Chapter 11 Permutations, Combinations, and the Binomial Theorem

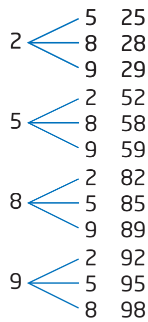
11.1 Permutations, pages 524 to 527

1. a)

Position 1	Position 2	Position 3
Jo	Amy	Mike
Jo	Mike	Amy
Amy	Jo	Mike
Amy	Mike	Jo
Mike	Jo	Amy
Mike	Amy	Jo

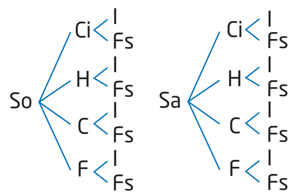
6 different arrangements

b)



12 different two-digit numbers

c) Use abbreviations: Soup (So), Salad (Sa), Chili (Ci), Hamburger (H), Chicken (C), Fish (F), Ice Cream (I) and Fruit Salad (Fs). 16 different meals



2. a) 56 **b)** 2520 **c)** 720 **d)** 4

3. Left Side $= 4! + 3!$
$= 4(3!) + 3!$
$= 5(3)!$
Right Side $= (4 + 3)!$
$= 7!$
Left Side $\neq$ Right Side

4. a) $9! = (9)(8)(7)(6)(5)(4)(3)(2)(1)$
$= 362\ 880$

b) $\dfrac{9!}{5!4!} = \dfrac{(9)(8)(7)(6)(5!)}{(5!)(4)(3)(2)(1)}$
$= 126$

c) $(5!)(3!) = (5)(4)(3)(2)(1)(3)(2)(1)$
$= 720$

d) $6(4!) = 6(4)(3)(2)(1)$
$= 144$

e) $\dfrac{102!}{100!2!} = \dfrac{(102)(101)(100!)}{100!(2)(1)}$
$= (51)(101)$
$= 5151$

f) $7! - 5! = (7)(6)(5!) - 5!$
$= 41(5!)$
$= 4920$

5. a) 360 **b)** 420 **c)** 138 600
d) 20 **e)** 20 **f)** 10 080

6. 24 ways

7. a) $n = 6$ **b)** $n = 11$ **c)** $r = 2$
d) $n = 6$

8. a) 6 **b)** 35 **c)** 10

9. a) Case 1: first digit is 3 or 5; Case 2: first digit is 2 or 4
b) Case 1: first letter is a B; Case 2: first letter is an E

10. a) 48 **b)** 240 **c)** 48

11. a) 5040 **b)** 2520 **c)** 1440 **d)** 576

12. 720 total arrangements; 288 arrangements begin or end with a consonant.

13. No. The organization has 25 300 members but there are only 18 000 arrangements that begin with a letter other than O followed by three different digits.

14. 20

15. $266\frac{2}{3}$ h

16. a) 5040 **b)** 1440 **c)** 3600

17. a) 3360 **b)** 360

18. a) AABBS **b)** Example: TEETH

19. 3645 integers contain no 7s

20. a) 17 576 000
b) Example: Yes, Canada will eventually exceed 17.5 million postal communities.

21. a) 10^{14}
b) Yes, $10^{14} = 100\ 000\ 000\ 000\ 000$, which is 100 million million.

22. a) $r = 3$ **b)** $r = 7$ **c)** $n = 4$ **d)** $n = 42$

23. ${}_nP_n = \dfrac{n!}{(n-n)!} = \dfrac{n!}{0!}$ and ${}_nP_n = n!$, so $0! = 1$.

24. The number of items to be arranged is less than the number of items in each set of arrangements.

25. 63 **26.** 84 **27.** 737 **28.** 15 **29.** 10

30. Example: Use the numbers 1 to 9 to represent the different schoolgirls.

Day 1	Day 2	Day 3	Day 4
1 2 3	1 4 7	1 4 9	1 6 8
4 5 6	2 5 8	2 6 7	2 4 9
7 8 9	3 6 9	3 5 8	3 5 7

31. 24 zeros; Determine how many factors of 5 there are in 100!. Each multiple of 5 has one factor of 5 except 25, 50, 75, and 100, which have two factors of 5. So, there are 24 factors of 5 in 100!. There are more than enough factors of 2 to match up with the 5s to make factors of 10, so there are 24 zeros.

32. a) EDACB or BCADE **b)** 2
c) None. Since F only knows A, then F must stand next to A. However, in both arrangements from part a), A must stand between C and D, but F does not know either C or D and therefore cannot stand next to either of them. Therefore, no possible arrangement satisfies the conditions.

