Function Transformations

***Pre-Calculus 12*, pages 4–5**

Suggested Timing
45–60 min

Blackline Masters
BLM 1–1 Chapter 1 Prerequisite Skills
BLM U1–1 Unit 1 Project Checklist

Planning Notes

Read the Key Terms with students. Ask them which terms they are already able to define and have them give an example to illustrate their understanding. Then have a discussion about what they know about transformations.

In section 1.1, students examine the graphs of functions and tables of values to determine the effects of moving functions left, right, up, and down. They also learn how to analyse a graph and its transformed graph, and determine the equation based on the translation that was applied. In section 1.2, students look at reflections and stretches. They consider functions that have been reflected in the *x*-axis and *y*-axis. They determine the graph of a function given the equation of a transformed function, or find the equation of a transformed function given its graph. In section 1.3, students explore combined transformations. They consider the equations and graphs of these transformations, and the order in which transformations must occur. Finally, in section 1.4, students look at inverses of relations.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter. Encourage students to use a summary method of their choice.

Unit Project

In the Project Corner at the end of section 1.3, students consider functions identified in a rural image and identify the types of transformations they see. Consider working as a class, or have students discuss the image in small groups.

Meeting Student Needs

- Consider having students complete the questions on **BLM 1–1 Chapter 1 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out to students **BLM U1–1 Unit 1 Project Checklist**, which provides a list of all the requirements for the Unit 1 project.
- Provide students with a checklist containing the learning outcomes for this unit. Discuss specific terms to ensure students understand what they will learn.
- To help students prepare for the chapter, consider preparing worksheets of quick activities involving concepts of translations, reflections, and rotations from previous grades.
- Students can discuss patterns they see in the opening photos (temple near Halebid Kamat, India, Montreal building, tiger moth, and Navajo blanket).

Enrichment

Encourage students to collect examples from nature that reflect mathematical shapes that they encounter in the student resource. Ask them to look for designs that are repeated, reflected, stretched, or transformed. Suggest that they visually match portions of the shapes with functions they have seen in their studies.

Gifted

Einstein's work on the photoelectric effect and light travelling in discrete particles rather than waves is mathematical in nature. Ask students to research the concept of waves versus particles. They could also speculate about the mathematics of modelling pure energy that travels in waves and/or particles. Ask, "How might formula creation be affected by the fact that waves of light have no mass, but particles appear to have mass by definition?"

Career Link

Ask students what they know about physics. Ask
- Have you studied physics? What do you study in your physics courses?
- How do you think the study of physics affects our daily lives?
- How do you think physics might affect life in the future?
- What are some professions that make use of physics?

Suggest that students go online to research the historical development of physics, the current areas of development, and how the future might be shaped by the work physicists do.

1.1 Horizontal and Vertical Translations

***Pre-Calculus 12*, pages 6–15**

Suggested Timing

60–90 min

Materials

- grid paper
- ruler

Blackline Masters

Master 3 Centimetre Grid Paper

BLM 1–2 Section 1.1 Extra Practice

Mathematical Processes for Specific Outcomes

RF2 Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

☑ Communication (C)

☑ Connections (CN)

☑ Reasoning (R)

☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–3, 5, 6, 8, 10–12
Typical	#5, 7–12, 13 or 14, C1, C2, C4
Extension/Enrichment	#15–19, C2–C4

Planning Notes

Before beginning this section, consider helping students reactivate their knowledge of evaluating expressions. You may also want to revisit the process of creating a table of values, substituting values for the input variable to see the resulting output value. Students are required to create a table of values in the Investigate.

Investigate Vertical and Horizontal Translations

For this investigation, students could work individually or in pairs, drawing the graphs on grid paper. Alternatively, you could have students set up a coordinate axis on the classroom floor. Then one student stand on the grid, while other students direct him/her to move according to the translation rules being explored.

Help students recognize the patterns of the transformations in the Investigate. When students add and subtract 3 from the function, ask

- Which points change?
- Why does only the y-coordinate change?
- What important parts of the graph are affected by this transformation? Does either intercept change? Does the vertex change?

Then ask students to consider what happens when 3 is added to and subtracted from the variable. Ask

- How is the transformation different from the one in the first part of the investigation?
- Why do you think that only the x-coordinate changes?
- What important parts of the graph are affected by this transformation? Does either intercept change? Does the vertex change?

Meeting Student Needs

- Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within the section.
- You may suggest that students explore the work of artists who use transformations in their works. (e.g., M. C. Escher)
- Display various floor tiles, pieces of linoleum, and wall paper samples in the classroom as examples of transformations.
- Some students will benefit from having the graphs prepared prior to the investigation. They can then focus on the ideas conveyed in each section of the Investigate.
- Discuss the results of the Investigate as a class.
- Some students may prefer to write an Exit Slip describing the Key Ideas learned from the investigation, while others may want to create a poster listing the key ideas, including examples and figures.

Enrichment

Drag racers can accelerate from a standing start to over 330 mph (531 km/h) over a quarter mile (0.4 km). Ask students to estimate speeds along the course of the racetrack, and then create a graph of velocity versus time. Ask them to sketch how the graph would change if the final speed was increased by the addition of power or the reduction of friction. What mathematical terms describe the changes?

Gifted

Challenge students to look for and identify three frieze patterns in nature or art. If possible, illustrate the patterns using graphic overlays on the images. Some students may wish to create video overlays, with the graphs of functions being repeated over the patterns on the images.

Common Errors

- When the graph of x^2 is translated to become $f(x) = (x - 3)^2$, some students think that the graph moves to the left.

Rx Suggest that students think about the point $x = 0$. The value of x has to be 3 for $f(x - 3)$ to equal $f(0)$. Since x has to be $+3$, this point on the graph moves three units right. Likewise, the graph of each value of x in $f(x - 3)$ is three units to the right compared to values of x in $f(x)$. Use a table of values to support this exploration.

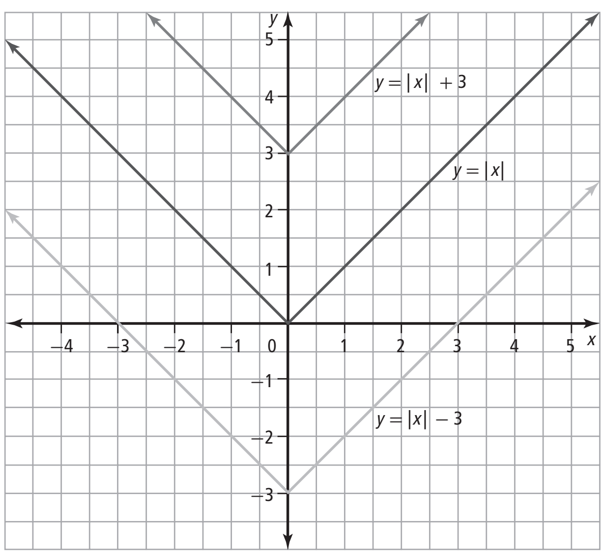
Answers

Investigate Vertical and Horizontal Translations

1. a)

x	$f(x)$	$f(x) + 3$	$f(x) - 3$
−3	3	6	0
−2	2	5	−1
−1	1	4	−2
0	0	3	−3
1	1	4	−2
2	2	5	−1
3	3	6	0

b)



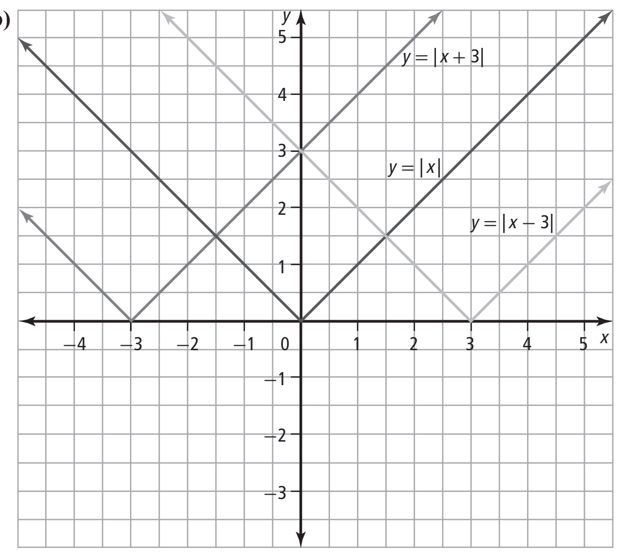
2. a) The graphs are translated vertically by $+3$ and -3 respectively.
b) The variable k indicates the height of the translated graph in relation to the original.

3. No. The k variable affects the vertical translation of the graph, regardless of the value of $f(x)$.

4. a)

x	$f(x)$	$f(x + 3)$	$f(x - 3)$
−9	9	6	12
−6	6	3	9
−3	3	0	6
0	0	3	3
3	3	6	0
6	6	9	3
9	9	12	6

b)



5. a) The functions are all horizontal translations of the original function.
b) The h variable indicates the distance the function is translated right or left.

6. No. The h variable affects the x-coordinate before the function is applied.

7. When $k > 0$, the transformed function shifts upward. When $k < 0$, the transformed function shifts downward.

8. When $h < 0$, the graph of the function shifts left $|h|$ units. When $h > 0$, the graph is transformed h units right.

9. The variables h and k do not affect the shape of the function; they only affect the position of the graph in relation to the y-axis and x-axis. These variables do not affect the orientation except for the position. The variable h has a direct effect on the x-intercepts in that they will be translated by h units. The same is true for k and the y-intercepts. Although k has an effect on the x-intercepts, it is impossible to determine the exact effect unless the function is known. The same is true for h and the y-intercepts. Since h translates the function left and right, the domain is also changed in this way. The same is true for k and the range.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • Encourage students to respond for both positive and negative values of k and h. • You may wish to orally review the responses as a class. It is important to determine if students understand if the parameters affect the shape of the graph. Ask students to describe whether the graph of a function will change shape as a result of a transformation.

