Polynomial Functions

Opener

Pre-Calculus 12, pages 104–105

Suggested Timing
45–60 min

Blackline Masters
BLM 3–1 Chapter 3 Prerequisite Skills
BLM U1–1 Unit 1 Project Checklist

Planning Notes

In this chapter, students look at how to graph polynomials based on a variety of transformations being applied, and how to determine the equation of a polynomial given its graph.

Section 3.1 introduces the characteristics of polynomial functions. Students examine what makes a function a polynomial, and look at matching polynomials with their graphs.

Section 3.2 deals with the remainder theorem, which is the building block of the factor theorem. Students look at long division of polynomials and the shorthand version, called *synthetic division*.

In section 3.3, students work with a special case of the remainder theorem, when the remainder is zero. The factor theorem, coupled with synthetic division, is key in factoring polynomials of any degree.

Section 3.4 deals with the graph of polynomial functions. Students learn to analyse various parameters within the polynomial function to understand key points on the graph. Students also apply their knowledge and skills to determine the equation of a given graph of a polynomial.

Begin the chapter by reading out the Key Terms. Ask students which terms they can already define. Can students give an example of each to illustrate their understanding? What do they know about polynomials? Ask them what they recall from Chapter 1, since transformations are very important in this chapter. Do they remember what it means to factor? What types of polynomials can they factor? Are they comfortable using graphing technology to graph functions and find the important parts? Can they determine the zeros of a function? Can they use the table and window functions? These are items that you may wish to review with students prior to beginning the chapter.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter. Encourage students to use a summary method of their choice.

Meeting Student Needs

- Consider having students complete the questions on **BLM 3–1 Chapter 3 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out **BLM U1–1 Unit 1 Project Checklist**, which provides a list of all the requirements for the Unit 1 Project.
- Provide students with a copy of the student learning outcomes for the unit.

Enrichment

Ask students to consider how the development of faster processors may reduce the number and severity of vehicle accidents. Have them speculate how polynomials such as $d = \frac{1}{2}at^2$, $d = vt$, and $f = ma$ (d is distance, v is velocity, a is acceleration, t is time, f is force, and m is mass) can be used to create safer software for these processors. Challenge students to describe a vehicle whose processor package makes the vehicle effectively collision proof.

Gifted

Have students focus on the polynomials in the Enrichment section above. Ask how the polynomials can be used to avoid rear-end collisions.

Career Link

Discuss with students what they know about computer engineering. Ask
- What do you know about computer engineering?
- What do you think computer engineers study?
- How do you think computer engineers might use mathematics and, in particular, polynomial functions?

Encourage students to go online to research careers in computer engineering and learn how computer technology is being integrated into an increasing range of products and services.

3.1 Characteristics of Polynomial Functions

***Pre-Calculus 12*, pages 106–117**

Suggested Timing

90–120 min

Materials

- graphing technology

Blackline Masters

BLM 3–2 Section 3.1 Investigate Graphs of Polynomial Functions
BLM 3–3 Section 3.1 Extra Practice

Mathematical Processes for Specific Outcomes

RF12 Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5).

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–4, 6, 7
Typical	#1–5, 7–9, 10, C1–C4
Extension/Enrichment	#9–13, C1–C4

Planning Notes

Review with students how to graph functions on their calculator. Window settings are also worth reviewing so students can see the end behaviour of the graph.

