$ is
  $PF_2 = \sqrt{(x - c)^2 + (y - 0)^2}$.
  $PF_1 + PF_2 = k$
  $\sqrt{(x + c)^2 + (y - 0)^2} + \sqrt{(x - c)^2 + (y - 0)^2} = k$
  Rewrite the above equality as
  $\sqrt{x^2 + 2xc + c^2 + y^2} = k - \sqrt{x^2 - 2xc + c^2 + y^2}$ ①
  By squaring both sides, simplifying, and substituting $c^2 = a^2 - b^2$, equation ① simplifies to $b^2x^2 + a^2y^2 = a^2b^2$.
  Divide each side by a^2b^2 to obtain $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$.
- This represents an ellipse with foci and vertices on the x-axis. Since $a > b$, it is wider along the x-axis than along the y-axis, so the **major axis** is the x-axis and the **minor axis** is the y-axis.

- The equation of an ellipse with foci and vertices on the y-axis may be developed in a similar way to the above, and is $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1$, where $a > b$.
- This ellipse is wider along the y-axis than along the x-axis, so the y-axis is the major axis and the x-axis is the minor axis.



Ellipse: Key Characteristics		
	$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, a > b$	$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1, a > b$
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Intercepts	$y = \pm b$	$x = \pm b$
Major axis and length	x-axis, length is $2a$	y-axis, length is $2a$
Minor axis and length	y-axis, $2b$	x-axis, $2b$
Foci	$(\pm c, 0)$ where $c^2 = a^2 - b^2$	$(0, \pm c)$ where $c^2 = a^2 - b^2$
Distance between foci	$2c$	$2c$

Example 1

Identify the key characteristics of each ellipse: $\frac{x^2}{100} + \frac{y^2}{16} = 1$, $\frac{x^2}{4} + \frac{y^2}{36} = 1$

Solution

	$\frac{x^2}{100} + \frac{y^2}{16} = 1$	$\frac{x^2}{4} + \frac{y^2}{36} = 1$
Centre	(0, 0)	(0, 0)
Vertices	$(\pm 10, 0)$	$(0, \pm 6)$
Intercepts	$y = \pm 4$	$x = \pm 2$
Major axis and length	x-axis, length is $2(10) = 20$	y-axis, length is $2(6) = 12$
Minor axis and length	y-axis, length is $2(4) = 8$	x-axis, length is $2(2) = 4$
Foci	$c^2 = 100 - 16$ $= 84$ $c = 2\sqrt{21}$ The foci are: $(\pm 2\sqrt{21}, 0)$	$c^2 = 36 - 4$ $= 32$ $c = 4\sqrt{2}$ The foci are: $(0, \pm 4\sqrt{2})$
Distance between foci	$4\sqrt{21}$	$8\sqrt{2}$

Example 2

Determine an equation of an ellipse with foci at (–4, 0) and (4, 0) and a constant sum of focal radii 10.

Solution

Let $P(x, y)$ be any point on the ellipse.

$PF_1 + PF_2 = 10$

$\sqrt{(x + 4)^2 + y^2} + \sqrt{(x - 4)^2 + y^2} = 10$

Write the equation so that there is a radical on each side.

$\sqrt{x^2 - 8x + 16 + y^2} = 10 - \sqrt{x^2 + 8x + 16 + y^2}$

Square both sides.

$x^2 - 8x + 16 + y^2 = 100 - 20\sqrt{x^2 + 8x + 16 + y^2} + x^2 + 8x + 16 + y^2$

Simplify.

$5\sqrt{x^2 + 8x + 16 + y^2} = 4x + 25$

Square both sides again.

$25x^2 + 200x + 400 + 25y^2 = 16x^2 + 200x + 625$

Simplify.

$9x^2 + 25y^2 = 225$

Divide each side by 225.

The equation of the ellipse is $\frac{x^2}{25} + \frac{y^2}{9} = 1$.

A

1. For each of the sets of information given below, state the equation of an ellipse with centre (0, 0).

   a) The vertices are $(\pm 7, 0)$; y-intercepts are ± 5.

   b) The y-axis is the major axis with length 16; the x-axis is the minor axis with length 6.

   c) $a^2 = 100$, $c^2 = 64$, and the major axis is the x-axis.