Example 1

For many students, the direction of a horizontal translation moves opposite to what they expect. For example, the graph of $y = (x - 5)^2$ translates the graph of $f(x) = x^2$ five units to the right instead to the left. To help students understand why this is, ask them to compare the inputs and outputs of the first and second table. How are they the same? different? Then ask students to compare the inputs and outputs of the first and third tables. Ask, "What do you have to do to the x-values in table 1 to make them match the values of the x-coordinates in table 3?"

Make the same point algebraically. Remind students that for the horizontal translation, the y-coordinates must be the same for both functions. Given $f(x) = x^2$, illustrate how the point (x, y) maps onto the point $(x + h, y)$ for the graph of $g(x) = (x - 5)^2$:

$$\begin{aligned} f(x) &= x^2 \\ f(2) &= (2)^2 \\ &= 4 \end{aligned}$$

With the value of $f(x) = 4$, find the value of x when $g(x) = 4$:

$$\begin{aligned} g(x) &= (x - 5)^2 \\ 4 &= (x - 5)^2 \\ 2 &= x - 5 \\ 7 &= x \end{aligned}$$

The point (2, 4) on the graph of $f(x)$ maps onto the point (7, 4) on the graph of $g(x)$. It has been shifted five units to the right.

Example 2

Ask students to explain why the graph moved four units to the right. Then ask, "What would the equation be to make the graph shift four units to the left?" Similarly, ask why the graph moved three units up. What would have to change to move the graph down?

Have a class discussion about how mapping notation works. Some students find this concept confusing. Ask students to describe to you what it means to map points.

If students are having difficulty, consider working through this Example using an alternative method. Start with the table of values for the base function $y = |x|$ and apply the mapping $(x, y) \rightarrow (x + 4, y + 3)$ to each point to create a table of values for $y = |(x - 4)| + 3$. Then, ask students to plot the function from the table of values.

Example 3

Discuss with students the concept of key points on a graph. Ask them how the points for $f(x)$ are chosen. Then, ask them if they can suggest three other points that satisfy $f(x)$ and explain how they would determine the corresponding image points. Finally, ask students to describe the transformation that has occurred from $f(x)$ to $g(x)$.

Meeting Student Needs

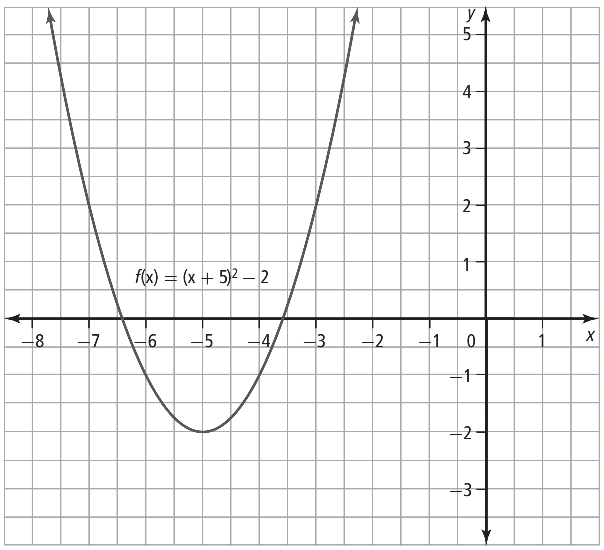
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Encourage students to work in pairs to compare the tables in Example 1. They should identify the relationship between the change in the equation and the points in the table, and the resulting graph. Students may want to use three different coloured pens to sketch the functions. Arrows may be drawn to demonstrate the transformation on each key point.
- Students may wish to compose a short poem or chant to help them remember the effect of h and k on the original equation.
- For Example 3, students may wish to enter the answer into graphing technology to determine if the given points are found on the graph of the equation.
- Students may wish to create a bookmark that lists the Key Ideas for easy reference when working through the chapter.

Answers

Example 1: Your Turn

The first function is a vertical translation of $y = x^2$ down by 1 unit because $k = -1$. The second function is a horizontal translation of $y = x^2$ left by 3 units because $h = 3$.

Example 2: Your Turn



Example 3: Your Turn

a) The graph has been translated up by 6 units and right by 4 units: $y - 6 = f(x - 4)$.

b) The graph has been translated up by 3 units and left by 8 units: $y - 3 = f(x + 8)$.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • This question is directly linked to work done in the Investigate. If students are struggling, they likely did not understand the point of the investigation. In this case, revisit the process followed in the Investigate before returning to the Example. • Encourage students to include a discussion of the shape of the graph in their responses.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • It may be easier for students to verbalize what each parameter does before making the combined move and sketch. • Some students may realize that the value of h is opposite to what appears in the brackets and the value of k is the actual value of the term following the brackets. For example, for $(x + 2) + 5$, $h = -2$ and $k = 5$.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • If students are having difficulty, draw their attention to the Did You Know? and ask them to describe the similarities between p and q, and the parameters h and k. • Some students may find it easier to use a subtraction rule to establish the translation or mapping rule. That is, they can use image points minus original points. For example, for the translation $(3, 3) \rightarrow (7, 6)$, the mapping rule can be determined by $7 - 3 = 4$ and $6 - 3 = 3$, so $(x, y) \rightarrow (x + 4, y + 3)$

Check Your Understanding

For #3, ask students if they can describe the transformation in words. Suggest that they consider how the transformation affects each point. Then ask, "How can you use mapping notation to represent what you have described in words?"

For #6, ask students which variable, x or y, changes when a vertical translation is applied, and which variable has changed in this question. Then ask, "If no translation was made, what would the value of y be when $x = 4$? How does this differ from the given point?"

Similar to #6, in #7 ask students which variable changes if a horizontal translation has been applied. Ask, "If no translation had been made, what would the x-value be when $y = 16$? How can you use this information to determine what horizontal translation has been applied?"

In #9, students may need to be reminded how to write domain and range using the proper notation. Remind them that domain and range deal with possible restrictions or limitations on the variables. Ask them if domain deals with the independent or dependent variable. Then ask what limitations may exist for the variables in this scenario.

For #10, students can explore whether the order of horizontal and vertical translations matters. Consider discussing this question as a class.

If you assign #14, consider collecting students' work and displaying it around the classroom. You could get students to examine each other's work and see if they can identify the four translations that were used.

For #17, ask students

- How can the zeros help you determine the transformation that has occurred?
- What are the original zeros of the function?
- How can you have two zeros and then end up with two zeros?
- How do you determine how much the graph has been translated if there are two zeros?

Meeting Student Needs

- Provide **BLM 1–2 Section 1.1 Extra Practice** to students who would benefit from more practice.
- Some students may wish to create a series of posters to illustrate the horizontal and vertical transformations on a given equation and include a written explanation. These could be turned in for assessment.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–3, 5, 6, 8, and 10–12. Students who have no problems with these questions can go on to the remaining questions.	• In #1 students are asked to demonstrate their understanding of the parameters h and k in general terms. Students must be able to complete these questions correctly before moving on to #2, where they must identify the values of image points. (Some students may find it easier to complete #3 before #2.) • Students use the opposite processes in #5 and 6, where they are given the points on an image and must give the equation of the transformed graph. Suggest that students draw a graph of the original function before attempting to find the image points. They can then work backward. • The table in #8 is an excellent assessment tool. Students who successfully complete this question should have sufficient understanding to complete the questions that follow.
Assessment *as* Learning	
Create Connections Have all students complete C1, C2, and C4	• Students are asked to write, in words, their understanding of the effects of the parameters h and k in C1. • You may wish to have students work in pairs for C2. Review the process of completing the square with students before assigning this question. • Ensure that students are able to provide the explanations in C4, and demonstrate why this assertion is true. Have them explain their answers to a partner or take them up as a class. This understanding is important for later in the chapter.

Reflections and Stretches

1.2

***Pre-Calculus 12*, pages 16–31**

Suggested Timing
90–120 min

Materials
- grid paper
- ruler
- graphing technology
- Mira™

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 1–3 Section 1.2 Extra Practice

Mathematical Processes for Specific Outcomes

RF3 Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

RF5 Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the:
- *x*-axis
- *y*-axis
- line $y = x$.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–6, 8–10
Typical	#1–5, 7–13, C1–C5
Extension/Enrichment	#12, 13–16, C1–C5

Planning Notes

Before beginning the Investigate, revisit the coordinate system with students. Ask them to identify the four quadrants. You might also ask them to recall the definition and requirements of a function. Define *conjecture* for students, as this is not a common word and is not defined in the student resource.

Investigate Reflections and Stretches of Functions

When students are working on step 1, ask them what it means to plot A′ using the x-axis as a mirror line. Which coordinate stays the same? changes? Then ask how far point A is from the x-axis before and after the reflection. Have a similar discussion regarding reflecting points and functions in the y-axis. Students could use a Mira™ to see if their reflection is done correctly. Suggest that they test the relationship they stated in step 1c) by graphing points in other quadrants.

For step 2, ask students, "When you reflect the function over the x-axis, will you reflect the entire shape or do it point by point? Will the function still be a function after the reflection?" Then discuss whether reflections change the shape, size, or orientation of objects. How are points that lie on the line of reflection affected?

Before students work on step 3, you may wish to revisit how to graph an equation using technology, including how to restrict the domain. Then discuss the concept of restricting domains. Ask
- What does the function $y = x^2 + 2x$ look like when the domain is restricted?
- Will the domain remain restricted after the reflection occurs?
- What will the new domain be?

Help students recognize the patterns of the transformations in Part B of the Investigate. When they multiply the function by 2 and then by $\frac{1}{2}$, ask:
- Which point changed? How did it change?
- Why did only the y-coordinate change?
- In step 7, when you sketch the graph, does the vertex change? the intercepts?