C1 a) ${}_aP_b = \dfrac{a!}{(a-b)!}$ is the formula for calculating the number of ways that b objects can be selected from a group of a objects, if order is important; for example, if you have a group of 20 students and you want to choose a team of 3 arranged from tallest to shortest.
b) $b \le a$

C2 By the fundamental counting principle, if the n objects are distinct, they can be arranged in $n!$ ways. However, if a of the objects are the same and the remaining b objects are the same, then the number of different arrangements is reduced to $\dfrac{n!}{a!b!}$ to eliminate duplicates.

C3 a) $\dfrac{(n+2)(n+1)n}{4}$ **b)** $\dfrac{7+20r}{r(r+1)}$

C5 a) 362 880 **b)** 5.559 763… **c)** 6.559 763
d) Example: The answer to part c) is 1 more than the answer to part b). This is because $10! = 10(9!)$ and $\log 10! = \log 10 + \log 9! = 1 + \log 9!$.

11.2 Combinations, pages 534 to 536

1. a) Combination, because the order that you shake hands is not important.
b) Permutation, because the order of digits is important.
c) Combination, since the order that the cars are purchased is not important.
d) Combination, because the order that players are selected to ride in the van is not important.

2. ${}_5P_3$ is a permutation representing the number of ways of arranging 3 objects taken from a group of 5 objects. ${}_5C_3$ is a combination representing the number of ways of choosing any 3 objects from a group of 5 objects. ${}_5P_3 = 60$ and ${}_5C_3 = 10$.

3. a) ${}_6P_4 = 360$ **b)** ${}_7C_3 = 35$
c) ${}_5C_2 = 10$ **d)** ${}_{10}C_7 = 120$

4. a) 210 **b)** 5040

5. a) AB, AC, AD, BC, BD, CD
b) AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC
c) The number of permutations is 2! times the number of combinations.

6. a) $n = 10$ **b)** $n = 7$ **c)** $n = 4$ **d)** $n = 5$

7. a) Case 1: one-digit numbers, Case 2: two-digit numbers, Case 3: three-digit numbers
b) Cases of grouping the 4 members of the 5-member team from either grade: Case 1: four grade 12s, Case 2: three grade 12s and one grade 11, Case 3: two grade 12s and two grade 11s, Case 4: one grade 12 and three grade 11s, Case 5: four grade 11s

8. Left Side $= {}_{11}C_3$
$= \dfrac{11!}{(11-3)!3!}$
$= \dfrac{11!}{8!3!}$

Right Side $= {}_{11}C_8$
$= \dfrac{11!}{(11-8)!8!}$
$= \dfrac{11!}{3!8!}$

Left Side = Right Side

9. a) ${}_5C_5 = 1$
b) ${}_5C_0 = 1$; there is only one way to choose 5 objects from a group of 5 objects and only one way to choose 0 objects from a group of 5 objects.

10. a) 4 **b)** 10

11. a) 15 **b)** 22

12. Left Side

$$= {}_nC_{r-1} + {}_nC_r$$
$$= \frac{n!}{(n-(r-1))!(r-1)!} + \frac{n!}{(n-r)!r!}$$
$$= \frac{n!}{(n-r+1)!(r-1)!} + \frac{n!}{(n-r)!r!}$$
$$= \frac{[n!(n-r)!r!] + [n!(n-r+1)!(r-1)!]}{(n-r+1)!(r-1)!(n-r)!r!}$$
$$= \frac{n!(n-r)!r(r-1)! + n!(n-r+1)(n-r)!(r-1)!}{(n-r+1)!(r-1)!(n-r)!r!}$$
$$= \frac{n!(n-r)!(r-1)![r + (n-r+1)]}{(n-r+1)!(r-1)!(n-r)!r!}$$
$$= \frac{n!\cancelto{1}{(n-r)!}\cancelto{1}{(r-1)!}(n+1)}{(n-r+1)\cancelto{1}{(r-1)!}\cancelto{1}{(n-r)!}r!}$$
$$= \frac{n!(n+1)}{(n-r+1)!r!}$$
$$= \frac{(n+1)!}{(n-r+1)!r!}$$

Right Side $= {}_{n+1}C_r$
$$= \frac{(n+1)!}{(n+1-r)!r!}$$

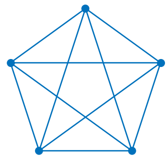
Left Side = Right Side

13. 20 different burgers; this is a combination because the order the ingredients is put on the burger is not important.

14. a) 210

b) combination, because the order of toppings on a pizza is not important

15. a) Method 1: Use a diagram.
Method 2: Use combinations.
$_5C_2 = 10$, the same as the number of combinations of 5 people shaking hands.



b) 10

c) The number of triangles is given by $_{10}C_3 = \dfrac{10!}{(10-3)!3!} = \dfrac{10!}{7!3!}$. The number of lines is given by $_{10}C_2 = \dfrac{10!}{(10-2)!2!} = \dfrac{10!}{8!2!}$. The number of triangles is determined by the number of selections with choosing 3 points from 10 non-collinear points, whereas the number of lines is determined by the number of selections with choosing 2 points from the 10 non-collinear points.