Investigate Graphs of Polynomial Functions

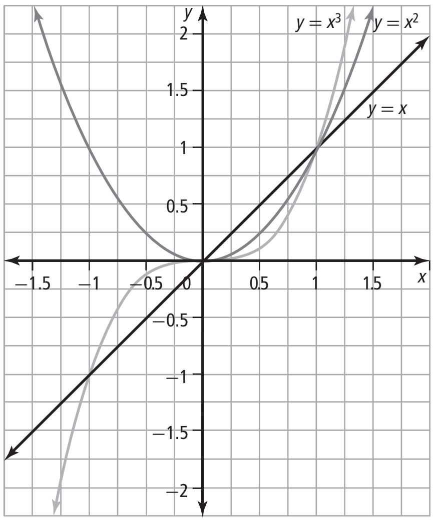
Have students graph each equation in the set and label their sketch with the equation. Make sure they do all the equations in set A before moving on to the equations in set B. This will make it easier to compare the graphs to one another later in the investigation. Ask students how the degree of the function is related to the type of function. Ask them to describe and sketch the characteristic shape for each type of graph in set A. Ask them to predict what will happen to graphs in set B once the leading coefficient is negative. Ask students to summarize the effect of adding or subtracting a constant to each of the functions as in set C. In set D, ask students what they know about the graphs before they graph them. Can students predict the end behaviour of a graph based solely on the degree? Ask them what the difference between an odd degree and even degree is.

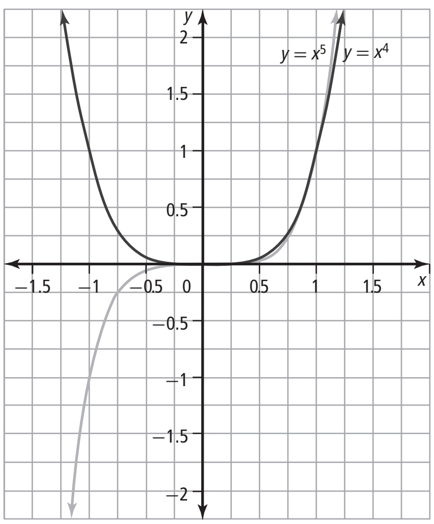
Meeting Student Needs

- To save time, students could enter all the equations in their graphing calculator and graph them one at a time by turning the function on and off. Once the equations are entered, adjusting them set by set will be easier.
- Some students may benefit from having a handout with the graphs already sketched.
- Provide students with **BLM 3–2 Section 3.1 Investigate Graphs of Polynomial Functions** to help them organize their answer to step 2 of the investigation.
- Students may find it helpful to organize their responses to #5 to 8 in a chart using headings such as: Degree, Sign of Leading Coefficient, Constant, x-intercepts.
- Students may be interested to learn that there are some French beekeepers in western Canada as indicated in the Weblink on page 59 in this Teacher's Resource.

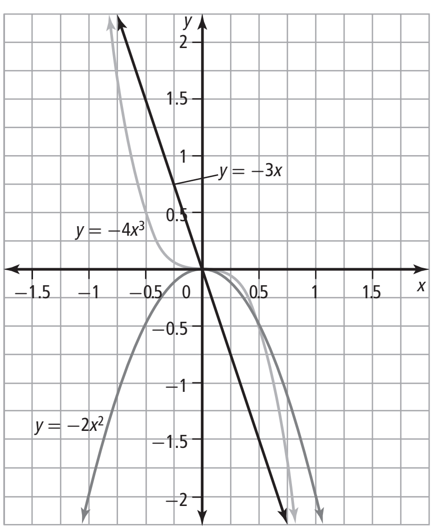
Investigate Graphs of Polynomial Functions