Ask a similar series of questions when students vary the value of b in step 10. Ask why it is the x-coordinate that changes in this step.

Note that the apparent differences between interpreting horizontal and vertical translations in the form $y = f(x - h) + k$ were resolved by rewriting the function as $y - k = f(x - h)$. Similarily, it might be helpful for students to see the same treatment for horizontal and vertical stretches: $y = af(bx)$ could be written as $\frac{1}{a}y = f(bx)$ or $\frac{y}{a} = f(bx)$.

Meeting Student Needs

- Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within the section.
- Students may wish to open a document in a word processor and insert a piece of clip art. They could then experiment to see what happens if they stretch the picture horizontally and vertically. They could then see what happens when they flip the picture horizontally and vertically representing reflections in the x-axis and y-axis.
- Some students will benefit from a quick review of transformations performed in elementary mathematics. On a coordinate plane, sketch a capital E in quadrant I. Ask students to reflect the E into quadrant II, then quadrant III, and finally in quadrant IV. Guide them to recognize the axes as lines of reflection for this series of transformations.
- Students can work in groups of four. Two members of the group are responsible for completing each part of the Investigate. They can then display and discuss the results with the other members of the group. The group can then complete the Reflect and Respond questions together.
- For an alternative approach to part B, have students create a table of values to compare output values for functions involving stretches, for given input values. Encourage students to graph the functions on the same set of coordinate axes. Ask them to describe how the graphs relate to the graph of $y = f(x)$.

Enrichment

A skipping rope tied to a post at one end and held at the other can create wave patterns that travel to the post and are then reflected back. Ask students to compare this type of reflection with mathematical reflections. Do mathematical reflections have anything in common with the rope's motion that might help students develop a mathematical interpretation of the motion of the rope?

Gifted

Discuss with students how transmissions from a satellite are reflected by a parabolic satellite dish. Or, have them consider the path that reflected sound waves travel down the outer ears of a bat. Ask students how the term *reflected* in these cases similar to and different from the mathematical concept of reflection?

Common Errors

- Some students think that the x-coordinate of a point changes when the point is reflected in the x-axis, and that the y-coordinate of a point changes when a point is reflected in the y-axis.

R_x Suggest that students plot a point and reflect it in the x-axis. They can then draw a line segment connecting the original point and the reflected point. Guide them to notice that the line segment is parallel to the y-axis and passes through one point on the x-axis. This shows that the x-coordinate stays the same and the y-coordinate changes. Repeat this exercise reflecting the point in the y-axis.

Answers

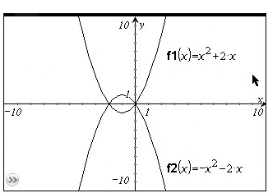
Investigate Reflections and Stretches of Functions

1. c) their y-components are opposites of each other **d)** yes

2. a) Find the negative of the y-components for each point and plot the x-component at that value.

b) All the y-components take the opposite sign, so the function is multiplied by -1.

3.



$y = -x^2 - 2x$

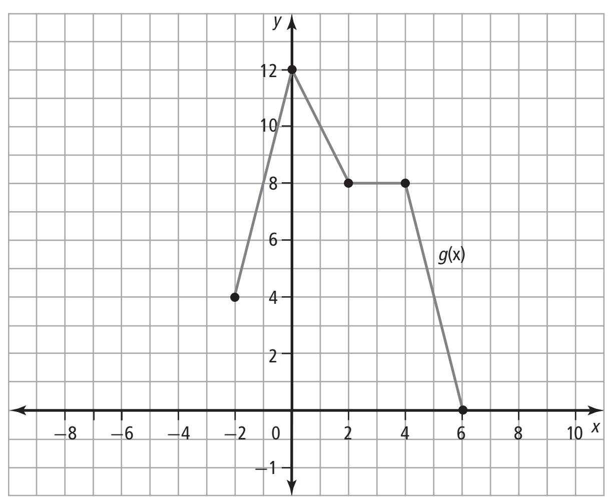
4. This step is basically the same as for the x-axis mirror line, but with a few differences. First, the points will take the opposite sign of the x-coordinate. Instead of multiplying the function by -1, you simply replace x with $-x$. So, the function for the mirror image is $y = x^2 - 2x$.

5.

	Reflection in	Verbal Description	Mapping	Equation of Transformed Function
Function $y = f(x)$	x-axis	mirrored over the x-axis	$(x, y) \to (x, -y)$	$y = -f(x)$
	y-axis	mirrored over the y-axis	$(x, y) \to (-x, y)$	$y = f(-x)$

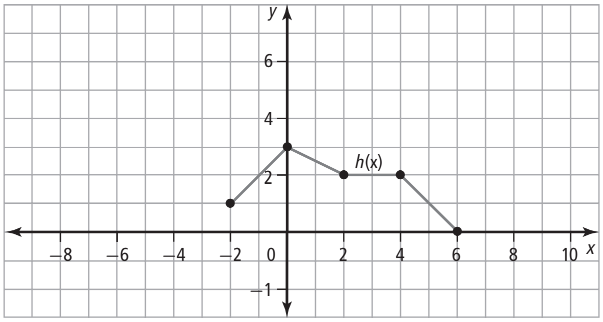
6. d) When the y-coordinate is multiplied by 2, the image point is twice as far from the x-axis as is the original point. When the y-coordinate is multiplied by a factor of $\frac{1}{2}$, the image point is half the distance from the x-axis as is the original point.

7. a)



Answers

b)



8. $y = 2f(x)$ and $y = \frac{1}{2}f(x)$.

9. The statement means that the function has become steeper by a factor of 2. In the second case it is vertically stretched by a factor of $\frac{1}{2}$.

10. a) Instead of being vertically stretched, the function would stretch horizontally.

b) $y = f(2x)$ and $y = f\left(\frac{1}{2}x\right)$

11.

	Stretch About	Verbal Description	Mapping	Equation of Transformed Function
Function $y = f(x)$	*x*-axis	vertical stretch	$(x, y) \rightarrow (x, ay)$	$y = af(x)$
	y-axis	horizontal stretch	$(x, y) \rightarrow (\frac{1}{b}x, y)$	$y = f(bx)$

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • If students are having difficulty answering these questions, encourage them to make a table of values, and compare several points from the original to the same points in the image. Doing so may assist them in seeing the pattern produced by the transformation. • You may need to coach students in the terminology of "about the *x*-axis" or "about the *y*-axis." Explain its meaning, using a visual example if needed.

Example 1

Ask students to examine the function notation before they examine the graph. Ask

- What happens if a function $y = f(x)$ is transformed to $y = -f(x)$?
- Which coordinate changes, x or y? Does the other coordinate change?
- What is the implications of this for the graph?

Have a similar discussion about what happens when the function $y = f(x)$ is transformed to $y = f(-x)$.

Have students work in pairs to come up with ways they can remember whether $y = f(-x)$ and $y = -f(x)$ reflect in the x-axis or y-axis. They can then share their strategies with the class.

Example 2

Ask students to examine the function notation before they examine the graph. Ask:

- What do you think will happen if a function $y = f(x)$ is transformed to $y = af(x)$? Which coordinate changes, x or y? Which stays the same?
- Why do the x-coordinates stay the same?
- What distance is affected?

Discuss with students the effects of the transformations on domain and range. Ask students what happens to the domain and range of $y = f(x)$ when the function is transformed to $y = 2f(x)$ and to $y = \frac{1}{2}f(x)$. For each, ask what type of point would be an invariant point.

Example 3

Ask students to examine the function notation before they examine the graph. Ask:

- What do you think will happen if a function $y = f(x)$ is transformed to $y = f(bx)$? Which coordinate is changing in this scenario, x or y? Which stays the same?
- Why do the y-coordinates stay the same?
- What distance is affected by this change?

Students may have difficulty with horizontal stretches and the factor by which functions are stretched horizontally. Ask:

- If $y = f(x)$ is transformed to $y = f(2x)$, is the graph double the distance from the x-axis or the y-axis?
- What distance has changed?
- Why did the distance not double?
- By how much has it changed?
- How does this change affect the domain and range?

Go through a similar line of questions for the transformation $y = f\left(\frac{1}{2}x\right)$.

Spend some time with the Example before students go on to the Your Turn question. Students need to discover and become comfortable with the fact that the horizontal stretch factor has the opposite effect of what they might think. Also discuss the invariant point for this scenario.

Example 4

Before working through this Example, suggest that students consider the types of things they would look at before trying to write the equation of a transformation.

For example, ask:

- How can you tell if a function has been reflected or stretched?
- If the graph has been reflected, how can you determine the line of reflection?
- If the graph has been stretched, how can you determine if the graph has been stretched horizontally or vertically?
- What key points might help?

For each of these questions, ask students to identify the part of the equation that will be affected.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- A short activity using an image and a Mira™ may benefit some students. The image, when reflected, should clearly remain the same shape but have a different orientation on the page. Ask students where the Mira™ would need to be placed in order to create at least one invariant point.
- Add the Key Ideas to the bookmark created in section 1.1.

Common Errors

- Some students confuse the effect the a parameter has on the graph of the function.

R$_x$ Have students graph x^3, $3x^3$, and $\frac{1}{3}x^3$ on the same set of axes. Ask them to compare how far the points on the different graphs are from the x-axis. For example, points on the graph of $3x^3$ are three times farther from the x-axis as are points on the graph of x^3. The graph of $3x^3$ is tall and narrow in comparison to x^3.

On the other hand, points on the graph of $\frac{1}{3}x^3$ are $\frac{1}{3}$ as far from the x-axis as are the points on x^3. So, in comparison its graph is short and wide.