16. Left Side $= {}_nC_r$

$$= \frac{n!}{(n-r)!r!}$$

Right Side $= {}_nC_{n-r}$

$$= \frac{n!}{(n-(n-r))!(n-r)!}$$
$$= \frac{n!}{(n-n+r)!(n-r)!}$$
$$= \frac{n!}{r!(n-r)!}$$

Left Side = Right Side

17. a) 125 970 **b)** 44 352 **c)** 1945

18. a) 2 598 960 **b)** 211 926 **c)** 388 700

19. a) 525 **b)** 576

20. a) $\dfrac{40!}{20!20!}$ **b)** 116 280

21. a) $\dfrac{52!}{39!13!} \times \dfrac{39!}{26!13!} \times \dfrac{26!}{13!13!} \times \dfrac{13!}{0!13!}$

b) $\dfrac{52!}{13!13!13!13!} = \dfrac{52!}{(13!)^4}$ **c)** 5.364×10^{28}

22. 90

23. a) 36 **b)** 1296

24. a) $_5C_2 = 10$, $10 \div 3 = 3$ Remainder 1. $_{15}C_6 = 5005$, and $5005 \div 3 = 1668$ Remainder 1.

b) yes, remainder 3 **c)** 7; 0, 1, 2, 3, 4, 5, 6

d) Example: First, I would try a few more cases to try to find a counterexample. Since the statement seems to be true, I would write a computer program to test many cases in an organized way.

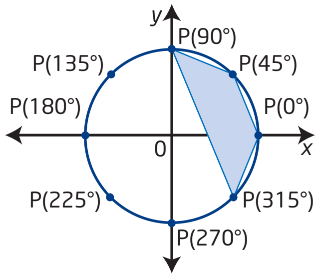
C1 No. The order of the numbers matters, so a combination lock would be better called a permutations lock.

C2 a) $_aC_b = \dfrac{a!}{(a-b)!b!}$ is the formula for calculating the number of ways that b objects can be selected from a group of a objects, if order is not important; for example, if you have a group of 20 students and you want to choose a team of any 3 people.

b) $a \geq b$ **c)** $b \geq 0$

C3 Example: Assuming that the rooms are the same and so any patient can be assigned to any of the six rooms, this is a combinations situation. Beth is correct.

C4 Step 1: Example:



Step 2: Number of each type of quadrilateral:
Squares: 2
Rectangles: 4
Parallelograms: 0
Isosceles trapezoids: 16

Step 3: Example: In the case drawn in Step 1, because of the symmetry of the given points on the unit circle, many of the possible quadrilaterals are the same. In general, there will be $_8C_4$ or 70 possible quadrilaterals.

11.3 The Binomial Theorem, pages 542 to 545

1. a) 1 4 6 4 1 **b)** 1 8 28 56 70 56 28 8 1

c) 1 11 55 165 330 462 462 330 165 55 11 1

2. a) $_2C_0$ $_2C_1$ $_2C_2$ **b)** $_4C_0$ $_4C_1$ $_4C_2$ $_4C_3$ $_4C_4$

c) $_7C_0$ $_7C_1$ $_7C_2$ $_7C_3$ $_7C_4$ $_7C_5$ $_7C_6$ $_7C_7$

3. a) $\dfrac{3!}{2!1!}$ **b)** $\dfrac{6!}{3!3!}$ **c)** $\dfrac{1!}{0!1!}$

4. a) 5 **b)** 8 **c)** $q + 1$

5. a) $1x^2 + 2xy + 1y^2$ **b)** $1a^3 + 3a^2 + 3a + 1$

c) $1 - 4p + 6p^2 - 4p^3 + 1p^4$

6. a) $1a^3 + 9a^2b + 27ab^2 + 27b^3$

b) $243a^5 - 810a^4b + 1080a^3b^2 - 720a^2b^2 + 240ab^4 - 32b^5$

c) $16x^4 - 160x^3 + 600x^2 - 1000x + 625$

7. a) $126a^4b^5$ **b)** $-540x^3y^3$ **c)** $192\ 192t^6$

d) $96x^2y^2$ **e)** $3072w^2$

8. All outside numbers of Pascal's triangle are 1's; the middle values are determined by adding the two numbers to the left and right in the row above.

9. a) 1, 2, 4, 8, 16

b) 2^8 or 256

c) 2^{n-1}, where n is the row number

10. a) The sum of the numbers on the handle equals the number on the blade of each hockey stick.

b) No; the hockey stick handle must begin with 1 from the outside of the triangle and move diagonally down the triangle with each value being in a different row. The number of the blade must be diagonally below the last number on the handle of the hockey stick.

