6.1 The Exponential Function and Its Inverse

Student Text Pages
310 to 322

Suggested Timing
75–150 min

Tools
- grid paper
- computer
- *The Geometer's Sketchpad®*
- graphing calculator (optional)

Related Resources
- G–1 Grid Paper
- G–2 Placemat
- G–3 Four Quadrant Grids
- T–2 *The Geometer's Sketchpad®* 4
- BLM 6–2 Section 6.1 Practice

Teaching Suggestions

- Depending on the needs of the students and their rate of progress, this lesson could be split into two 80-min period sessions, as follows:
  - Day 1: **Investigate 1**, **Parts A** and **B**, followed by **Example 1**, **Communicate Your Understanding** C1 and then questions 1 through 4.
  - Day 2: **Investigate 2**, followed by **Example 2** and the balance of the **Communicate Your Understanding** and exercise questions.

  If needed, use **T–2 *The Geometer's Sketchpad®* 4** to support this activity.
- In **Investigate 1**, **Part A**, students explore the finite differences of an exponential function to discover a repeating pattern in each successive set of differences. Graphing calculators or spreadsheet software can be used, but are not required.
- In **Investigate 1**, **Part B**, students explore the instantaneous rate of change of an exponential function and compare the results to those found in **Part A**. *The Geometer's Sketchpad®* (GSP) is recommended for this part as it makes it easy to construct and measure the slopes of secants that approximate tangent lines at various points along the curve.
- Technology tips for **Investigate 1**, **Part B**:
  - From the **Graph** menu, choose **Show Grid**.
  - Select the *unit point* on the x-axis to change the intervals for both the x- and y-axes. Select the *origin* to move the entire Cartesian plane.
  - From the **Graph** menu, choose **New Function**. With the function highlighted, from the **Graph** menu choose **Plot Function**.
  - To change the look of the graph from a thin curve to a thick curve or even change its colour, use the **Display** menu.
  - To construct the points, select the function curve then from the **Construct** menu choose **Point on Function Plot**. Once the points are constructed, select both points, then from the **Construct** menu choose **Segment**.
  - To measure the slope, select the segment, then from the **Measure** menu choose **Slope**.
  - Before dragging points to their desired locations, select each point first then from the **Measure** menu choose **Coordinates**. This allows students to know when points are exactly where they should be to retrieve desired results for the intervals requested by the table.
- In **Example 1**, a graphing calculator or graphing software could be used to verify the result.
- **Investigate 2** provides an opportunity for students to explore the inverse of an exponential function using a dynamic geometry approach: a point is constructed on an exponential curve from which an image point is generated by reflecting in the line $y = x$. By then dragging the constructed point all along the curve, the image of the inverse is traced out, providing a quick and powerful visual image of what students will later learn to identify as the logarithm function. In step 8, students can either change the base manually, or alternatively they could change it dynamically using a slider or a parameter and the Motion Controller. If needed, use **T–2 *The Geometer's Sketchpad®* 4** to support this activity.
- Students should by now understand that the graph of the inverse of a function can be found by reflecting in the line $y = x$. Accordingly, graphing technology can be used to expedite the solution to **Example 2**.

Ongoing Assessment

Use **Assessment Masters A–1** to **A–7** to remind students about the Math Processes Expectations and how you may be assessing their integrated use of them.