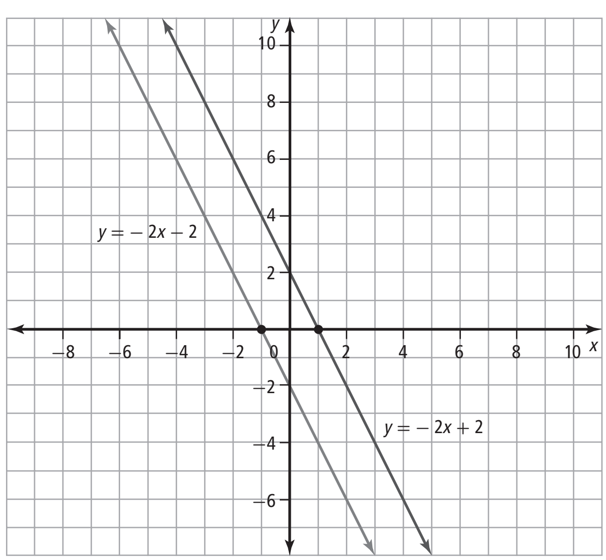
- Some students confuse the effect the b parameter has on the graph of the function.

R$_x$ Have students build a table of values to see the effect of changing the b parameter. Ask students to consider the functions $y = f(x)$ and $y = f(2x)$. For $y = 2$, $x = 2$ for the first function and $x = 1$ for the second function. Therefore, each value of x in $f(2x)$ is half the distance from the y-axis as are the values of x in $f(x)$.

Answers

Example 1: Your Turn

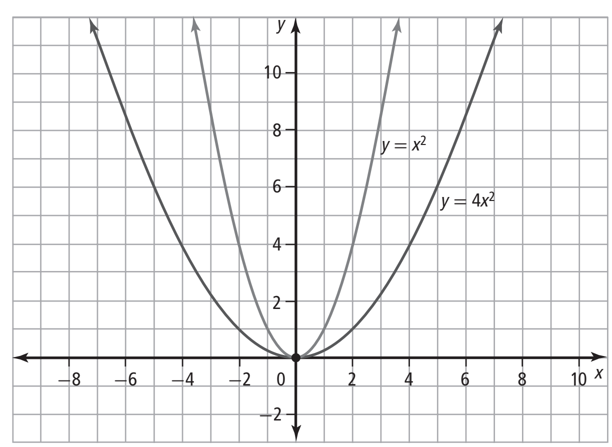
a)



c) The functions are mirrors of the original over the x-axis and y-axis. The first function has an invariant point at $(-1, 0)$. The second function has an invariant point at $(0, 2)$.

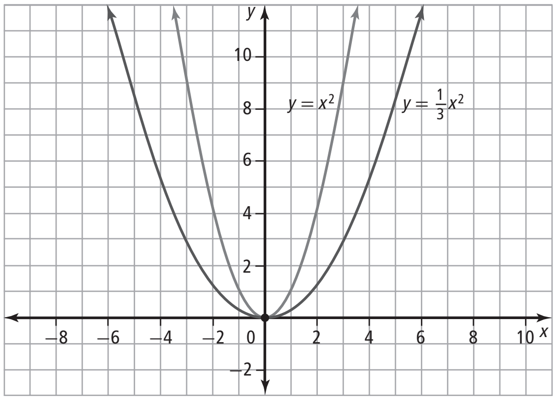
Example 2: Your Turn

a)



The transformation is a vertical stretch of 4. The domain is $\{x \mid x \in R\}$. The range is $\{y \mid y \geq 0, y \in R\}$. There is an invariant point at the origin.

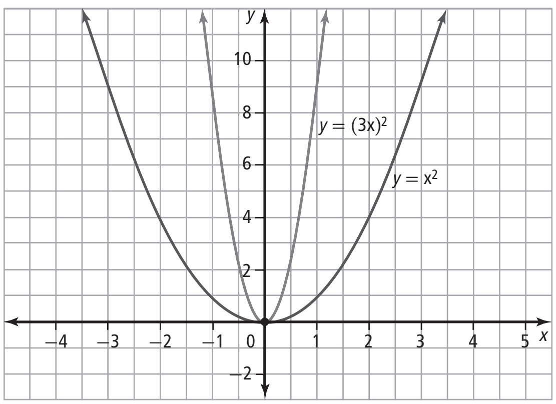
b)



The transformation is a vertical stretch of $\frac{1}{3}$. The domain is $\{x \mid x \in R\}$. The range is $\{y \mid y \geq 0, y \in R\}$. There is an invariant point at the vertex, (0, 0).

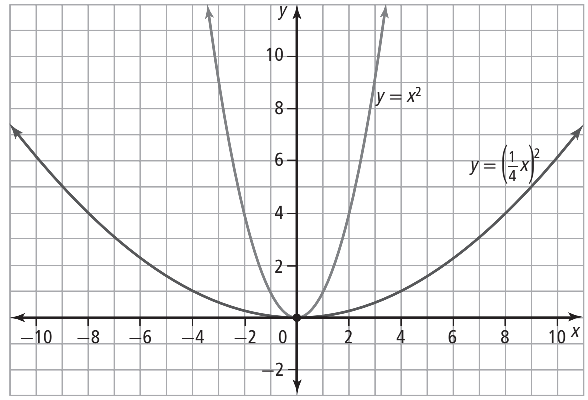
Example 3: Your Turn

a)



The transformation is a horizontal stretch of $\frac{1}{3}$. The domain is $\{x \mid x \in R\}$. The range is $\{y \mid y \geq 0, y \in R\}$.

b)



The transformation is a horizontal stretch of 4. The domain is $\{x \mid x \in R\}$. The range is $\{y \mid y \geq 0, y \in R\}$.

Example 4: Your Turn

$g(x) = 4x^2$ or $g(x) = (2x)^2$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • If students are having difficulty, encourage them to make a table of values and identify several points from the original graph and the image. This may help them see the pattern produced. • Point out to students that the distances to the axis do not change from the original to the image.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • If students are having difficulty, encourage them to make a table of values and identify several points from the original graph and the image. This may assist them in visually seeing the pattern produced. • Ask students whether the domain and range change for these images.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Ask students whether they believe the domain and range will change for the new graph. • Ask students to explain, in general, what a fractional value versus an integral value for the parameter *b* does. • Have students identify the different ways to show a mapping.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Some students may find it easier to verbally describe the pattern they see from a table of values before writing an equation.

Check Your Understanding

For #1, ask students

- How do you obtain the points in the second column?
- What do you do to points in the second column to obtain points in the third column?
- What happens when points in the second column are negative? positive?
- How do you obtain the points in the fourth column?
- What combination of columns do you use to graph?
- How many functions are there?

Students might want to graph each function in a different colour on the same grid and include a legend so they can easily differentiate between the functions.

Before graphing the functions in #3 and 4, challenge students to state the domain and range of the function, and then state the domain and range of the reflected function. Ask, "What is the effect on domain and range when reflecting a function in the x-axis? the y-axis?"

For #7, ask students to consider how they can tell, looking at a graph, what parameters have been changed. Ask, "How can you recognize a reflection? a vertical stretch? a horizontal stretch?"

For #8, suggest that students set up a table of values to help them with this question. They should consider what coordinates are being changed and how they are being changed.

Before students begin #9, ask, "When you replace the y-value, as you will in part c), d), and f), what must you do to have an equation in the form $y =$?"

For #11, discuss what the domain and range of each function is. Ask, "What restrictions are there for each function within the context of the question?"

For each part in #14, ask students what effect each function has on the coordinates in $y = f(x)$. Would the zeros be changed or remain the same?

In #16, students may need to be reminded that when you reflect an object, each point and its corresponding point is an equal distance from the line of reflection. Ask them to look at key points on the graph and determine how far each one is from the line of reflection. They can then use this information to reflect the points.

Meeting Student Needs

- Provide **BLM 1–3 Section 1.2 Extra Practice** to students who would benefit from more practice.
- Take students to the gymnasium or a large space with a tiled floor to practice "walking transformations." Have a graph taped onto the floor with key points on the intersection of tiles. Students can then walk through various transformations either provided by you or other students. If there are five key points available, have five students make the transformation. Switch to five other students for a second transformation, beginning from the original graph.
- Students may create a wall mural using transformations of an original function. These could be "works of art." For example, the lines of the function can be solid black, and the closed spaces shaded various colours. Ensure that the students can write the equations of the transformed graphs.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–6 and 8–10. Students who have no problems with these questions can go on to the remaining questions.	• Ensure that students know the difference between a reflection, addressed in #1, and a stretch, addressed in #2. You may need to revisit the definition of *invariant points*. • Some students may need to be coached to understand the terminology "about the x-axis." Ask them to verbalize or sketch an example before assigning #6. • If students are having difficulty with the Apply questions, suggest they first set up a table, and then use the pattern and coordinates to sketch the image graph.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3. Students who have completed these questions can move on to complete the remaining ones.	• C1 asks students to make an important distinction that illustrates their understanding of the horizontal stretches. Have students compare their answers with those of a classmate, or discuss responses as a class. • If students are having difficulty with C2, suggest they sketch a graph for each that models the invariant points in the correct place. Have them use their graphs to respond to the question. Have selected students share their responses with the class. • C3 could have more than one correct answer. Have students work in pairs to complete this question.

Combining Transformations

1.3

***Pre-Calculus 12*, pages 32–43**

Suggested Timing
90–120 min

Materials
- grid paper
- ruler

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 1–4 Section 1.3 Extra Practice

Mathematical Processes for Specific Outcomes
RF4 Apply translations and stretches to the graphs and equations of functions.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–7, 9, two of 10–14
Typical	#1–12, C1, C3
Extension/Enrichment	#15–18, C1–C4

Planning Notes

Before beginning this section, have students revisit the transformations they have learned to this point. Student groups could discuss transformations and then present a quick refresher or summary to the class. Transformations of reflections, translations, and stretches could be illustrated with the Aboriginal designs and patterns found in the beadworks, star blankets, and paintings.