2. Sketch a graph of each ellipse in question 1 and identify the key characteristics.

B

3. **i)** Determine the foci of each ellipse below.

   ii) For each ellipse, state which is the major axis and which is the minor axis.

   a) $\frac{x^2}{16} + \frac{y^2}{64} = 1$

   b) $\frac{x^2}{100} + \frac{y^2}{81} = 1$

   c) $\frac{x^2}{4} + \frac{y^2}{49} = 1$

   d) $\frac{x^2}{25} + \frac{y^2}{36} = 1$

   e) $4x^2 + 25y^2 = 100$

   f) $16x^2 + 4y^2 = 100$

4. Complete this chart for each ellipse, with centre (0, 0).

	$\frac{x^2}{121} + \frac{y^2}{64} = 1$	$\frac{x^2}{16} + \frac{y^2}{81} = 1$	$\frac{x^2}{144} + \frac{y^2}{169} = 1$	$\frac{x^2}{18} + \frac{y^2}{44} = 1$
Vertices				
Intercepts				
Major axis and length				
Minor axis and length				
Foci				
Distance between foci				

5. **a)** Find an equation of each ellipse described below, with centre (0, 0).

   i) The length of the major axis is 10; the length of the minor axis is 8.

   ii) The length of the minor axis is 6; one vertex is (−5, 0).

   iii) One intercept is (5, 0); one vertex is (0, −7).

   b) For each of the above, is there another possible equation that satisfies the given conditions? Justify your answer.

☆**6.** **a)** Determine the equation of an ellipse with foci at $(-8, 0)$ and $(8, 0)$ and constant sum of focal radii 20.

b) Identify the key characteristics of the ellipse you found in part a).

c) Sketch a graph of the ellipse.

7. **a)** Determine an equation of an ellipse with foci at $(0, -7)$ and $(0, 7)$ and constant sum of focal radii 18.

b) Identify the key characteristics of the ellipse you found in part a).

c) Sketch a graph of the ellipse.

☆**8.** A pool has the shape of an ellipse. The major axis has length 10 m and the minor axis has length 8 m.

a) Write an equation of the ellipse.

b) Find the width of the pool at a point on the major axis that is 2 m from the centre.

C

9. Ellipses can be long and narrow or nearly circular. The **eccentricity**, ***e***, of an ellipse is a measure of the amount of **elongation** of an ellipse. The value of e is determined by the formula $e = \frac{c}{a}$, where $0 < e < 1$. The closer the value of e is to 1, the more elongated the ellipse. Determine the eccentricity of each ellipse in question 4.

10. Find an equation of an ellipse with centre $(0, 0)$, vertices $(\pm 6, 0)$, and eccentricity $\frac{2}{3}$.

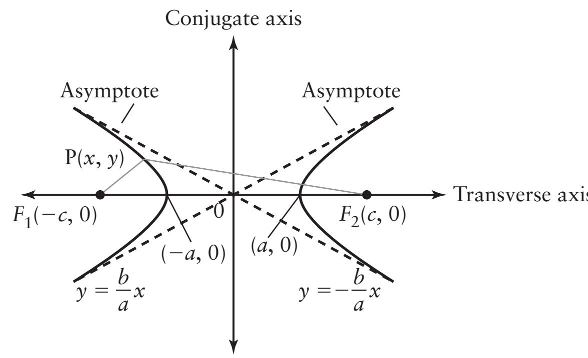
11. An equation of an ellipse with centre (h, k), and major axis the x-axis, is $\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$.

a) Determine the key characteristics of the ellipse $\frac{(x-2)^2}{4} + \frac{(y+3)^2}{9} = 1$.

b) Sketch a graph of this ellipse.

KEY CONCEPTS

- A hyperbola is defined as the set of all points for which the **difference** of the distances to two fixed points, F_1 and F_2, is constant, that is, $|PF_1 - PF_2| = k$.
- The general equation of a hyperbola with centre (0, 0) and foci $F_1(-c, 0)$ and $F_2(c, 0)$ on the x-axis may be developed from the basic definition in a fashion similar to the one used to develop the equation of an ellipse. Note that $c^2 = a^2 + b^2$.



  The distance from point $P(x, y)$ to $F_1(-c, 0)$ is $PF_1 = \sqrt{(x + c)^2 + (y - 0)^2}$.