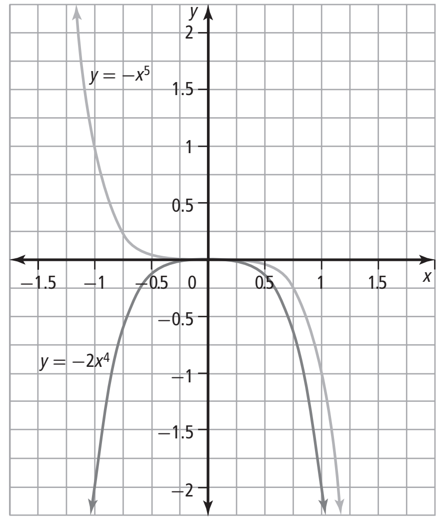
1. Set A



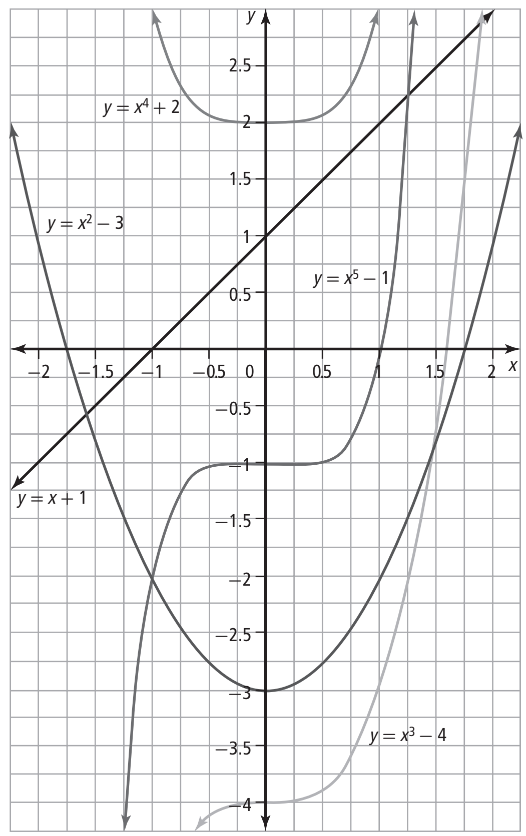


Set B

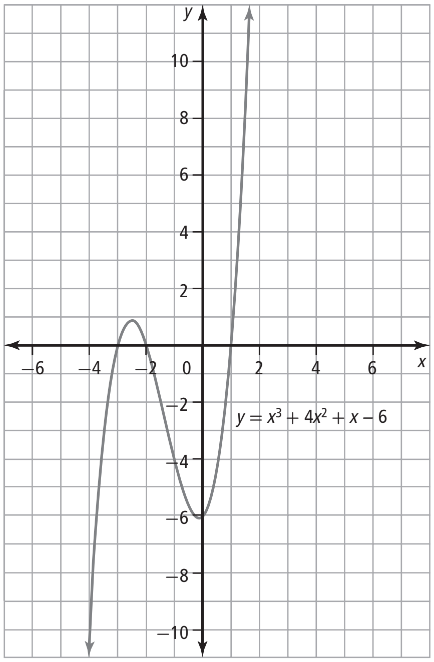


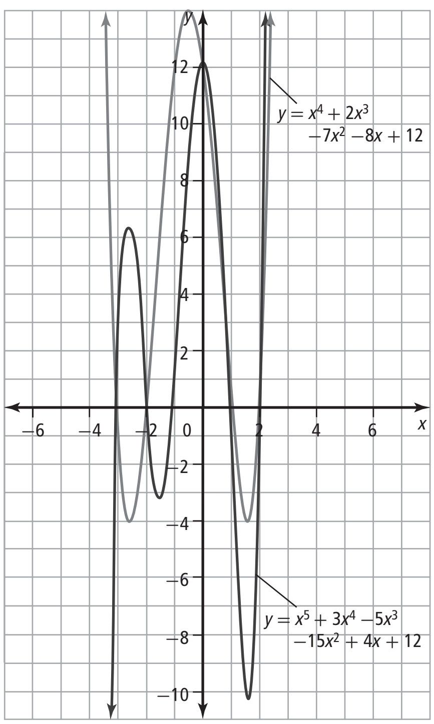


Set C



Set D





2. Example:

Set	Function	End Behaviour	Degree	Constant Term	Leading Coefficient	Number of *x*-intercepts
A	Linear	up into Q1 and down into Q3	1	0	1	1
	Quadratic	up into Q1 and up into Q2	2	0	1	1
	Cubic	up into Q1 and down into Q3	3	0	1	1
	Quartic	up into Q1 and up into Q2	4	0	1	1
	Quintic	up into Q1 and down into Q3	5	0	1	1
B	Linear	up into Q2 and down into Q4	1	0	−3	1
	Quadratic	down into Q3 and down into Q4	2	0	−2	1
	Cubic	up into Q2 and down into Q4	3	0	−4	1
	Quartic	down into Q3 and down into Q4	4	0	−2	1
	Quintic	up into Q2 and down into Q4	5	0	−1	1
C	Linear	up into Q1 and down into Q3	1	1	1	1
	Quadratic	up into Q1 and up into Q2	2	−3	1	2
	Cubic	up into Q1 and down into Q3	3	−4	1	1
	Quartic	up into Q1 and up into Q2	4	2	1	0
	Quintic	up into Q1 and down into Q3	5	−1	1	1
D	Quadratic	up into Q1 and up into Q2	2	−2	1	2
	Cubic	up into Q1 and down into Q3	3	−6	1	3
	Quartic	up into Q1 and up into Q2	4	12	1	4
	Quintic	up into Q1 and down into Q3	5	12	1	5