Investigate the Order of Transformations

An alternative approach to step 1 is to ask students what the equation of the graph would be if they stretched the graph vertically. Then ask, "If you added a horizontal stretch to your equation, how would it change?" Students could then do the reverse and see if they end up with the same equation. Have students discuss why it does not matter in which order the stretches are performed.

Another approach to the Investigate is to divide students into groups and give each group a different function. When groups complete the investigation, have them compare their results. Doing this will help students generalize their results.

Consider asking students to write the mapping notation of functions before and after combined transformations have occurred. Some students will recognize that the mapping $(x, y) \rightarrow \left(\frac{1}{b}x + h, ay + k\right)$ can also be written as $(x, y) \rightarrow \left(\frac{1}{b}(x + bh), a\left(y + \frac{k}{a}\right)\right)$, thus changing the order of applying the transformations.

A student who looks at the function $y = 2\sqrt{x} + 6$ and describes the transformations on $y = \sqrt{x}$ as a vertical translation of six units up and vertical stretch of factor two is as accurate as a student who interprets the transformations as a vertical stretch of factor two followed by a vertical translation of six upward. Another approach is to translate the axes, do the expansion/compression, and then do the reflection around the translated origin. Some students may find this alternative approach easier.

Meeting Student Needs

- Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within the section.
- Invite a quilter to display some quilts with transformations in the classroom. You may even have students "create a quilt" by inviting each student to create one square. They can then, with the assistance of the quilter, put the squares together. They could raffle the quilt and use the proceeds for graduation expenses or donate them to an organization of their choice.
- Students can work in partners or groups of three. Each student in the group could transform the graphs in a different order. They can then compare their final results. Consider having half of the groups complete the stretches section and the other half complete the combined transformation section. Conclude with a class discussion of the investigation.

Enrichment

The population of certain arctic animals, such as the snowshoe hare, follows a pattern similar to an inverted parabola. At the beginning of the cycle, the population often grows because the rise in numbers is so quick that predator populations are not sufficient to keep the growth in check. However, decline is almost certain because of starvation and disease. Have students create graphs that show what might happen if, in response to a sudden surge in population, a new predator was introduced to the hare's domain. How would this affect the natural cycle? Ask students to explain why they would support or oppose the introduction of this new predator.

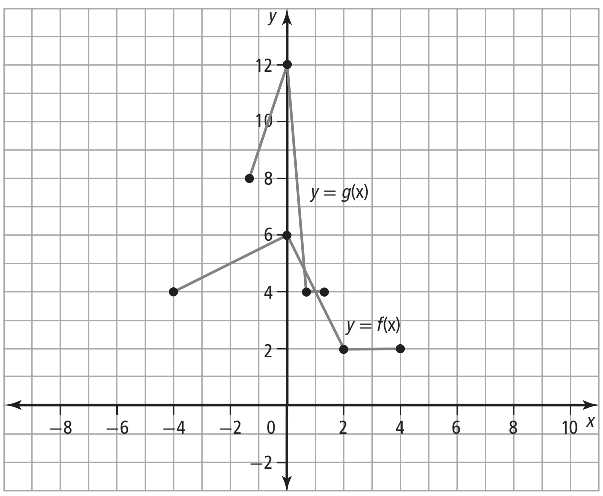
Gifted

Ask students to explore how graphing mathematical functions furthers the understanding of the causes and effects of global warming. How might the math of these patterns help protect the planet? Ask students to give one specific example of a graph that informs the public and creates positive change.

Answers

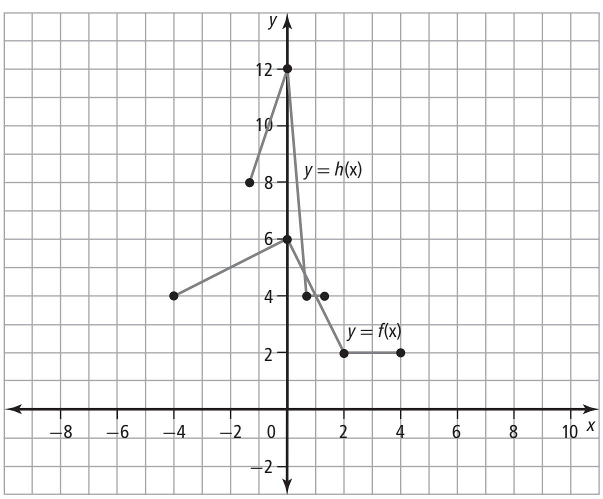
Investigate the Order of Transformations

1. b)



$y = 2f(x)$; $y = 2f\left(\frac{1}{3}x\right)$

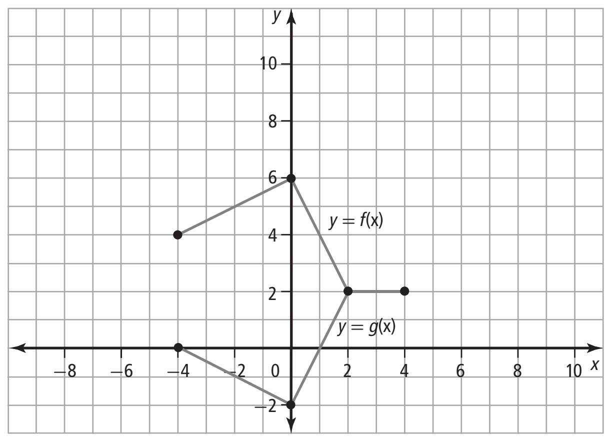
c)



$y = f\left(\frac{1}{3}x\right)$, $y = 2f\left(\frac{1}{3}x\right)$

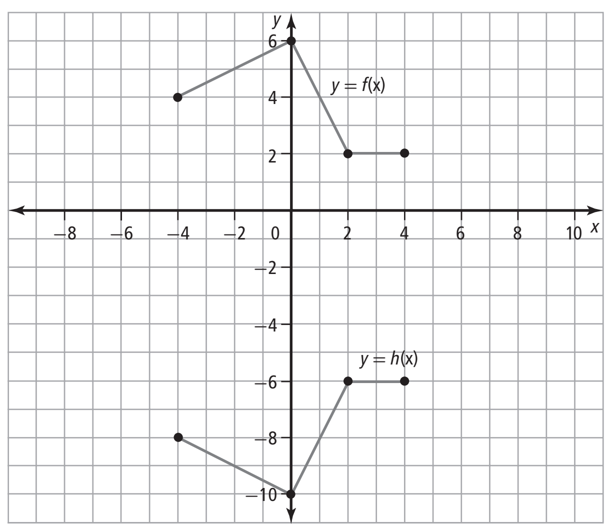
2. No.

3. b)



$y = -f(x)$; $y = -f(x) + 4$

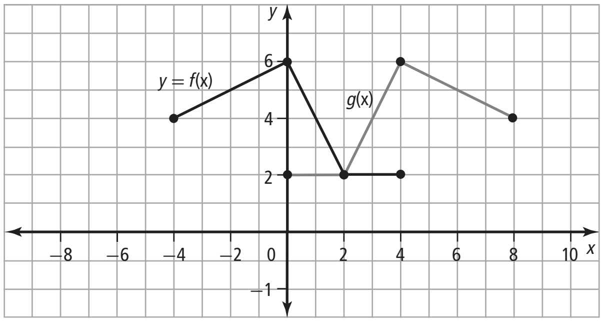
c)



$y = f(x) + 4$; $y = -f(x) - 4$

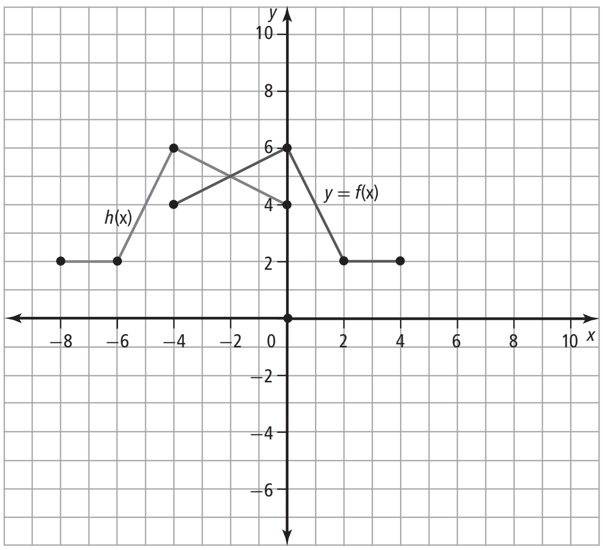
4. Yes. The vertical translation moved the graph away from the x-axis when done first. When done after the reflection in the x-axis, the vertical translation moved the graph closer to the x-axis.

5. b)



$y = f(-x)$; $y = f(-(x - 4))$

c)



$y = f(x - 4)$; $y = f(-x - 4)$

6. Yes. Since the reflection in the y-axis depends on the distance from the y-axis, doing the horizontal translation before the reflection changes the final result.

7. a) No, the order would not matter since the vertical stretch is independent of the constant term that translates the function vertically.

8. All reflections should be done first so that later translations are correct.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • Encourage students to try performing the vertical stretch first and then the vertical translation, and to try reversing the order of these before predicting whether order affects the answer. • Discuss these Reflect and Respond questions as a class. • Later, you may wish to discuss that transformations should be done in alphabetical order: *a*, *b*, *h*, *k*.