- The **Communicate Your Understanding** questions are designed to verify that students understand the key features of an exponential function and its inverse. They should realize that the exponential graph—and its inverse—collapse to lines when the base equals 1.
- Students should recognize for **questions 1** and **2** that exponential functions have finite differences that increase by a common ratio.
- The results to **questions 5** through **8** can be checked using graphing technology, such as the DrawInv feature of the TI-84 Plus graphing calculator.
- For **question 10**, students will need to recall the difference between average rate of change and instantaneous rate of change.
- **Question 17** helps students demonstrate that they are reflecting upon the comparison of the two graphs while showing that they have communicating skills and can make connections with mathematical procedures.
- Students can use graphing technology to check their results to **questions 11** through **18**.
- **Question 21** provides an opportunity to assess students' reasoning and communicating skills.
- Students can solve and/or check **question 22** using more than one method (e.g., algebraically, graphically, numerically).
- Encourage students to explore **question 23** using GSP or a graphing calculator. Go to *www.mcgrawhill.ca/books/functions12* and follow the links to this GSP file.
- For **question 24**, students will need to recall the vertical line test for functions.
- Technology tips for **question 24**:
  - To graph the inverse of this function, from the **Graph** menu, choose **New Function**. From the **Equation** menu, select $x = f(y)$. Next, select the parameter b and press ^, then y, and click **OK**. From the **Graph** menu, choose **Plot Function**.
  - To change the value of the parameter b, either double-click it or right-click on it choosing **Edit Parameter**.
  - To animate the parameter b, right-click it and select **Animate Parameter**. This will generate a **Motion Controller** where students can select the speed by which the value of the parameter changes.
- Use **BLM 6–2 Section 6.1 Practice** for remediation or extra practice.

Investigate Answers (pages 310–313)

Investigate 1

Part A

1. a), 2. a), 3. c), 4.

x	y	$\Delta_1 y$	$\Delta_2 y$	$\Delta_3 y$
0	1			
1	2	1		
2	4	2	1	
3	8	4	2	1
4	16	8	4	2
5	32	16	8	4
6	64	32	16	8

1. b) The y-value is multiplied by 2 as x increases by 1.

2. b) Answers may vary. First differences also differ by a multiple of 2.

3. a) The pattern is the same, but one step below.

b) They will be the same, but another step below.

c) Yes.

5. Steps 1a), 2a), 3c), and 4:

x	y	$\Delta_1 y$	$\Delta_2 y$	$\Delta_3 y$
0	1			
1	3	2		
2	9	6	4	
3	27	18	12	8
4	81	54	36	24
5	243	162	108	72
6	729	486	324	216

Step 1b): The y-value is multiplied by 3 as x increases by 1.
Step 2b): Answers may vary. First differences also differ by a multiple of 3.
Step 3a): Answers may vary. Sample answer: $\Delta_1 y$ is 2 times y_i, where $\Delta_1 y = y_{i+1} - y_i$, $i = 1, 2, 3, \ldots$.
Step 3b): $\Delta_2 y$ is 2 times $\Delta_1 y$, where $\Delta_2 y = \Delta 1 y_{i+1} - \Delta_1 y_i$, $i = 1, 2, 3, \ldots$.
Step 3c) Yes.

6. Rate of change is proportional to the function's value.

Part B

2. c) Answers may vary.

3. b), 4., 5.

Interval		Average Rate of Change, m_{AB}	Instantaneous Rate of Change at A, m_A	Instantaneous Rate of Change at B, m_B
A	B			
$x = 0$	$x = 1$	1	0.7	1.4
$x = 1$	$x = 2$	2	1.4	2.8
$x = 2$	$x = 3$	4	2.8	5.6
$x = 3$	$x = 4$	8	5.6	11.2
$x = 4$	$x = 5$	16	11.2	22.4

c) Same.

6. The average rates of change are the same as the first differences, since each slope calculation is between consecutive x-values that differ by 1 unit.

7. a) $m_B = 2 \times m_A$

b) 1.05 Answers may vary. Sample answer: No. This value is only a guess based on linear interpolation for a curve.

c) Answers may vary. Sample answer: 0.98

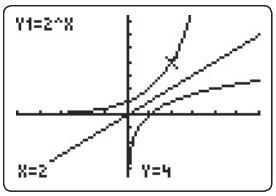
8. Answers may vary. Sample answer: The changes in the instantaneous rate of change along the curve do not increase linearly.