3. Example: Set A has only the basic function with coefficients of 1 and no constants. Therefore, the graphs are all the basic shapes. Set B shows how the leading coefficient changes the steepness and orientation of the graphs. Set C demonstrates changing constants and how they change the position of the graph. Set D shows the functions with many coefficients and a constant, demonstrating how the size and shape of the graphs can be transformed.

4. $y = x$: The cubic graphs from sets A, C, and D are similar since they tend to increase. The quintic graphs from sets A, C, and D are similar since they also tend to increase.
$y = -x$: The cubic graph and the quintic graph from set B are similar to this graph since they tend to decrease.
$y = x^2$: The quartic graphs from sets A and C are similar since they have minimum values and open upward. The quartic graph from set D is less similar but does have a minimum value.
$y = -x^2$: The quartic graph from set B opens downward and has a maximum value.

5. a) All three functions have an odd degree. Their graphs have end behaviour that is opposite, i.e., either positive on the left side and negative on the right side, or negative on the left side and positive on the right side.

b) Both types of functions have an even degree. Their graphs have end behaviour that is the same on both sides, i.e., either positive on both the left and right sides, or negative on both the left and right sides.

c) If the degree of the function is odd, the end behaviour is opposite on the left and right sides of the graph. If the degree of the function is even, the end behaviour is the same on the left and right sides of the graph.

6. The leading coefficient dictates the steepness as well as the orientation with respect to the *x*-axis.

7. The constant term determines where the graph will intersect the *y*-axis.

8. The maximum number of *x*-intercepts is determined by the degree of the function. The minimum number of *x*-intercepts is zero for functions with an even degree. The minimum number of *x*-intercepts is one for functions with an odd degree.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to make a comparison chart and complete #5 as a class. • Students may need to be prompted for #6 if they are having difficulty understanding what is asked for. Ask them to describe how the answers in set B differ. Similarly for #7, ask how sets C and D differ. • It is important that students have a good understanding for their response in #8. Consider having several students give responses or completing #8 as a class.

Example 1

Ask students what makes a function a polynomial. What are the key characteristics? For the function in part a), you may wish to ask students how to change from radical to exponential form.

For part a) of the Your Turn, ask students to write a reciprocal function in exponential form. Students may struggle with this concept and need review.

Example 2

Ask students why the number of x-intercepts cannot be more than the degree of the function. How can you quickly determine the y-intercept of the function? Why do functions with an odd degree not have a maximum or minimum value?

Example 3

Ask students to explain how to determine the volume of an object given its width. Do they know another way to determine the value using graphing technology? Can they use their table function? Could they use a system of equations to determine the answer? Which method do they prefer? Ask students why the least volume would be a width of zero. Within the context of this question, why is $w = 0$ unlikely? Ask them why negative values are not appropriate. Can you determine the maximum volume? Why is this impossible to do?

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Students may think that $g(x) = \sqrt{5}x + 3$ is not a polynomial.

Rx Review with students that the variables must have integral exponents but the coefficients may contain radicals.

- Students may think that $g(x) = \sqrt{x^6}$ is not a polynomial.

Rx Ask students to place this function in exponential form. Ask them if the exponent $\frac{6}{2}$ can be simplified. Then, they will see that the function is indeed a polynomial.

Answers

Example 1: Your Turn

a) No. It is a rational function.

b) Yes. The degree is 5; the leading coefficient is −2; the constant term is 4.

c) Yes. The degree is 4; the leading coefficient is −4; the constant term is 3.

d) No. This is a square root function.