Example 1

Ask students to describe in words what the functions $y = 3f(2x)$ and $y = f(3x + 6)$ mean. Ask

- What transformations have occurred?
  - If it is a stretch, is it a horizontal or a vertical stretch?
  - If it is a translation, is it a horizontal or vertical translation?
- Why is the horizontal stretch about the y-axis and not the x-axis?
- Why is the vertical stretch about the x-axis and not the y-axis?

For the Your Turn question, ask students what the effect of a b value of $\frac{1}{2}$ will have.

Example 2

Ask students in what order the transformations should be applied.

Example 3

Students could be asked again to state the order that they need to follow to determine the equation. Ask students why the vertex of the function is useful in determining if there has been a translation, but is not useful in determining if a stretch has occurred. Students might want to graph each transformation one at a time, using different colours to help them see the progression from $f(x)$ to $g(x)$. If students are struggling finding the stretch, they could solve an equation similar to (original value)(stretch factor) = (final value), and solve for stretch factor.

If students are still having difficulty, consider providing an alternative explanation for finding a and b. For the a parameters, provide students with the following directions:

- Look at the original function, $f(x)$. The height of this function is four units, from $y = 0$ to $y = 4$.
- Look at the image function $g(x)$. The height of this function is eight units, from $y = 2$ to $y = 10$.
- Recognize that $g(x)$ is double the height of $f(x)$, so there is a vertical stretch by a factor of 2. Therefore, $a = 2$.

For the b parameter, provide students with the following directions:

- Look at the original function, $f(x)$. The width of this function is eight units, from $x = -4$ to $x = 4$.
- Look at the image function $g(x)$. The width of this function is two units, from $x = -8$ to $x = -6$.
- Recognize that $g(x)$ is one quarter the width, so there is a horizontal stretch by a factor of one quarter. Therefore, $b = 4$.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Display the equations $y - k = af(b(x - h))$ or $y = af(b(x - h)) + k$ on the board. Label the function of each variable. Some students will benefit from seeing the complete process summarized before different parts discussed separately. Keep this summary displayed for the remainder of the chapter.

Common Errors

- Students may confuse the effects of parameter b and think that a function such as $y = 3f(2x)$ has been horizontally stretched about the y-axis by a factor of 2.

R_x Go over this material with students again and use a table of values to show that while it appears that the x-values have been doubled, they have actually been halved.

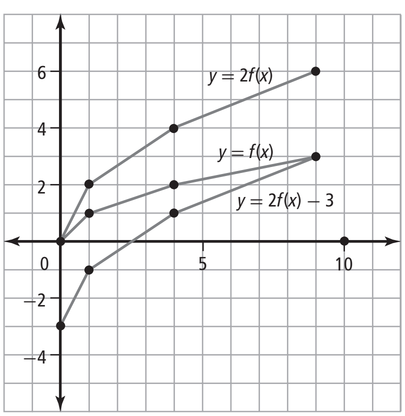
- Some students have difficulty graphing the image graph that involves several transformations.

R_x Suggest that students who are having this difficulty work through the questions by doing one transformation at a time.

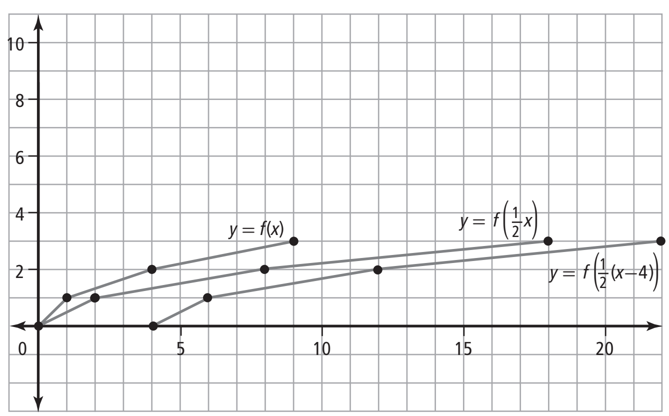
Answers

Example 1: Your Turn

a) The function is stretched vertically by a factor of 2 and is then translated down by 3.



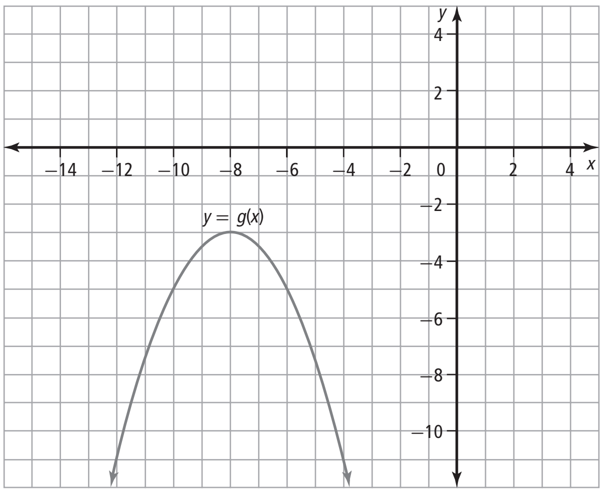
b) The function is stretched horizontally about the y-axis by a factor of 2. It is then translated 4 units to the right.



Answers

Example 2: Your Turn

The transformations are a reflection over x-axis, a vertical stretch by 2, a horizontal stretch by 2, a translation left by 8, and a translation down by 3; $g(x) = -2f\left(\frac{1}{2}(x + 8)\right)^2 - 3$.



Example 3: Your Turn

$y = \frac{3}{4}(x - 3)^2 - 4$; Locate the vertex and it will show the translations. The function translates 3 units to the left and 4 units down. The function is reflected over the x-axis, thus the negative sign. Then it is just a matter of solving for the scaling factors. In the vertical direction, 4 units becomes 3 units, so there is a vertical stretch by a factor of $\frac{3}{4}$.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Have partners compare their ideas on which transformations should be applied so that they can reach a consensus. • Have students compare their final graph with another group, comparing the order in which the transformations were done.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Remind students to remember the alphabet order that can be used to apply the transformations in the correct order. • Have students compare their results with a partner or have groups share their responses and discuss differences that have arisen. Ask students to identify why differences may have occurred.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Some students may benefit from orally describing what they see as a difference between the two graphs before beginning to write an equation. Help them link their oral description to the possible values for the parameters a, b, h, and k in creating $g(x)$. • Some students may wish to use graphing technology to verify their reasoning.

Check Your Understanding

For #4, encourage students to create a table of values for the original function and then for the transformed function. Doing so might make it easier for them to see the transformations when they compare each point by point.

For #6, ask students what order they should follow to show how the point has been transformed. Suggest that they try different orders to see if they have the same end result.

For #9, ask students what has to be done to the equations in parts a) and e) to make graphing them easier.

Suggest that students create a table of values in #10 to compare key points on each of the functions. Doing so might help students see the transformations. Ask, “Which of the transformations is the easiest to identify?”

Ask students what transformations have occurred in #11. Students could graph the transformations one at a time to help answer this question. They could start by graphing the stretches, then graph any reflections, and finish with translations. Taking a step-by-step approach like this might help struggling students.

Ask students why they think the range is restricted in #14. Ask them what the restricted range of the transformed function will be.

Ask students what transformations are involved in #16. Ask students how they can apply those transformations to an unknown value. What order must they follow to ensure the correct value? Encourage students to do these questions one step at a time.

For C1, you may need to explain the window setting in steps 1 and 2. The window settings x: [−3, 3, 1] y: [−3, 3, 1] indicate that the window extends from −3 to 3 along both the x-axis and y-axis, with units of 1.

Meeting Student Needs

- Provide **BLM 1–4 Section 1.3 Extra Practice** to students who would benefit from more practice.
- Students may chose to repeat the "walking transformation" activity discussed in section 1.2.
- Traditional and contemporary items created and constructed by Aboriginal peoples display all aspects of transformations. Suggest that students research topics such as star blankets, beadwork, Aboriginal paintings, snowshoes, and birch bark canoes. How are transformations apparent in these objects? Suggest that students make a presentation of their findings.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–7, 9, and two of 10–14. Students who have no problems with these questions can go on to the remaining questions.	• Consider doing #1, 2, and 3; these are excellent assessment for learning questions. Use these questions to decide whether students are ready to proceed with additional questions. Use any specific graphs that were incorrectly identified as a tool to revisit students' understanding of the parameters. • Use #4, 5, or 6 as follow-up assessment questions if extensive coaching was required in the earlier questions. • Excellent assessment indicators are provided in #7 and 9 as a follow-up to coaching and individual assistance. Have students discuss and compare both results with a partner, or discuss the results as a class.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C3.	• You may wish to have students work in pairs or small groups for C1, the Mini Lab. Have groups share their thinking with the class. • Some students may require coaching to work through completing the square for C3. Suggests that they identify the parameters and how they will affect the graph of $y = x^2$.

Inverse of a Relation

1.4

***Pre-Calculus 12*, pages 44–55**

Suggested Timing

60–90 min

Materials

- grid paper
- ruler
- Mira™

Blackline Masters

Master 3 Centimetre Grid Paper

BLM 1–5 Section 1.4 Extra Practice

Mathematical Processes for Specific Outcomes

RF5 Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the:
- x-axis
- y-axis
- line $y = x$.

✓ Communication (C)
✓ Connections (CN)
✓ Reasoning (R)
✓ Visualization (V)

RF6 Demonstrate an understanding of inverses of relations.

✓ Communication (C)
✓ Connections (CN)
✓ Reasoning (R)
✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2–8, 10–12
Typical	#2–7, 9–11, 13–15, one of 16–18, C2–C4
Extension/Enrichment	#15, 19–21, C1–C4

Planning Notes

Begin this topic by showing students graphically that the inverse of a function is actually a reflection in the line $y = x$. This makes it easier for many students to then move to an algebraic approach for determining an inverse. You can also use a graphic approach to revisit the definition and criteria for a function. Understanding this concept is very important when it comes to inverses. You may also want to have students recall the order of operations, and how to rearrange and solve equations for different variables, stating the domain and range.

