3.2 Reciprocal of a Quadratic Function

Student Text Pages
157 to 167

Suggested Timing
150 min

Tools
- grid paper
- graphing calculator

OR
- computer
- *The Geometer's Sketchpad®*

Related Resources
- G–1 Grid Paper
- T–2 *The Geometer's Sketchpad®* 4
- BLM 3–3 Section 3.2 Practice

Teaching Suggestions

- The expectations of this course specify that technology is to be used to investigate reciprocals of quadratic functions. Graphing software, such as *The Geometer's Sketchpad®*, or graphing calculators are appropriate tools.
- Technology tips for the **Investigate**, using the TI-83 Plus or TI-84 Plus:
  - Find the minimum or maximum of the reciprocal of a quadratic function by pressing 2nd [CALC] and selecting **3:minimum** or **4:maximum**.
  - Analyse the table of values when answering questions that deal with the behaviour of the graph from both the left and right of each asymptote.
  - Students can draw in the asymptotes. When viewing the graph, press 2nd [DRAW] and select **4:Vertical** (vertical asymptote) and **3:Horizontal** (horizontal asymptote). A vertical line or horizontal line will appear for you to move using the arrow keys to the desired location, then press ENTER to draw it in.
- Technology tips for the **Investigate**, using *The Geometer's Sketchpad®*:
  - Create a point on the function by selecting the graph of the function, from the **Construct** menu, choose **Point on Function**. Select the point, from the **Measure** menu, choose **Coordinates**. Select the coordinates of the point and go to the **Graph** menu and choose **Tabulate** to create a table of values for that point. By doing this, students can analyse the function and its behaviour from both the left and right of each asymptote.
  - Students can draw in the asymptotes. For example, for a vertical asymptote, create a point on the x-axis where the vertical asymptote will be, select it and the x-axis, go to the **Construct** menu and choose **Perpendicular Line**. Change the look of asymptotes drawn by selecting them and going to the **Display** menu to choose **Line Width** and even **Color**.
- In the **Investigate**, encourage students to communicate through the use of words, numbers, and graphs. This is a good introduction to the lesson and may take 40–60 min. Use **T–2 *The Geometer's Sketchpad®* 4** to support this activity.
- Compare the restrictions on the domain to the x-intercepts of a quadratic function. For the reciprocal of a quadratic function, the asymptotes occur at the roots of the quadratic relation.
- In **Example 2**, the rates of change charts can easily be done through the use of the spreadsheet capabilities of graphing calculators. Eventually, the students will understand the behaviour of the function between the asymptotes and will not require these charts. Demand that they provide reasoning for their decisions, though.
- **Example 2** requires reasoning and connecting skills to establish where the slope is increasing and decreasing, and then the selecting of tools and computational strategies along with representing skills to graph the function.
- **Example 3** illustrates a reciprocal function that does not have a vertical asymptote. At this point in the course, you can stress the importance of formal algebraic reasoning and an appropriate rationale behind making any claim about asymptotes. For example:

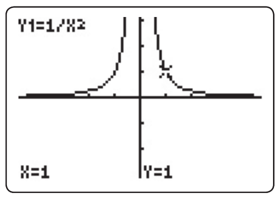
  $$x^2 + 1 = 0$$
  $$x^2 = -1$$
  $$x = \sqrt{-1}$$

  x is undefined, so there is no vertical asymptote.

- In graph such as those in **Example 3**, tell the students that the point at which the slope changes from decreasing to increasing is called an inflection point, and that how to find its coordinates will be covered in calculus.
- **Question 9** is a good example of the use of the students' investigation skills.
- **Question 11** requires students to work backward to find the equation for the function. Suggest that they set up a system of equations that need to be solved.
- **Question 12** allows students to reason out how to write the function required and to use connecting skills to write the function in part a). Representing skills will be necessary to sketch a graph of this function. Finally, students will reflect on the reasonableness of the model.
- **Questions 13** and **14** are good applications of the topic.
- In **question 17**, suggest that students compare their methods. Some may use algebraic methods and others may use graphing techniques.
- Use **BLM 3–3 Section 3.2 Practice** for remediation or extra practice.