11. a) 13 **b)** $220x^9y^3$ **c)** $r = 6$, $_{12}C_6 = 924$

12. a) $(x + y)^4$ **b)** $(1 - y)^5$

13. a) No. While $11^0 = 1$, $11^1 = 11$, $11^2 = 121$, $11^3 = 1331$, and $11^4 = 14\ 641$, this pattern only works for the first five rows of Pascal's triangle.

b) m represents the row number minus 1, $m \leq 4$.

14. a) $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$, $(x - y)^3 = x^3 - 3x^2y + 3xy^2 - y^3$; the signs for the second and fourth terms are negative in the expansion of $(x - y)^3$

b) $(x + y)^3 + (x - y)^3$
$= x^3 + 3x^2y + 3xy^2 + y^3 + x^3 - 3x^2y + 3xy^2 - y^3$
$= 2x^3 + 6xy^2$
$= 2x(x^2 + 3y^2)$

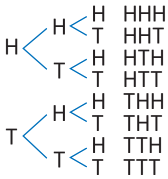
c) $2y(3x^2 + y^2)$; the expansion of $(x + y)^3 - (x - y)^3$ has coefficients for x^2 and y^2 that are reversed from the expansion of $(x + y)^3 + (x - y)^3$, as well as the common factors $2x$ and $2y$ being reversed.

15. a) Case 1: no one attends, case 2: one person attends, case 3: two people attend, case 4: three people attend, case 5: four people attend, case 6: all five people attend

b) 32 or 2^5

c) The answer is the sum of the terms of the sixth row of Pascal's triangle.

16. a)



b) HHH + HHT + HTH + HTT + THH + THT + TTH + TTT
$= H^3 + 3H^2T + 3HT^2 + T^3$

c) H^3 represents the first term of the expansion of $(H + T)^3$ and $3H^2T$ represents the second term of the expansion of $(H + T)^3$.

17. a) $\frac{a^3}{b^3} + 6\left(\frac{a^2}{b^2}\right) + 12\left(\frac{a}{b}\right) + 8$ or $\frac{a^3}{b^3} + \frac{6a^2}{b^2} + \frac{12a}{b} + 8$

b) $\frac{a^4}{b^4} - 4\left(\frac{a^4}{b^3}\right) + 6\left(\frac{a^4}{b^2}\right) - 4\left(\frac{a^4}{b}\right) + a^4$
$= a^4\left(\frac{1}{b^4} - \frac{4}{b^3} + \frac{6}{b^2} - \frac{4}{b} + 1\right)$

c) $1 - 3x + \frac{15}{4}x^2 - \frac{5}{2}x^3 + \frac{15}{16}x^4 - \frac{3}{16}x^5 + \frac{1}{64}x^6$

d) $16x^8 - 32x^5 + 24x^2 - 8x^{-1} + x^{-4}$

18. a) $5670a^4b^{12}$ **b)** the fourth term; it is $-120x^{11}$

19. a) 126 720 **b)** the fifth term; its value is 495

20. $m = 3y$

21. Examples:
Step 1: The numerators start with the second value, 4, and decrease by ones, while the denominators start at 1 and increase by ones to 4.
For the sixth row:
$1 \times 5 = 5$, $5 \times \frac{4}{2} = 10$, $10 \times \frac{3}{3} = 10$, $10 \times \frac{2}{4} = 5$, $5 \times \frac{1}{5} = 1$.
Step 2: The second element in the row is equal to the row number minus 1.
Step 3: $\times \frac{20}{1}$; $\times \frac{19}{2}$, $\times \frac{18}{3}$, and so on to $\times \frac{3}{18}$, $\times \frac{2}{19}$, $\times \frac{1}{20}$

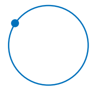
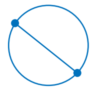
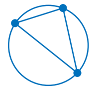
22. a) Each entry is the sum of the two values directly below it.

b) $\frac{1}{6}$ $\frac{1}{30}$ $\frac{1}{60}$ $\frac{1}{60}$ $\frac{1}{30}$ $\frac{1}{6}$
$\frac{1}{7}$ $\frac{1}{42}$ $\frac{1}{105}$ $\frac{1}{140}$ $\frac{1}{105}$ $\frac{1}{42}$ $\frac{1}{7}$

c) Examples: Outside values are the reciprocal of the row number. The product of two consecutive outside row values gives the value of the second term in the lower row.

23. Consider $a + b = x$ and $c = y$, and substitute in $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + x^3$.
$(a + b + c)^3$
$= (a + b)^3 + 3(a + b)^2c + 3(a + b)c^2 + c^3$
$= a^3 + 3a^2b + 3ab^2 + b^3 + 3(a^2 + 2ab + b^2)c + 3ac^2 + 3bc^2 + c^3$
$= a^3 + 3a^2b + 3ab^2 + b^3 + 3a^2c + 6abc + 3b^2c + 3ac^2 + 3bc^2 + c^3$

24. a)

Diagram	Points	Line Segments	Triangles	Quadrilaterals	Pentagons	Hexagons
	1					
	2	1				
	3	3	1			
	4	6	4	1		
	5	10	10	5	1	
	6	15	20	15	6	1

b) The numbers are values from row 1 to row 6 of Pascal's triangle with the exception of the first term.

c) The numbers will be values from the 8th row of Pascal's triangle with the exception of the first term: 8, 8 28 56 70 56 28 8 1.

