4.2 Trigonometric Ratios and Special Angles

Student Text Pages
211 to 219

Suggested Timing
60–75 min

Tools
- scientific calculator
- graphing calculator
- computer algebra system
- compasses
- grid paper
- protractor
- computer
- *The Geometer's Sketchpad®*

Related Resources
- G–1 Grid Paper
- G–4 Protractor
- T–2 *The Geometer's Sketchpad®* 4
- T–4 The Computer Algebra System (CAS) on the TI-89 Calculator
- BLM 4–4 Summary of Special Angles and Trigonometric Ratios
- BLM 4–5 Section 4.2 Practice

Teaching Suggestions

- While working through **Investigate 1**, Method 1, be prepared to deal with the differences in scientific calculators. There is no standard way of entering keystrokes for operations with trigonometric ratios. Students will need to become familiar with their own scientific calculator when it comes to changing the mode from degrees to radians and vice versa.
- **Investigate 1**, Method 3, makes use of a computer algebra system (CAS). Be sure that students know the difference between exact and approximate displays of the answer, since they may not have seen this before. You can refer to the Technology Appendix on page 506. If needed, use **T–4 The Computer Algebra System (CAS) on the TI-89 Calculator** to support this method.
- For **Investigate 2**, let students draw their own unit circle. Students will have had experience working with this tool in grade 11 when determining trigonometric ratios of special angles using degree measure. The objective is to become familiar enough with the wealth of information available from this tool that students will be able to quickly sketch and use it, even in a time-constrained setting such as a test or examination.
- The table of special angles and their trigonometric ratios will be used extensively in the rest of this chapter and the next one on trigonometric functions. Once it is completed, encourage students to have it available on their desktops at all times. Colour coding by quadrant may be of use to some students. Use **BLM 4–4 Summary of Special Angles and Trigonometric Ratios** to support **Investigate 2**.
- You can use *The Geometer's Sketchpad®* to determine trigonometric ratios for special angles as an alternative method for **Investigate 2**.
- Before working through **Example 1**, you can perform a simulation using a short length of string, a kite held up by a student, and a protractor. You can paste a printout of equivalent radian measures over the degree scale on a standard large wooden or plastic protractor. In preparation for **question 10** in **Connect and Apply**, ask students to predict what would happen if the string were lengthened or shortened between the two measurements, and then demonstrate.
- As students consider **Communicate Your Understanding** question C1, ensure that they understand how to use the unit circle to explain why the tangent ratio sometimes returns undefined results.
- You can add some interest to question C2 by reminding students of the "toothpicks and striped bed sheets" method of determining the value of π, also known as Buffon's Needle. Go to *www.mcgrawhill.ca/books/functions12* and follow the links to find out more.
- **Question 10** gives the students the opportunity to employ inductive reasoning by investigating and making conjectures about the horizontal movement of a kite and the change in its vertical distance. Students will also select the tools that are needed to develop the expressions required and will make connections to do so, showing that they can use learning from other areas of mathematics to understand the particular concepts required in this question.
- You can create a visual aid for **question 14**. Use *The Geometer's Sketchpad®* to print a faceplate in radian measure, and paste it over the existing face of an inexpensive alarm clock. This can be used to verify student answers.

- Consider employing the home use licence for *The Geometer's Sketchpad*® so that students can do questions such as **question 15** as homework.
- You can prepare students for **question 22** by asking them to compare the value of $\frac{\pi}{36}$ to $\sin\frac{\pi}{36}$.
- Use **BLM 4–5 Section 4.2 Practice** for remediation or extra practice.

Investigate Answers (pages 211–213)

Investigate 1

METHOD 2

1. b) $\sin 45^\circ \doteq 0.7071$

2. a) $45^\circ = \frac{\pi}{4}$

c) 0.7071; this matches the answer in step 1b).

3. $\cos\frac{\pi}{4} \doteq 0.7071$, $\tan\frac{\pi}{4} = 1$

4. $\csc\frac{\pi}{4} \doteq 1.4142$

5. $\sec\frac{\pi}{4} \doteq 1.4142$, $\cot\frac{\pi}{4} = 1$

6. $\sin 1.5 \doteq 0.9975$, $\cos 1.5 \doteq 0.0707$, $\tan 1.5 \doteq 14.1014$, $\csc 1.5 \doteq 1.0025$, $\sec 1.5 \doteq 14.1368$, $\cot 1.5 \doteq 0.0709$

7. It is important to check the angle mode of a calculator to ensure accurate results. For example, $\sin 1.5 \doteq 0.9975$ while $\sin 1.5^\circ \doteq 0.0262$. These are not equal because the angles are measured in different units.