Investigate the Inverse of a Function

In step 1, you could give students the mapping notation $(x, y) \rightarrow (y, x)$ to complete the table.

In step 2, students could place a Mira™ along the line $y = x$ and see what they observe.

Ask students, "If you can draw the inverses of functions using reflections, can you draw them using other types of transformations?" Have students try several functions and their inverses to see if other transformations can be used.

Ask students if the inverse of a function is always a function. Ask
- If the inverse of a function is not a function, can you restrict the domain of the original function so that the inverse is a function?
- Will the inverse of a linear function also be a linear function? Is this true of quadratic functions?
- Will the inverse of a function be of the same degree as the original function?

You might also talk to students about proper function notation. When the inverse of f is not a function, it is denoted y^{-1}, read "y inverse."

Meeting Student Needs

- Discuss the outcome(s) and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within the section.
- Encourage students to sketch each graph in a different colour. The line for $y = x$ should be sketched with a solid black line.
- Review how to write the equation of a straight line. ($y = mx + b$)
- Students could work in pairs to write the equation of the inverse of various given equations. They can then work together to discover the method that can be used for all inverse equations.

Enrichment

Provide students with this scenario: A grade 8 student is asked to complete the sentence, 4% of 100 is the same as 8% of ___. The student tries to solve the question using the proportion $\frac{4}{100} = \frac{8}{?}$. Ask students to explain why this strategy is inappropriate and why the word inverse might be used in the explanation.

Gifted

The formula $d = vt$ is used to find the distance travelled by an object travelling at the constant velocity for a period of time. An inverse could be created by interchanging d and t. Ask students if this would make sense in terms of the purpose of the original formula.

Common Errors

- Some students interchange x and y to find the inverse, but when they do they also interchange the signs. For example, they make the following inversion: $(3, -4) \rightarrow (4, -3)$.

R_x Remind students that the x-coordinate and y-coordinate are interchanged in their entirety. Students could graph $(3, -4)$ and $(4, -3)$ and place a Mira™ on the line $y = x$ to see that the points are not reflected in this line. So, the correct inversion is $(3, -4) \rightarrow (-4, 3)$.

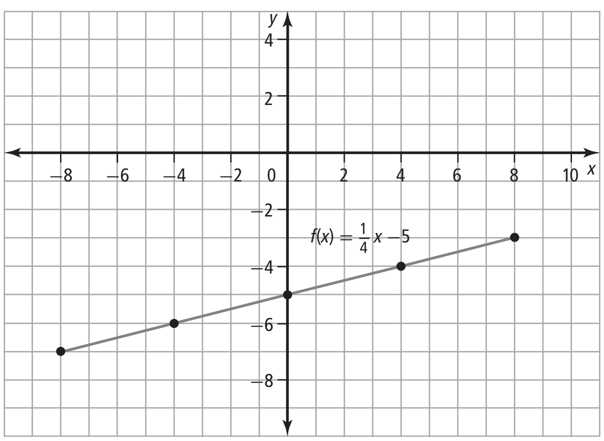
Answers

Investigate the Inverse of a Function

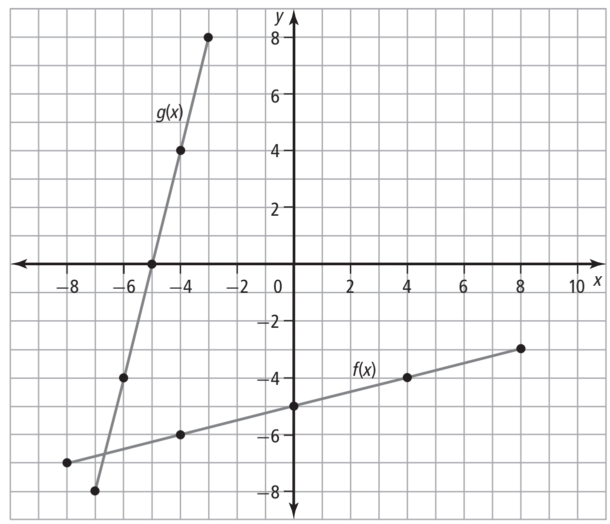
1. a)

Key Points on the Graph of $f(x)$	Image Points on the Graph of $g(x)$
(−8, −7)	(−7, −8)
(−4, −6)	(−6, −4)
(0, −5)	(−5, 0)
(4, −4)	(−4, 4)
(8, −3)	(−3, 8)

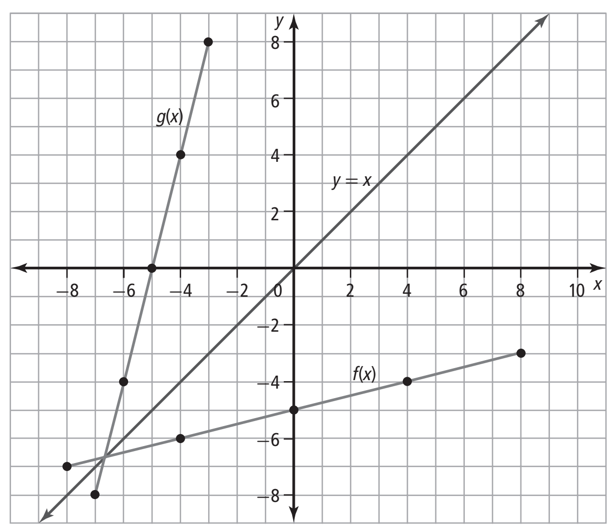
b)



c)



2. a)



b) The distances are the same for both functions.
c) a reflection in the line $y = x$

Answers

3. a) The coordinates of the ordered pairs are reversed.

b) $y = 4x + 20$

c) Yes the inverse is a function. For any point there is only one y component for a single x component.

4. The function must be reflected over the line $y = x$.

5. Not all inverses are functions. For example, the inverse of a quadratic has two solutions that are square root functions. Together these two functions form a relation, not a function.

6. a) Replace the x with y and vice versa. Solve for y and the function will form.

b) $y = \frac{x}{3} - \frac{2}{3}$

7. a) The slope of the inverse function is the reciprocal slope of the original function.

b) The x-intercepts of the original function become the y-intercepts of the inverse function. The same is true for the y-intercepts of the original function and the x-intercepts of the inverse function.

8. You could determine the x-intercepts and y-intercepts, and then find if they follow the conjecture from step 7. You can also use the conjecture that the slope is the inverse.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • Remind or lead students to realize that the inverse is a reflection in the line $y = x$. Make a list of suggestions that students have for testing for a function. • Discuss the results of #7 as a class. This is an important question. Encourage students to summarize the discussion in their graphic organizers.

Example 1

Revisit the definition of a function and the vertical line test with students. Discuss that if the inverse of a function fails the vertical line test, it is not a function. Ask, "Why does checking if the function passes or fails a horizontal line test also help determine if the inverse is a function?"

Example 2

Ask students why the domain of $f(x)$ in this example needed to be restricted to make it a function. Ask

- If the domain was not restricted, what notation could you use to state the inverse?
- Once the domain is restricted, can you use function notation?
- Can you see another way to restrict the domain to make the inverse a function?

Example 3

There are several methods to help students find the inverse of a function. Students with strong algebraic skills will like the methods shown in the student resource. Before working through the solution, have a discussion about the effects of reversing the variables, and solving for y. Ask why this strategy results in the inverse function. Ask students what they notice about the dashed line $y = x$ in relation to the graph of the function and its inverse.

Meeting Student Needs

- Post examples of inverse relations and inverse functions around the classroom.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Prepare a set of graphs that demonstrate functions, one-to-one functions, and non-functions. Work with students to label the graphs accurately.
- Students may find it easier to label the domain and the range while viewing the graph of the function.
- Have students work in pairs. The first partner should instructs the second partner on how to write the equation of the inverse of a given function. Then the partners switch roles to work on a different function.

Common Errors

- Given a function, students may incorrectly find the inverse. For example, when finding the inverse of $f(x) = 3x + 2$, some students find $f(x)^{-1} = 3x - 2$ instead of $f(x)^{-1} = \frac{x - 2}{3}$.

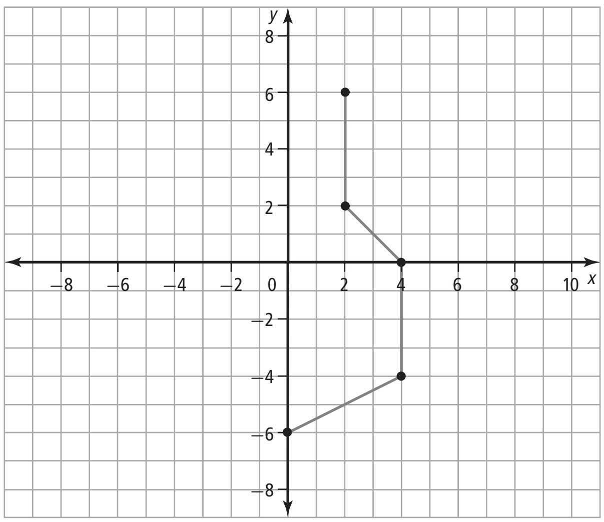
R_x Remind students that all operations are undone. If 2 is added to x, then 2 is subtracted from y. Since in this example x was multiplied by 3, y will be divided by 3. Students can follow the order of operations in the reverse order: add and subtract first, and then multiply and divide.