Example 2: Your Turn

a) The curve extends up into quadrant II and down into quadrant IV. There is a possibility of three x-intercepts, one y-intercept, and the graph has no maximum or minimum values.

b) C

Example 3: Your Turn

a) 1430 in.^{3}

b) Since this is a cubic, the smallest volume would be for a negative height, which is impossible. The smallest volume you could get would be when height is 0 in.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• For students having difficulty, provide a polynomial such as $y = 2x^2 + 4x - 1$ as a sample that you can prompt them through before they complete the Your Turn. • You may wish to have students refer to the example and complete one characteristic at a time. For example, identify the degree in the example. Then, identify the degree for #a)–d). • Direct students to Link the Ideas for a quick summary of each of the parameters.
Example 2 Have students do the Your Turn related to Example 2.	• Remind students of the work they did with transformations when describing the effects of the leading coefficient. • The chart provided before Example 2 should assist students in answering the questions. You may wish to have them verbalize the information in the Degree 3 column, and have them explain the meaning of the information to ensure they understand and can proceed.
Example 3 Have students do the Your Turn related to Example 3.	• Suggest to students that they could also use their graphing technology's table function in Method 1.

Check Your Understanding

For #3, ask students which characteristic of the graph they examine to determine whether the degree of the polynomial function is even or odd. What feature of the graph indicates whether the leading coefficient is positive or negative?

For #5, suggest to students to substitute in a few values for a, b, and n to help them in their answer.

For #6, ask students to think about parameters for a, b, and n within the context of the question. Why is the constant negative? What does that mean? Within the context of the question, the variable cannot equal which values? Why?

For #8, ask students if they can determine another equation to model the situation.

For #9, how can you determine the current population of the town? Ask students why the polynomial function might not be a realistic model for the town's population.

For #10, ask students to explain how to factor a cubic function. Can they see a common factor? What restrictions are implied in this question?

For #11, how does the negative leading coefficient change the graph? Which features of the graph change?

For #12, how does each parameter affect the function? What transformations occur for each value? Can students apply their knowledge to a cubic function?

Meeting Student Needs

- You may wish to have students complete the Project Corner at the end of section 3.1 with a partner. Encourage them to share their methods. This will assist students in working through the Unit 1 Project Wrap-Up.
- Students should complete the checklist of understanding provided in the opener of this section. This will guide them when deciding which questions they need to practice to gain a deeper understanding of the necessary outcomes for this section.
- Provide **BLM 3–3 Section 3.1 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–4, 6, and 7. Students who have no problems with these questions can go on to the remaining questions.	• Ensure that students can identify which factor indicates that the equation in #1 is not a polynomial function. • Remind students that the value of 1 can be placed in front of any variable as its leading coefficient. You may wish to ask students what x^0 might be associated with. • For #3, 4, 6, and 7, students may wish to refer to the tables provided after Example 1. • Students should be proficient at identifying the degree, leading coefficient, and constant term of polynomial functions, since these are covered in detail in grade 9.
Assessment *as* Learning	
Create Connections Have all students complete C1, C2, and C4.	• Students may find it easier to describe and summarize their answers for C1 and C2 with a visual. Encourage them to draw a graph if needed. C1 and C2 would be good responses to include in a summative graphic organizer for future reference. • Encourage students to use correct terminology in their descriptions, especially for C1.

The Remainder Theorem

3.2

***Pre-Calculus 12*, pages 118–125**

Suggested Timing

90–120 min

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BLM 3–4 Section 3.2 Extra Practice

Mathematical Processes for Specific Outcomes

RF11 Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients).

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2, 3a), b), 4a)–c), 5a)–c), 6, 8a), b), 9, 11
Typical	#2, 3c), d), 4d)–f), 5d)–f), 6, 7a), b), 8c), d), 10, 12, C1–C3
Extension/Enrichment	#7, 10, 13–17, C1–C3

Planning Notes

Use a visual demonstration to introduce this section. Display a rectangular prism at the front of the room. Write the prism's three dimensions on the whiteboard and ask students to calculate the volume. What process did they use? Then, use three monomials and ask students to calculate the volume. What polynomial resulted? Now, use a combination of monomials and binomials for dimensions. What polynomial resulted?