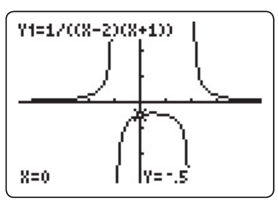
Investigate Answers (pages 157–158)

1. Window variables: $x \in [-5, 5]$, $y \in [-3, 3]$



a) $y = 0$
b) $x = 0$
c) No intercepts.
d) Domain: $\{x \in R, x \neq 0\}$, Range: $\{y \in R, y > 0\}$
e) As $x \to \pm\infty$, $f(x) \to 0$. As $x \to 0^-$, $f(x) \to +\infty$. As $x \to 0^+$, $f(x) \to +\infty$.
f) The function is positive for $x \in (-\infty, 0)$ and $x \in (0, +\infty)$.
g) The function is increasing for $x \in (-\infty, 0)$. The function is decreasing for $x \in (0, +\infty)$.

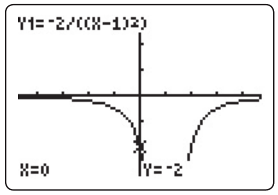
2. a) $\{x \in \mathbb{R}, x \neq -1, x \neq 2\}$
b) Window variables: $x \in [-5, 5]$, $y \in [-3, 3]$



c) As $x \to -1^-$, the function is positive and increasing toward $+\infty$. As $x \to -1^+$, the function is negative and decreasing toward $-\infty$.
As $x \to 2^-$, the function is negative and decreasing toward $-\infty$. As $x \to 2^+$, the function is positive and increasing toward $+\infty$.
d) i) positive, increasing slope as $x \to -1^-$ ii) negative, increasing slope as $x \to +\infty$
e) Answers may vary. Sample answer: Since $g(x)$ is symmetrical, the maximum point occurs right in the middle of the interval (–1, 2).
f) $\left(\frac{1}{2}, -\frac{4}{9}\right)$
g) $\left\{y \in \mathbb{R}, y \leq -\frac{4}{9}, y > 0\right\}$
h) Answers may vary. Sample answer:
The function $g(x)$ has two vertical asymptotes, a U-shaped graph between the two asymptotes, a maximum point, and positive and negative values.

3. **a)** $\{x \in \mathbb{R}, x \neq 1\}$

b) Window variables: $x \in [-5, 5]$, $y \in [-3, 3]$



c) As $x \to 1^-$, the function is negative and decreasing toward $-\infty$. As $x \to 1^+$, the function is negative and decreasing toward $-\infty$.

d) **i)** negative, decreasing slope as $x \to 1^-$. **ii)** positive, decreasing slope as $x \to +\infty$.

e) $\{y \in \mathbb{R}, y < 0\}$

f) Answers may vary. Sample answer:
The function $h(x)$ has one vertical asymptote and only negative values.

4. **a)** **i)** $x = -2$, $x = -1$, $y = 0$

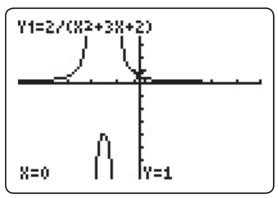
ii) Domain: $\{x \in \mathbb{R}, x \neq -1, x \neq -2\}$, Range: $\{y \in \mathbb{R}, y > 0, y \leq -8\}$

iii) For $x < -2$, the slope is positive and increasing. For $-2 < x < -1.5$, the slope is positive and decreasing. For $-1.5 < x < -1$, the slope is negative and decreasing. For $x > -1$, the slope is negative and increasing.

iv) As $x \to -2^-$, the function is positive and increasing to $+\infty$. As $x \to -2^+$, the function is negative and decreasing to $-\infty$. As $x \to -1^-$, the function is negative and decreasing to $-\infty$. As $x \to -1^+$, the function is positive and increasing to $+\infty$. As $x \to \pm\infty$, the function is positive and approaching 0. As $x \to -\infty$, the function is positive and approaching 0.

v) y-intercept 1

b), c) Window variables: $x \in [-5, 5]$, $y \in [-15, 10]$, Yscl=2



5. Answers may vary. Sample answer:
Graphs of reciprocals of linear functions have one vertical asymptote, two branches, and no maximum or minimum point.
Graphs of reciprocals of quadratic functions have one or two vertical asymptotes, have two or three branches, and can have a minimum or maximum point.

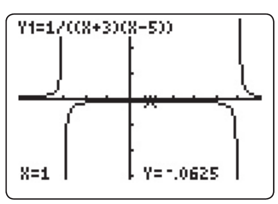
Communicate Your Understanding Responses (page 164)

C1. **a)** negative, increasing slope

b) negative, decreasing slope

c) positive, increasing slope

C2. Answers may vary. Sample answer:
Window variables: $x \in [-5, 6]$, $y \in [-3, 3]$



C3. For $x < -5$, the function is negative and decreasing. For $-5 < x < -2$, the function is positive and decreasing. For $-2 < x < 1$, the function is positive and increasing. For $x > 1$, the function is negative and increasing.
The asymptotes have equations $x = -5$, $x = 1$, and $y = 0$.
The y-intercept is 2. There is no x-intercept.