25. a) 2.7083…

b) The value of e becomes more precise for the 7th and 8th terms. The more terms used, the more accurate the approximation.

c) 2.718 281 828

d) $15! = \left(\frac{15}{e}\right)^{15}\sqrt{2\pi(15)} \approx 1.300 \times 10^{12}$;
on a calculator $15! \approx 1.3077 \times 10^{12}$

e) Using the formula from part d),
$50! = \left(\frac{50}{e}\right)^{50}\sqrt{2\pi(50)}$
$\approx 3.036\ 344\ 594 \times 10^{64}$;
using the formula from part e),
$50! = \left(\frac{50}{e}\right)^{50}\sqrt{2\pi(50)}\left(1 + \frac{1}{12(50)}\right)$
$\approx 3.041\ 405\ 168 \times 10^{64}$; using a calculator $50! = 3.041\ 409\ 32 \times 10^{64}$, so the formula in part e) seems to give a more accurate approximation.

C1 The coefficients of the terms in the expansion of $(x + y)^n$ are the same as the numbers in row $n + 1$ of Pascal's triangle. Examples: $(x + y)^2 = x^2 + 2xy + y^2$ and row 3 of Pascal's triangle is 1 2 1; $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$ and row 4 of Pascal's triangle is 1 3 3 1.

C2 Examples:

a) Permutation: In how many different ways can four different chocolate bars be given to two people? Combination: Steve has two Canadian pennies and two U.S. pennies in his pocket. In how many different ways can he draw out two coins? Binomial expansion: What is the coefficient of the middle term in the expansion of $(a + b)^4$?

b) All three problems have the same answer, 6, but they answer different questions.

C3 Examples:

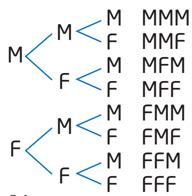
a) For small values of n, it is easier to use Pascal's triangle, but for large values of n it is easier to use combinations to determine the coefficients in the expansion of $(a + b)^n$.

b) If you have a large version of Pascal's triangle available, then that will immediately give a correct coefficient. If you have to work from scratch, both methods can be error prone.

C4 Answers will vary.

Chapter 11 Review, pages 546 to 547

1. a)



b) 3

2. a) 81 **b)** 32

3. a) 24 **b)** 360 **c)** 60 **d)** $\frac{12!}{2!3!2!}$

4. a) 48 **b)** 24 **c)** 72

5. a) 5040 **b)** 288 **c)** 144

6. a) 1 160 016 **b)** $8.513\,718\,8 \times 10^{11}$

c) about 270 000 years

7. a) $\frac{n+1}{n-1}$ **b)** $\frac{x^2 + x + 1}{x}$

8. a) 210 **b)** 63

9. a) 120 **b)** 5040 **c)** 200 **d)** 163 800

10. a) 15

b) amounts all in cents: 1, 5, 10, 25, 6, 11, 26, 15, 30, 35, 16, 31, 36, 40, 41

11. a) $n = 8$, ${}_8C_2 = 28$

b) $n = 26$, ${}_{26}C_3 = 2600$ and $4({}_{26}P_2) = 4(650) = 2600$

12. 2520

13. a) Permutation: How many arrangements of the letters AAABB are possible?
Combination: How many ways can you choose 3 students from a group of 5?

b) Yes, ${}_5C_2 = \frac{5!}{(5-2)!2!} = \frac{5!}{3!2!}$ and ${}_5C_3 = \frac{5!}{(5-3)!3!} = \frac{5!}{2!3!}$.

14. a) 1 3 3 1

b) 1 9 36 84 126 126 84 36 9 1

15. Examples: Multiplication: expand, collect like terms, and write the answer in descending order of the exponent of x.

$$(x + y)^3 = (x + y)(x + y)(x + y)$$
$$= x^3 + 3x^2y + 3xy^2 + y^3$$

Pascal's triangle: Coefficients are the terms from row $n + 1$ of Pascal's triangle. For $(x + y)^3$, row 4 is 1 3 3 1.