Answers

Example 1: Your Turn

a) The original relation is a function, but the horizontal test indicates that the inverse is a relation.

b)



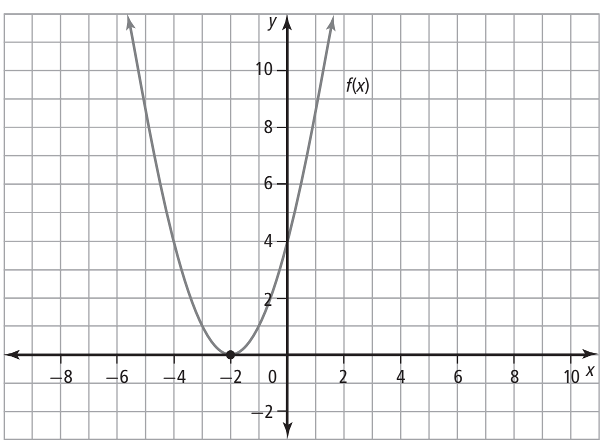
c) original function: domain is $-6 \leq x \leq 6$, range is $0 \leq y \leq 4$, x-intercept is $(-6, 0)$, y-intercept is $(0, 4)$

inverse relation: domain is $0 \leq x \leq 4$, range is $-6 \leq y \leq 6$, x-intercept is $(4, 0)$, y-intercept $(0, -6)$

d) invariant point: $(2, 2)$

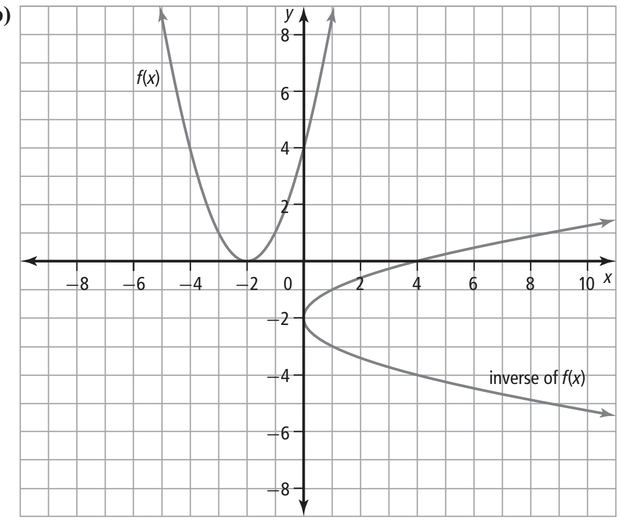
Example 2: Your Turn

a)



Based on the horizontal line test, the inverse is not a function.

b)



c) To keep the inverse a function, the domain of $f(x)$ would have to be restricted to either values greater than or equal to -2 or less than or equal to -2.

Example 3: Your Turn

$y = 3x - 8$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Coach students that the inverse is a reflection in the line $y = x$. The images are the same distance from (or perpendicular to) this line. • Some students will see the inverse as simply switching x and y. Ensure that students understand what this switch does to a function. • Ask students how the inverse will affect the domain and range. It is important that they link this understanding.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Revisit with students how to check for a function (i.e., mapping and vertical line test). • Some students may wish to sketch the line $y = x$ into their graphs to help plot and visualize the inverse.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Coach students to replace $f(x)$ with y so they can interchange x and y. Remind students to write the new equation in terms of y.

Check Your Understanding

Ask students how they will approach #1. Ask

- If you draw the line $y = x$, what can you do with all the points?
- How can you remember the order in which to connect the points once you have reflected them in the line $y = x$?
- Can you do this question using a table of values? How might a table of values help?

Ask students why a horizontal line test can help determine if the inverse of a function is a function. Ask, "Is there another way you could do this?"

For #5, ask students if they can answer the question without first determining the inverse algebraically. Ask them how the order of operations can help in answering this question.

For #8, ask students how the x-intercept and y-intercept can help them graph the inverse of a linear function. Ask, "Do you need any other information?"

If students are having difficulty with #9, ask

- Why does the domain need to be restricted to produce a function?
- Can you use the restricted domain of the function to help you find the range of the inverse?
- What part of the function could you remove to let it pass the horizontal line test?
- How can this help you find the restrictions for the domain of the inverse?

For #10, ask students if they can tell by just looking at the equation which function's inverse is not a function. Ask, "Do quadratic functions ever have an inverse that is a function?"

You could show students an alternative way to approach #13. This approach might be suitable for gifted students. Introduce students to the idea of composition of functions, which involve replacing the range of one function with the domain of the other. This can be an introduction to composite functions, which students will work with in Chapter 10. The rule you can give students is, "If $f(g(x)) = x$ and $g(f(x)) = x$, then $f(x)$ and $g(x)$ are inverses of each other." For example:

$f(x) = x - 4$ and $g(x) = x + 4$

$$\begin{aligned} f(g(x)) &= f(x + 4) \\ &= (x + 4) - 4 \\ &= x \end{aligned}$$

$$\begin{aligned} g(f(x)) &= g(x - 4) \\ &= (x - 4) + 4 \\ &= x \end{aligned}$$

Since $f(g(x)) = x$ and $g(f(x)) = x$, $f(x)$ and $g(x)$ are inverses of each other.

For #14, ask students how they can restrict the domain of the function so that it will only produce half of the parabola. Ask students how finding the vertex might help them do this. Once they have restricted the domain of the function, discuss with them how this helps restrict the range of the inverse so it is also a function.

For #18, students could use the formula to determine their own ring size. Discuss with students how to determine the circumference of their finger. Consider inviting a local jeweller to visit the classroom and talk to students about how to size rings and how rings are made bigger or smaller. The jeweller can probably bring in (or provide you with) a sizing tool that students could use to see if their calculations were correct.

For #19, ask

- If an interval is increasing for the function, will it be increasing for the inverse?
- What keys points should you use to divide your function into useable intervals?

Meeting Student Needs

- Provide **BLM 1–5 Section 1.4 Extra Practice** to students who would benefit from more practice.
- Students can add the Key Ideas to the bookmarks that they created earlier in the chapter. Each student should now have a complete list of the key ideas for the chapter.

- Prepare copies of the graphs for #1 to 3 for students to use.
- For #6, students could explain orally or write instructions on how to identify the inverse of a function.
- For #10, review the process of completing the square with the students prior to having them rewrite the given equations. Briefly discuss the significance of the values for a, p, and q.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do 1, 2–8, and 10–12. Students who have no problems with these questions can go on to the remaining questions.	• Students start off with the reflection line of $y = x$ in #1. Remind them to watch that the original points and image points are at right angles to the reflection line, and that the distances on both sides are the same. • Encourage students to prove in more than one way whether the image in #3 is a function. • Students need to be comfortable with finding and writing the inverse in #4 and 5. In #4, students need to be aware that they are asked to ensure the inverse is a function and therefore must develop a domain that would allow this. • You can use #6, 7, 8, and 12 to check for students' understanding. • Some students may need coaching or a review of how to write an equation in slope-intercept form, ($y = mx + b$).
Assessment *as* Learning	
Create Connections Have all students complete C2–C4.	• C2 gives students an opportunity to describe in words how to find the inverse. Encourage them to explain more than one method. • C3 allows students to demonstrate their understanding of testing for a relation. Encourage them to describe more than one method. • You may wish to brainstorm answers to C3 as a class. Have students record possible approaches in their notebook. • For C4, you may have students work in groups of three. Discuss responses to step 4 as a class.

Chapter 1 Review and Practice Test

1

***Pre-Calculus 12*, pages 56–59**

Suggested Timing
60–90 min

Materials
- graphing technology
- grid paper
- ruler

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 1–2 Section 1.1 Extra Practice
BLM 1–3 Section 1.2 Extra Practice
BLM 1–4 Section 1.3 Extra Practice
BLM 1–5 Section 1.4 Extra Practice
BLM 1–6 Chapter 1 Study Guide
BLM 1–7 Chapter 1 Test

Planning Notes

Have students who are not confident discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test. You may wish to provide students with **BLM 1–6 Chapter 1 Study Guide**, which links the achievement indicators to the questions in the Chapter 1 Practice Test in the form of self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1–12.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 1–2 Section 1.1 Extra Practice**, **BLM 1–3 Section 1.2 Extra Practice**, **BLM 1–4 Section 1.3 Extra Practice**, and **BLM 1–5 Section 1.4 Extra Practice**.
- Suggest that students create a visual organizer, such as a brain map or topic tree, showing the connection between the concepts in the chapter. Suggest they include examples of each concept in their diagram.
- Students should work through the complete list of outcomes provided at the beginning of the unit. Which outcomes and indicators do they know? Which outcomes and indicators require extra study and practice? Encourage students to partner with a student needing similar practice.
- Students may wish to rewrite their bookmark, trying to remember as many details as possible without looking at the original bookmark. They should repeat this process until they understand the material.

Enrichment

Encourage students to work with a partner to review the chapter. They could practise by selecting a function and a transformation, and then describing the new graph verbally without the use of technology or pencil and paper.

Gifted

Provide students with this scenario: Suppose a software engineer was writing a program to determine the inverses of a formula. Create a flowchart that shows a process for finding the inverse of functions where the inverse
- is a function and there are no restrictions
- is a function and there are restrictions
- is not a function and there are no restrictions
- is not a function and there are restrictions

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 1 Review The Chapter 1 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #1, 4–7, 9, 10, 12–17	• Students may wish to work in pairs. • Students completing the assigned questions can move forward and complete the remaining questions. • Encourage students to use **BLM 1–6 Chapter 1 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 1 Study Guide This master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. The minimum questions suggested are questions that students should be able to confidently answer. Encourage students to try additional questions. • Consider allowing students to use any summative charts, concept maps, or graphic organizers when completing the practice test.
Assessment *of* Learning	
Chapter 1 Test After students complete the practice test, you may wish to use **BLM 1–7 Chapter 1 Test** as a summative assessment.	• Before the test, coach students in areas in which they are having difficulty. • You may wish to have students refer to **BLM 1–6 Chapter 1 Study Guide** and identify areas they need reinforcement in before beginning the Practice Test.