METHOD 2

1. c) $\sin 45^\circ \doteq 0.7071$

2. a) $45^\circ = \frac{\pi}{4}$

c) $\sin\frac{\pi}{4} \doteq 0.7071$. This matches the answer in step 1c).

3. $\cos\frac{\pi}{4} \doteq 0.7071$, $\tan\frac{\pi}{4} = 1$

4. $\csc\frac{\pi}{4} \doteq 1.4142$

5. $\sec\frac{\pi}{4} \doteq 1.4142$, $\cot\frac{\pi}{4} = 1$

6. $\sin 1.5$ R 0.9975, $\cos 1.5$ R 0.0707, $\tan 1.5$ R 14.1014, $\csc 1.5$ R 1.0025, $\sec 1.5$ R 14.1368, $\cot 1.5$ R 0.0709

7. It is important to check the angle mode of a calculator to ensure accurate results. For example, $\sin 1.5 \doteq 0.9975$ while $\sin 1.5^\circ \doteq 0.0262$. These are not equal because the angles are measured in different units.

METHOD 2

1. c) $\sin 45^\circ = \frac{\sqrt{2}}{2}$

d) $\sin 45^\circ \doteq 0.7071$

2. a) $45^\circ = \frac{\pi}{4}$

c) $\sin\frac{\pi}{4} \doteq 0.7071$. This matches the answer in step 1d).

3. $\cos\frac{\pi}{4} \doteq 0.7071$, $\tan\frac{\pi}{4} = 1$

4. $\csc\frac{\pi}{4} \doteq 1.4142$

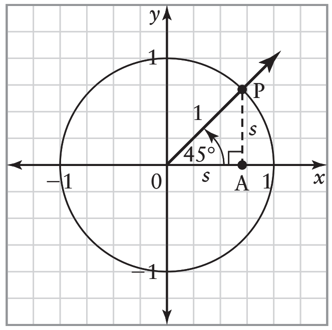
5. $\sec\frac{\pi}{4} \doteq 1.4142$, $\cot\frac{\pi}{4} = 1$

6. $\sin 1.5 \doteq 0.9975$, $\cos 1.5 \doteq 0.0707$, $\tan 1.5 \doteq 14.1014$, $\csc 1.5 \doteq 1.0025$, $\sec 1.5 \doteq 14.1368$, $\cot 1.5 \doteq 0.0709$

7. It is important to check the angle mode of a calculator to ensure accurate results. For example, $\sin 1.5 \doteq 0.9975$ while $\sin 1.5^\circ \doteq 0.0262$. These are not equal because the angles are measured in different units.

Investigate 2

1. a)–d)



The triangle is isosceles. Let the side length be s.

$s^2 + s^2 = 1$

$2s^2 = 1$

$s^2 = \frac{1}{2}$

$s = \frac{1}{\sqrt{2}}$

e) $\sin 45° = \frac{1}{\sqrt{2}}$, $\cos 45° = \frac{1}{\sqrt{2}}$, $\tan 45° = 1$

f) $45° = \frac{\pi}{4}$

1. g), 2.–5.

Special Angles and Trigonometric Ratios

θ (°)	θ (radians)	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0	0	1	0
30	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
90	$\frac{\pi}{2}$	1	0	Undefined
120	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
135	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1
150	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
180	π	0	-1	0
210	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
225	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1
240	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
270	$\frac{3\pi}{2}$	-1	0	Undefined

Differentiated Instruction

Use **jigsaw** to teach this section.

Common Errors

- The error explored in question C3, $\sin\frac{\pi}{2} = \frac{1}{2}\sin\pi$, is commonly made by students.

R_x Revisit this error from time to time, as appropriate, during lessons or take-up sessions.

Ongoing Assessment

Achievement Check, question 20, on student text page 219.