Combination: coefficients correspond to the combinations as shown:

$(x + y)^3 = {}_3C_0x^3y^0 + {}_3C_1x^2y^1 + {}_3C_2x^1y^2 + {}_3C_3x^0y^3$

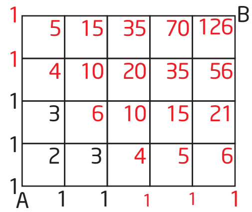
16. a) $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$

b) $x^3 - 9x^2 + 27x - 27$

c) $16x^8 - 32x^4 + 24 - \frac{8}{x^4} + \frac{1}{x^8}$

17. a) $36a^7b^2$ **b)** $-192xy^5$ **c)** $-160x^3$

18. a)



b) Pascal's triangle values are shown with the top of the triangle at point A and the rows appearing up and right of point A.

c) 126

d) There are 4 identical moves up and 5 identical moves right, so the number of possible pathways is $\frac{9!}{4!5!} = 126$.

19. a) 45 moves

b) 2 counters: 1 move; 3 counters: $1 + 2 = 3$ moves; 4 counters: $1 + 2 + 3 = 6$ moves; and so on up to 12 counters: $1 + 2 + 3 + \cdots + 10 + 11 = 66$ moves

c) 300 moves

Chapter 11 Practice Test, page 548

1. C **2.** D **3.** C **4.** B **5.** A **6.** C

7. a) 180

b) AACBDB, ABCADB, ABCBDA, BACBDA, BACADB, BBCADA

8. No, n must be a whole number, so n cannot equal -8.

9. a) 10 **b)** $\frac{5!}{2!3!}\left(\frac{4!}{2!2!}\right) = 60$

10. 69

11. Permutations determine the number of arrangements of n items chosen r at a time, when order is important. For example, the number of arrangements of 5 people chosen 2 at a time to ride on a motorcycle is ${}_5P_2 = 20$. A combination determines the number of different selections of n objects chosen r at a time when order is not important. For example, the number of selections of 5 objects chosen 2 at a time, when order is not important, is ${}_5C_2 = 10$.

12. $672x^9$

13. a) 420 **b)** 120

14. a) $n = 6$ **b)** $n = 9$

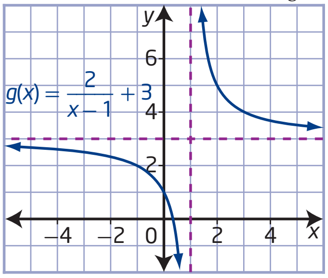
15. $y^5 - 10y^2 + 40y^{-1} - 80y^{-4} + 80y^{-7} - 32y^{-10}$

16. a) 24 **b)** 48 **c)** 18

Cumulative Review, Chapters 9–11, pages 550 to 551

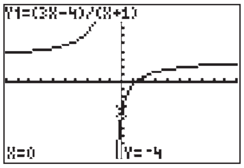
1. a) a vertical stretch by a factor of 2 about the x-axis and a translation of 1 unit right and 3 units up

b)



c) domain $\{x \mid x \neq 1, x \in \mathrm{R}\}$, range $\{y \mid y \neq 3, y \in \mathrm{R}\}$, x-intercept $\frac{1}{3}$, y-intercept 1, horizontal asymptote $y = 3$, vertical asymptote $x = 1$

2. a)



b) domain $\{x \mid x \neq -1, x \in \mathrm{R}\}$, range $\{y \mid y \neq 3, y \in \mathrm{R}\}$, x-intercept $\frac{4}{3}$, y-intercept -4, horizontal asymptote $y = 3$, vertical asymptote $x = -1$

3. a) The graph of $y = \frac{x^2 - 3x}{x^2 - 9}$ has a vertical asymptote at $x = -3$, a point of discontinuity at (3, 0.5), and an x-intercept of 0; C.

b) The graph of $y = \frac{x^2 - 1}{x + 1}$ has no vertical asymptote, a point of discontinuity at $(-1, -2)$, and an x-intercept of 1; A.

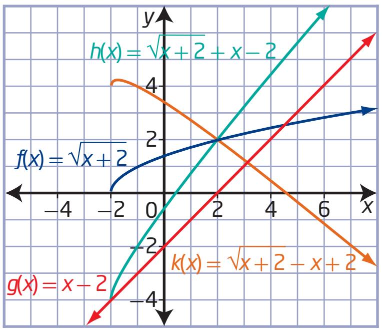
c) The graph of $y = \frac{x^2 + 4x + 3}{x^2 + 1}$ has no vertical asymptote, no point of discontinuity, and x-intercepts of -3 and -1; B.