9. b) Answers may vary. Sample answer: For values of $b > 1$, the graph of $y = b^x$ exhibits exponential growth and the instantaneous rate of change is continuously increasing. For values of b, where $0 < b < 1$, the graph of $y = b^x$ exhibits exponential decay and the instantaneous rate of change is continuously decreasing.

Investigate 2

2. b) i) Domain: $\{x \in \mathbb{R}\}$, Range: $\{y \in \mathbb{R}, y > 0\}$
ii) none **iii)** 1 **iv)** The function is positive for all values of x.
v) The function is increasing for all values of x. **vi)** $y = 0$

3. a)–c) Window settings: $x \in [-5, 6]$, $y \in [-4, 7]$



Differentiated Instruction

Use **placemat** to review exponential functions.

Start a **word wall/information wall** for terminology.

Common Errors

- When using *The Geometer's Sketchpad®*, some menu options do not appear to be available, because too many or too few items are selected.

R_x Have students get into the habit of deselecting (by clicking in the white space) after each construction, and then carefully selecting only the objects required for the next operation.

5. a) The same shape as the graph of $y = 2^x$, but reflected in the line $y = x$.

b) i) Domain: $\{x \in \mathbb{R}, x > 0\}$, Range: $\{y \in \mathbb{R}\}$ **ii)** 1 **iii)** none

iv) The function is negative for $x < 1$. The function is positive for $x > 1$.

v) The function is increasing for $x > 0$. **vi)** $x = 0$

6. a)

x	Calculation $2^x = y$	y	Inverse (y, x)
-4	$2^{-4} = \frac{1}{16}$	$\frac{1}{16}$	$\left(\frac{1}{16}, -4\right)$
-3	$2^{-3} = \frac{1}{8}$	$\frac{1}{8}$	$\left(\frac{1}{8}, -3\right)$
-2	$2^{-2} = \frac{1}{4}$	$\frac{1}{4}$	$\left(\frac{1}{4}, -2\right)$
-1	$2^{-1} = \frac{1}{2}$	$\frac{1}{2}$	$\left(\frac{1}{2}, -1\right)$
0	$2^0 = 1$	1	(1, 0)
1	$2^1 = 2$	2	(2, 1)
2	$2^2 = 4$	4	(4, 2)
3	$2^3 = 8$	8	(8, 3)
4	$2^4 = 16$	16	(16, 4)

7. a) The graph of the inverse of $y = 2^x$ is a reflection of the graph of $y = 2^x$ about the line $y = x$.

b) A reflection in the line $y = x$ results in the x- and y-coordinates being switched.

8. b) The graphs are reflections of one another in the line $y = x$.

Communicate Your Understanding Responses (page 318)

C1. a) The finite differences show a repeated pattern.

b) The graph is increasing (as in $y = b^x$, $b > 1$) or decreasing (as in $y = b^x$, $0 < b < 1$).

C2. a) Yes. Every value of x in the domain yields one y-value.

b) Yes. The graph of the inverse passes the vertical line test.

C3. a) Answers may vary. Sample answer:
The graphs of the function and its inverse have the same shape, are increasing, and have positive slope.

b) Answers may vary. Sample answer:
The graphs are reflections of one another. The graph of $y = 2^x$ has negative x-values and increasing slope, while the graph of its inverse has negative y-values and decreasing slope.

C4. a) It is a horizontal line, $y = 1$.

b) It is a vertical line, $x = 1$.

c) The x- and y-coordinates are switched.

Mathematical Process Expectations

Process Expectation	Selected Questions
Problem Solving	
Reasoning and Proving	7, 8, 17, 18, 21, 23
Reflecting	17, 18, 23
Selecting Tools and Computational Strategies	1, 3, 4, 10, 11, 21
Connecting	2–13, 15–22, 24
Representing	3–6, 9–11, 14, 19, 21, 24
Communicating	1, 3, 4, 9, 10, 17, 18, 21, 23, 24