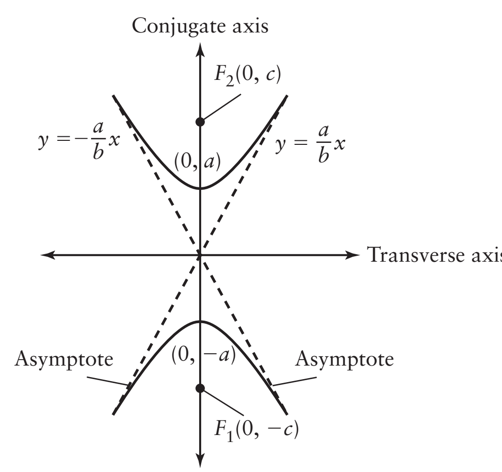
  The distance from point $P(x, y)$ to $F_2(c, 0)$ is $PF_2 = \sqrt{(x - c)^2 + (y - 0)^2}$.

  The sum of the distances is constant, i.e., $|PF_1 - PF_2| = k$ and so $\sqrt{(x + c)^2 + (y - 0)^2} - \sqrt{(x - c)^2 + (y - 0)^2} = k$.

  Rewrite the above equality as $\sqrt{x^2 + 2xc + c^2 + y^2} = k + \sqrt{x^2 - 2xc + c^2 + y^2}$ ①

  Further algebraic operations and simplification reduce equation ① to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.
- This represents a hyperbola with foci and vertices on the x-axis, which is called the **transverse axis**. The y-axis is the **conjugate axis**. The hyperbola with centre (0, 0) has two branches that intersect only one axis. The axis that is intersected is called the **transverse axis**, and the other axis (which is not intersected) is called the **conjugate axis**.

- This hyperbola has **asymptotes** with equations $y = \pm\frac{b}{a}x$ and no y-intercepts. The **asymptotes** of a hyperbola are two imaginary lines that the branches of this curve become closer and closer to but do not intersect. They determine the shape of the hyperbola. The equations of the asymptotes are based on the values of a and b.
- The equation of a hyperbola with foci and vertices on the y-axis is $\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$. In this case, the y-axis is the transverse axis and the x-axis is the conjugate axis. This hyperbola has asymptotes with equations $y = \pm\frac{a}{b}x$ and no x-intercepts.



Hyperbola: Key Characteristics		
	$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$	$\frac{x^2}{b^2} - \frac{y^2}{a^2} = -1$
Centre	(0, 0)	(0, 0)
Vertices	$(\pm a, 0)$	$(0, \pm a)$
Transverse axis and length	x-axis, $2a$	y-axis, $2a$
Conjugate axis and length	y-axis, $2b$	x-axis, $2b$
Foci	$(\pm c, 0)$ where $c^2 = a^2 + b^2$	$(0, \pm c)$ where $c^2 = a^2 + b^2$
Distance between foci	$2c$	$2c$
Asymptotes	$y = \pm\frac{b}{a}x$	$y = \pm\frac{a}{b}x$

Example 1

Identify the key characteristics of each hyperbola with centre (0, 0): $\frac{x^2}{25} - \frac{y^2}{9} = 1$, $\frac{x^2}{4} - \frac{y^2}{36} = -1$

Solution

	$\frac{x^2}{25} - \frac{y^2}{9} = 1$	$\frac{x^2}{4} - \frac{y^2}{36} = -1$
Vertices	$(\pm 5, 0)$	$(0, \pm 6)$
Transverse axis and length	x-axis, $2(5) = 10$	y-axis, $2(6) = 12$
Conjugate axis and length	y-axis, $2(3) = 6$	x-axis, $2(2) = 4$
Foci	$c^2 = 25 + 9$ $= 34$ $c = \sqrt{34}$ $\left(\pm\sqrt{34}, 0\right)$	$c^2 = 36 + 4$ $= 40$ $c = 2\sqrt{10}$ $\left(0, \pm 2\sqrt{10}\right)$
Distance between foci	$2\sqrt{34}$	$4\sqrt{10}$
Asymptotes	$y = \pm\frac{3}{5}x$	$y = \pm 3x$

Example 2

Determine an equation of a hyperbola with foci at (−5, 0) and (5, 0) and constant difference of focal radii 8.