4. a) 2 **b)** $-1, 9$ **c)** 0

5. a) $-0.71, 0.71$ **b)** 0.15, 5.52

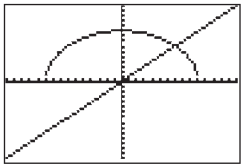
6. a) $h(x) = \sqrt{x+2} + x - 2$, $k(x) = \sqrt{x+2} - x + 2$

b)



c) $f(x)$: domain $\{x \mid x \geq -2, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$
$g(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
$h(x)$: domain $\{x \mid x \geq -2, x \in \mathrm{R}\}$, range $\{y \mid y \geq -4, y \in \mathrm{R}\}$
$k(x)$: domain $\{x \mid x \geq -2, x \in \mathrm{R}\}$, range $\{y \mid y \leq 4.25, y \in \mathrm{R}\}$

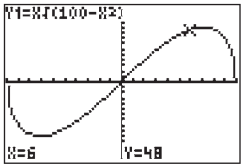
7. a)



$f(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
$g(x)$: domain $\{x \mid -10 \leq x \leq 10, x \in \mathrm{R}\}$, range $\{y \mid 0 \leq y \leq 10, y \in \mathrm{R}\}$

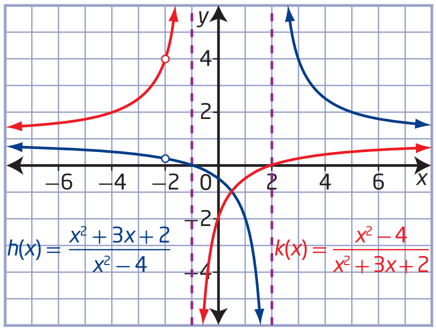
b) $h(x) = x\sqrt{100 - x^2}$

c)



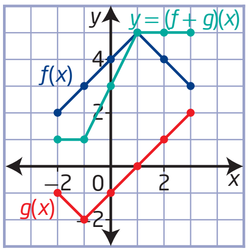
domain $\{x \mid -10 \leq x \leq 10, x \in \mathrm{R}\}$, range $\{y \mid -50 \leq y \leq 50, y \in \mathrm{R}\}$

8. a) $h(x) = \frac{x+1}{x-2}$, $x \neq -2, 2$; $k(x) = \frac{x-2}{x+1}$, $x \neq -2, -1$



b) The two functions have different domains but the same range; $h(x)$: domain $\{x \mid x \neq -2, 2, x \in \mathrm{R}\}$, range $\{y \mid y \neq 1, y \in \mathrm{R}\}$, $k(x)$: domain $\{x \mid x \neq -2, -1, x \in \mathrm{R}\}$, range $\{y \mid y \neq 1, y \in \mathrm{R}\}$

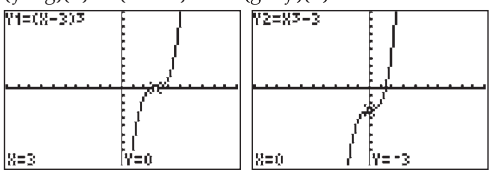
9.



10. a) -3 **b)** $\frac{1}{2}$

11. a) $(f \circ g)(x) = (x - 3)^3$ and $(g \circ f)(x) = x^3 - 3$

b)



c) The graph of $(f \circ g)(x) = (x - 3)^3$ is a translation of 3 units right of the graph of $f(x)$. The graph of $(g \circ f)(x) = x^3 - 3$ is a translation of 3 units down of the graph of $f(x)$.

12. a) $f(g(x)) = x$; domain $\{x \mid x \in \mathrm{R}\}$

b) $g(f(x)) = \csc x$; domain $\{x \mid x \neq \pi n, n \in \mathrm{I}, x \in \mathrm{R}\}$

c) $f(g(x)) = \frac{1}{x^2 - 1}$; domain $\{x \mid x \neq \pm 1, x \in \mathrm{R}\}$

13. 96 meals

14. 480 ways

15. 55

16. 525 ways

17. a) 103 680 **b)** 725 760

18. a) 3 **b)** 6 **c)** 5

19. Examples: Pascal's triangle:
$(x + y)^4 = 1x^4y^0 + 4x^3y^1 + 6x^2y^2 + 4x^1y^3 + 1x^0y^4$; the coefficients are values from the fifth row of Pascal's triangle.
$(x + y)^6 = 1x^6y^0 + 6x^5y^1 + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6x^1y^5 + 1x^0y^6$; the coefficients are values from the seventh row of Pascal's triangle.
Combinations: $(x + y)^4 = {}_4C_0x^4y^0 + {}_4C_1x^3y^1 + {}_4C_2x^2y^2 + {}_4C_3x^1y^3 + {}_4C_4x^0y^4$; the coefficients ${}_4C_0$, ${}_4C_1$, ${}_4C_2$, ${}_4C_3$, ${}_4C_4$ have the same values as in the fifth row of Pascal's triangle.
$(x + y)^6 = {}_6C_0x^6y^0 + {}_6C_1x^5y^1 + {}_6C_2x^4y^2 + {}_6C_3x^3y^3 + {}_6C_4x^2y^4 + {}_6C_5x^1y^5 + {}_6C_6x^0y^6$; the coefficients ${}_6C_0$, ${}_6C_1$, ${}_6C_2$, ${}_6C_3$, ${}_6C_4$, ${}_6C_5$, ${}_6C_6$ have the same values as the seventh row of Pascal's triangle.