DIFFERENTIATED INSTRUCTION

Use **inside-outside circle** to consolidate concepts of these two sections.

COMMON ERRORS

- Students may confuse the types of graphs of functions with or without asymptotes.

R_x Remind students that reciprocals of quadratics can be classified into three types:

- $f(x) = \frac{1}{(kx - a)^2}$, which has two vertical asymptotes and two branches
- $f(x) = \frac{1}{(kx - a)(mx - b)}$ or $f(x) = \frac{1}{ax^2 + bx + c}$ where $b^2 - 4ac > 0$, which has two vertical asymptotes and three branches
- $f(x) = \frac{1}{ax^2 + bx + c}$ where $b^2 - 4ac < 0$, which has no vertical asymptote

ONGOING ASSESSMENT

Achievement Check, question 15, on student text page 167.

For $x < -5$, the slope of the function is negative and decreasing. For $-5 < x < -2$, the slope of the function is negative and increasing. For $-2 < x < 1$, the slope of the function is positive and increasing. For $x > 1$, the slope of the function is positive and decreasing.

Domain: $\{x \in \mathbb{R}, x \neq -5, x \neq 1\}$

Range: $\{y \in \mathbb{R}, y < 0, y \geq 0.1\}$

The minimum point is $(-2, 0.1)$.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	
Reasoning and Proving	7, 10, 11, 13, 17, 18
Reflecting	11, 17
Selecting Tools and Computational Strategies	6–10, 13, 14, 18
Connecting	3–18
Representing	5, 8–15, 18
Communicating	7, 9, 10, 18

Achievement Check, question 15, student text page 167

This performance task is designed to assess the specific expectations covered in Sections 3.2. The following Math Process Expectations can be assessed.

- Reasoning and Proving
- Reflecting
- Representing
- Communicating
- Selecting Tools
- Connecting

Sample Solution

a) $f(x) = \frac{3}{x^2 - 25}$

$= \frac{3}{(x + 5)(x - 5)}$

$x \neq -5, x \neq 5$

b) domain: $\{x \in \mathbb{R}, x \neq -5, x \neq 5\}$, range: $\left\{y \in \mathbb{R}, y \leq -\frac{3}{25}, y > 0\right\}$

c) The vertical asymptotes have equations $x = 5$ and $x = -5$.
The horizontal asymptote has equation $y = 0$.

d) For the x-intercept, let $f(x) = 0$.

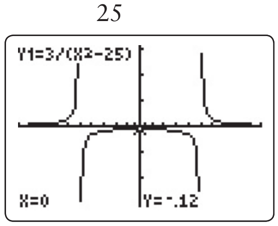
$0 = \frac{3}{x^2 - 25}$

Since there is no value of x that makes this statement true, there is no x-intercept.
For the y-intercept, let $x = 0$.

$f(0) = \frac{3}{0^2 - 25}$

$= -\frac{3}{25}$

e)



f) $x \to -5^-$ the function is positive and approaching $+\infty$; $x \to -5^+$ the function is negative and approaching $-\infty$

$x \to 5^-$ the function is negative and approaching $-\infty$; $x \to 5^+$ the function is positive and approaching $+\infty$

Level 3 Notes

Look for the following:

- Restrictions on x, domain and range, equations of asymptote(s), x- and y-intercepts are mostly accurate
- Proper form is evident in the answers given above
- Sketch of the graph of the function is mostly accurate
- Sketch is clearly labelled and most key features are identified
- End behaviour of the function is described fairly accurately

What Distinguishes Level 2

- Restrictions on x, domain and range, equations of asymptote(s), x- and y-intercepts are somewhat accurate
- Some proper form is evident in the answers given above
- Sketch of the graph of the function is somewhat accurate
- Sketch is somewhat labelled and some key features are identified
- End behaviour of the function is described with some accuracy

What Distinguishes Level 4

- Restrictions on x, domain and range, equations of asymptote(s), x- and y-intercepts contain only minor errors
- Proper form is evident in all the given answers
- Sketch of the graph of the function shows a high degree of accuracy
- Sketch is clearly labelled and all the key features are identified and described
- End behaviour of the function is described accurately and with justification