Special Angles and Trigonometric Ratios				
θ (°)	θ (radians)	sin θ	cos θ	tan θ
300	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
315	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1
330	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
360	2π	0	1	0

6. Results should match the table.

Communicate Your Understanding Responses (page 216)

C1. Evaluating $\tan\frac{\pi}{2}$ on a calculator results is an error. $\tan\frac{\pi}{2}$ is undefined. The denominator of the ratio is zero. $\tan\frac{3\pi}{2}$ is also undefined.

C2. $\frac{355}{113} \doteq 3.141\ 592\ 92$

$\sin(3.141\ 592\ 92) \doteq 0.000\ 000\ 267$

The answer should be zero. The approximation is not close enough to yield the correct value for the sine ratio within the limit of the accuracy of the calculator.

C3. $\sin\frac{\pi}{2} = 1$ $\qquad$ $\frac{1}{2}\sin\pi = \frac{1}{2} \times 0$

$= 0$

The two expressions are not equal. It is not possible to factor $\frac{1}{2}$ out of the argument of a sine ratio.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	15, 19, 23
Reasoning and Proving	10, 19–22
Reflecting	19
Selecting Tools and Computational Strategies	9–12, 15–22
Connecting	7–18, 20–22
Representing	15, 19
Communicating	10, 15, 20–22

Achievement Check, question 20, student text page 219

This performance task is designed to assess the specific expectations covered in Section 4.2.

The following Math Process Expectations can be assessed.

- Reflecting
- Representing
- Communicating
- Selecting Tools
- Connecting
- Reasoning and Proving
- Problem Solving

Sample Solution

a) $\textbf{L.S.} = 2\sin x \cos y$

$= 2\sin\frac{5\pi}{4}\cos\frac{3\pi}{4}$

$= 2\left(-\frac{1}{\sqrt{2}}\right)\left(-\frac{1}{\sqrt{2}}\right)$

$= 1$

b) $\textbf{R.S.} = \sin(x + y) + \sin(x - y)$

$= \sin\left(\frac{5\pi}{4} + \frac{3\pi}{4}\right) + \sin\left(\frac{5\pi}{4} - \frac{3\pi}{4}\right)$

$= \sin 2\pi + \sin\frac{\pi}{2}$

$= 0 + 1$

$= 1$

c) Emilio is correct.

d) $\textbf{L.S.} = 2\sin x \cos y$

$= 2\sin\frac{5\pi}{6}\cos\frac{\pi}{6}$

$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$= \frac{\sqrt{3}}{2}$

$\textbf{R.S.} = \sin(x + y) + \sin(x - y)$

$= \sin\left(\frac{5\pi}{6} + \frac{\pi}{6}\right) + \sin\left(\frac{5\pi}{6} - \frac{\pi}{6}\right)$

$= \sin\pi + \sin\frac{2\pi}{3}$

$= 0 + \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3}}{2}$

Since **L.S.** = **R.S.**, $x = \frac{5\pi}{6}$ and $y = \frac{\pi}{6}$ is also a solution.

e) The answers to parts c) and d) cannot be used to conclude that the equation will work for all values of x and y. A formal proof is required.

Level 3 Notes

Look for the following:

- Trigonometric expressions *n* left side of equation are determined using exact values
- Addition and subtraction formulas are applied to expand the trigonometric expressions in right side of equation
- Once expanded, the trigonometric ratios in the right side of equation are determined using exact values
- Answer to part d) is supported with justification using method of part a) and part b)
- Answers in parts a), b), and c) are mostly accurate
- Valid reasoning/justification is provided to support answer in part e)
- Evidence of proper form occurs in most of the solution steps

What Distinguishes Level 2

- Trigonometric expressions in left side of equation are in decimal form
- Addition and subtraction formulas are applied to expand the trigonometric expressions in right side of equation
- Once expanded, the trigonometric ratios in the right side of equation are in decimal form
- Answer to part d) is somewhat justified
- Answers in parts a), b), and c) are somewhat accurate
- Valid reasoning/justification is provided to support answer in part e)
- Evidence of proper form occurs in some of the solution steps

What Distinguishes Level 4

- Trigonometric expressions in left side of equation are determined using exact values
- Addition and subtraction formulas are applied to expand the trigonometric expressions in right side of equation
- Once expanded, the trigonometric ratios in the right side of equation are determined using exact values
- Answer to part d) is supported with justification using method of part a) and part b)
- Answers in parts a), b), and c) are consistently accurate with minor errors
- Valid reasoning/justification is provided to support answer in part e)
- Solution steps show a high degree of proper form