Solution

Let $P(x, y)$ be any point on the ellipse.

$|PF_1 - PF_2| = 8.$

$\sqrt{(x + 5)^2 + y^2} - \sqrt{(x - 5)^2 + y^2} = 8$

Write the equation so that there is a radical on each side.

$-\sqrt{x^2 - 10x + 25 + y^2} = 8 - \sqrt{x^2 + 10x + 25 + y^2}$

Square both sides.

$x^2 - 10x + 25 + y^2 = 64 - 16\sqrt{x^2 + 10x + 25 + y^2} + x^2 + 10x + 25 + y^2$

Simplify.

$4\sqrt{x^2 + 10x + 25 + y^2} = 5x + 16$

Square both sides again.

$16x^2 + 160x + 400 + 16y^2 = 25x^2 + 160x + 256$

Simplify.

$9x^2 - 16y^2 = 144$

Divide each side by 144.

The equation of the hyperbola is $\frac{x^2}{16} - \frac{y^2}{9} = 1$.

A

1. For each hyperbola below, state the transverse axis and the conjugate axis, and their lengths.

a) $\frac{x^2}{16} - \frac{y^2}{64} = -1$

b) $\frac{x^2}{100} - \frac{y^2}{81} = 1$

c) $\frac{x^2}{4} - \frac{y^2}{49} = 1$

d) $\frac{x^2}{25} - \frac{y^2}{36} = -1$

e) $4x^2 - 25y^2 = -100$

f) $16x^2 - 4y^2 = 100$

2. Sketch a graph of each hyperbola in question 1.

B

3. A hyperbola has centre (0, 0) and transverse axis on the y-axis. Write the equation of the hyperbola if:

a) $a = 4, b = 7$

b) $a = 3, b = 6$

c) $a = 8, b = 5$

d) the transverse axis has length 8 units and the conjugate axis has length 18 units

e) one vertex is (0, −3) and one asymptote is $y = -2x$.

4. A hyperbola has centre (0, 0) and transverse axis on the x-axis. Write the equation of the hyperbola if:

   a) $a = 10, b = 3$

   b) $a = 8, b = 6$

   c) $a = 9, b = 11$

   d) the transverse axis has length 14 units and the conjugate axis has length 10 units

   e) one vertex is (−4, 0) and one focus is (−5, 0)

5. State the key characteristics of each hyperbola you graphed in question 2.

6. Complete this chart for each hyperbola with centre (0, 0).

	$\frac{x^2}{81} - \frac{y^2}{64} = 1$	$\frac{x^2}{4} - \frac{y^2}{36} = -1$	$\frac{x^2}{144} - \frac{y^2}{121} = 1$	$\frac{x^2}{9} - \frac{y^2}{25} = -1$
Vertices				
Transverse axis and length				
Conjugate axis and length				
Foci				
Distance between foci				
Asymptotes				

☆7. **a)** Determine an equation of a hyperbola with foci at (−6, 0) and (6, 0) and constant difference of focal radii 10.

   b) Identify the key characteristics of the hyperbola you found in part a).

   c) Sketch a graph of the hyperbola.

8. **a)** Determine an equation of a hyperbola with foci at (0, −7) and (0, 7) and constant difference of focal radii 8.

   b) Identify the key characteristics of the hyperbola you found in part a).

   c) Sketch a graph of the hyperbola.

☆9. Determine the equation of a hyperbola with center (0, 0) that has one vertex at $(\sqrt{6}, 0)$ and passes through the point (9, 5).

C

10. The **eccentricity**, ***e***, of a hyperbola is defined the same way as for an ellipse, $e = \frac{c}{a}$. Determine the eccentricity of each ellipse in question 6 to two decimal places.

11. Find an equation of a hyperbola with centre (0, 0), where the transverse axis is the x-axis with length 8 and the eccentricity is 1.5.

12. The equation of a hyperbola with centre (h, k) and transverse axis on the x-axis is $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$.
Determine the key characteristics of the hyperbola $\frac{(x-2)^2}{4} - \frac{(y+3)^2}{9} = 1$. Sketch a graph of this hyperbola.

13. Write the equation $7x^2 - 2y^2 - 14x + 12y = 101$ in the form $\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$ and sketch the graph, indicating the key characteristics.