Investigate Polynomial Division

Ask students to describe the long division process step by step to ensure they understand how the values or expressions at each step are determined. For #1b), ask students if they are looking for groups of x or groups of $x + 3$. See if they can make the connection between the leading coefficient of $x + 3$ in #1b) and the 12 in #1a). For #4, how do you know which value to substitute? If the binomial is $x - 3$, do you substitute $x = 3$ or $x = -3$? If the binomial is x, what value would you substitute?

Meeting Student Needs

- Students could have a "relay" race. Divide the class into teams of four. On the board, write a long division question such as 154 236 396 ÷ 6. One student in each team writes out the question on a sheet of paper and completes the first step of the division. The sheet is passed to another team member to complete the second step, and so on until the question is answered. Ask students how they could predict that there would not be a remainder (divisibility rule for 6).
- An alternative activity is to write several questions on poster paper and display them in the classroom. Each student starts at one station. When instructed to begin, each student completes the first step and then moves to the second question. This process continues until all questions are answered. If used after the Investigation, the polynomials in #4 and #7a) may be used as well.
- Students may decide to make a comparison chart for #2c).

Common Errors

- When determining the remainder of $x^3 + 2x^2 - 5x - 6$ when it is divided by $x - 3$, some students may substitute $x = -3$ into the polynomial to determine the remainder.

R_x Have students equate the binomial to zero and solve for x. They are trying to find the potential zeros, so a factor of $x - 3$ has a zero of $x = 3$.

Answers

Investigate Polynomial Division

1. a) dividend: 327; divisor: 12; quotient: 27; remainder: 3
b) dividend: $x^2 + 7x + 17$; divisor: $x + 3$; quotient: $x + 4$; remainder: 5

2. a) Example: Start by choosing a number that multiplies the dividend so that it is less than or equal to 32. Multiply 12 to get 24. Then, subtract this number from 32 to get 8. Since 12 cannot divide 8, you drop the 7 and repeat the previous process until you can no longer drop numbers and divide.
b) Example: Follow steps similar to those in part a), but instead of putting the values directly beside each other you drop a plus sign between them. When you bring down a value, you also drop a plus sign between the values.
c) They are exactly same except that in part b), you add plus signs instead of nothing.

3. Example: An easy way to check is to multiply the quotient and the dividend. Then, add the remainder to the resulting number. If you do not get the divisor, either the long division is wrong or you multiplied it incorrectly.

4.

Polynomial Dividend	Binomial Divisor $x - a$	Quotient	Remainder	Substitute $x = a$ into the Polynomial
$x^3 + 2x^2 - 5x - 6$	$x - 3$	$x^2 + 5x + 10$	24	24
	$x - 2$	$x^2 + 4x + 3$	0	0
	$x - 1$	$x^2 + 3x - 2$	−8	−8
	$x + 1$	$x^2 + x - 6$	0	0
	$x + 2$	$x^2 - 5$	4	4

5. The remainder and a are the same.

6. To find a remainder, replace x with a.

7. a) i) −15 **ii)** −105 **iii)** 0

8. The remainder of $\frac{P(x)}{x - a}$ and $P(a)$ are the same.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Some students may need coaching with the long division process, since it often challenges students. You may wish to complete an example for #2b) that students could follow to write their response. • Remind students that they could solve #7a) with substitution. They should also realize that $P(a)$ is the same value as $\frac{P(x)}{x - a}$.

Example 1

Ask students why they think it is so important to write the polynomial in descending powers of x. Could they write it in ascending powers of x? If there are terms missing, what should they do? Why is the remainder written as a fraction of the divisor?

Example 2

When you divide the volume by the height, what are you solving for? How does the area of the base help you determine the missing dimensions? What would happen if the remainder were not zero? What would happen if the expression could not be factored? What would the restrictions on the variable be? Do the restrictions make sense in the context of this question? Are there additional restrictions?