20. a) $81x^4 - 540x^3 + 1350x^2 - 1500x + 625$
b) $\frac{1}{x^5} - \frac{10}{x^3} + \frac{40}{x} - 80x + 80x^3 - 32x^5$

21. a) 250 **b)** -56

22. a) ${}_{25}C_4$ **b)** 26 **c)** ${}_{25}C_3 = {}_{24}C_2 + {}_{24}C_3$

Unit 4 Test, pages 552 to 553

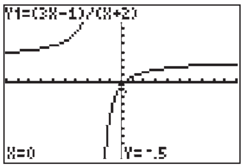
1. D **2.** B **3.** A **4.** B **5.** B **6.** D **7.** C

8. $\left(3, \frac{1}{7}\right)$

9. 0, 3.73, 0.27 **10.** $600x^2y^4$ **11.** -1

12. a) vertical stretch by a factor of 2 and translation of 1 unit left and 3 units down
b) $x = -1$ and $y = -3$
c) as x approaches -1, $|y|$ becomes very large

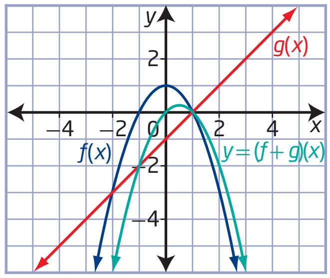
13. a)



b) domain $\{x \mid x \neq -2, x \in R\}$, range $\{y \mid y \neq 3, x \in R\}$, x-intercept $\frac{1}{3}$, y-intercept $-\frac{1}{2}$
c) $x = \frac{1}{3}$
d) The x-intercept of the graph of the function $y = \frac{3x - 1}{x + 2}$ is the root of the equation $0 = \frac{3x - 1}{x + 2}$.

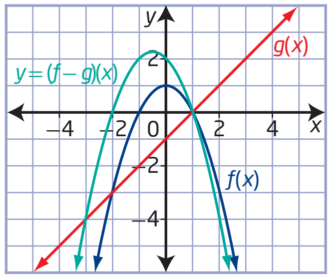
14. a) The graph of $f(x) = \frac{x - 4}{(x + 2)(x - 4)}$ has a vertical asymptote at $x = -2$, a point of discontinuity at $\left(4, \frac{1}{6}\right)$, y-intercept of 0.5, and no x-intercept.
b) The graph of $f(x) = \frac{(x + 3)(x - 2)}{(x + 3)(x - 1)}$ has a vertical asymptote at $x = 1$, a point of discontinuity at $(-3, 1.25)$, y-intercept of 2, and an x-intercept of 2.
c) The graph of $f(x) = \frac{x(x - 5)}{(x - 3)(x + 1)}$ has vertical asymptotes at $x = -1$ and $x = 3$, no points of discontinuity, y-intercept of 0, and x-intercepts of 0 and 5.

15. a)



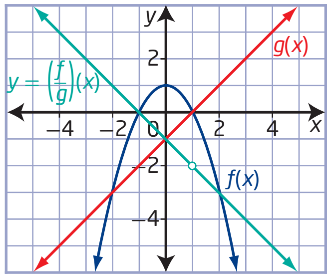
domain $\{x \mid x \in R\}$, range $\{y \mid y \leq 0.25, y \in R\}$

b)



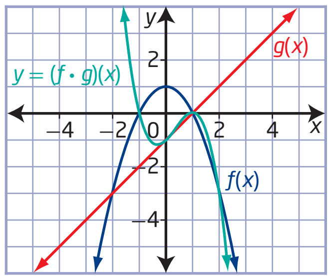
domain $\{x \mid x \in R\}$, range $\{y \mid y \leq 2.25, y \in R\}$

c)



domain $\{x \mid x \neq 1, x \in R\}$, range $\{y \mid y \neq -2, y \in R\}$

d)



domain $\{x \mid x \in R\}$, range $\{y \mid y \in R\}$

16. a) $h(x) = x - 3 + \sqrt{x - 1}$; $x \geq 1$
b) $h(x) = x - 3 - \sqrt{x - 1}$; $x \geq 1$
c) $h(x) = \frac{x - 3}{\sqrt{x - 1}}$; $x > 1$
d) $h(x) = (x - 3)\sqrt{x - 1}$; $x \geq 1$

17. a) 1 **b)** 1
c) $f(g(x)) = x^2 - 3$ **d)** $g(f(x)) = |x^2 - 3|$

18. a) $f(x) = 2^x$ and $g(x) = 3x + 2$
b) $f(x) = \sqrt{x}$ and $g(x) = \sin x + 2$

19. a) 21 **b)** 13 **c)** 10

20. a) 24 **b)** 232 more
c) There are fewer ways. Because the letter C is repeated, half of the arrangements will be repeats.

21. a) 60 **b)** 81

22. 4, -4