Example 3

Ask students to explain why the value that is used is +3. Why would you not use −3? If you did use −3, could you adapt the division process to account for that? Try the adapted process for −3 to see if you arrive at the same answer. Is there another way to check that does not involve long division?

Example 4

Ask students to explain why the value −4 is used for the remainder theorem but +4 is used for synthetic division.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Provide students with several questions to practice subtraction. For example, use $3x - (-10x)$ and $5x^3 + 2x^3 - (5x^3 - 7x^2)$. What is the correct process? Do students remember that to subtract means to "add the opposite"? Have them practice several questions, calculating the difference mentally.
- For Example 2, have students work through a simpler question involving numbers only. Give the students a volume such as 432 in.^{3} with a height of 4 in. What are the possible values for length and width? Ask them to describe the method used.

- Encourage students to make the connection that a number is a factor of another number when it divides evenly with a remainder of zero. For example, 2 and 3 are factors of 6 because $6 \div 2 = 3$, with a remainder of zero, and $6 \div 3 = 2$, with a remainder of zero. The same is true for polynomials.

Common Errors

- Students may confuse the zeros used for the remainder theorem with the values used for synthetic division. For example, in dividing $2x^3 + 3x^2 - 4x + 15$ by $x + 3$, students might want to use +3 for the remainder theorem and −3 for synthetic division.

R_x Use the zeros for each method to avoid confusion. Use $x = -3$ for the remainder theorem and for synthetic division, but add the terms instead of subtracting.

Answers

Example 1: Your Turn

a) $\frac{x^4 - 2x^3 + x^2 - 3x + 4}{x - 1} = x^3 - x^2 - 3 + \frac{1}{x - 1}$

b) $x \neq 1$

Example 2: Your Turn

The height and width could be $x + 8$ or $x - 1$.

Example 3: Your Turn

$x^2 + 9x + 15$; remainder 34

Example 4: Your Turn

−298

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Remind students to divide until the remainder is of a lesser degree than the divisor. • You may wish to have a student show their verification at the board to ensure that students who are having difficulty have a model to follow. Ensure they understand the importance of verifying and stating any restrictions on the variable.
Example 2 Have students do the Your Turn related to Example 2.	• Remind students to divide until the remainder is of a lesser degree than the divisor. • You may wish to have a student show their verification at the board to ensure that students who are having difficulty have a model to follow. Ensure they understand the importance of verifying and stating any restrictions on the variable. • It is important that students check the divisor to determine whether it can be further factored.
Example 3 Have students do the Your Turn related to Example 3.	• Remind students that the polynomial needs to be written in descending order and that a zero is to be used for the coefficient of any missing powers. • Students may need additional coaching in determining the answer when completing the division using synthetic division. They must learn to work from right to left, beginning with the remainder.
Example 4 Have students do the Your Turn related to Example 4.	• Allow students to select the method they prefer when verifying their answer. Have them verbalize the reasons for their choice. • Most students prefer synthetic division because it is faster and takes less room, but ensure that they are able to solve long division questions also (or vice versa). Students should be able to identify an error made in the process of either method.

Check Your Understanding

For #8, ask students which method will help to determine the value of *k*. Can you use the remainder theorem? Can you use synthetic division? If the remainder is given, where does it fit into each method?

In #9, if the remainders are the same, ask students if they can determine an equation to show that. Do you need to know what the remainder is to solve the question? Can you determine the remainder?

In #10, students should determine that there are two values for *k*. Why are there two possible values?

For #11, ask students if they can sketch a diagram to represent the rectangle and the dimensions that are given. How can you determine the width of the rectangle? Is there another way to solve this problem?

In #13, if the cylinder's volume is divided by its height, what does the result represent? How will this answer help you determine the volume of the new container? How many possible values does *x* have?

For #14, ask students what they create if they determine the equation for each remainder. How can you solve a system of linear equations? Which method is best? How can you verify your answer?

In #16, what value would you use to start the division process? Can you check your remainder using the remainder theorem? What value would you substitute?

In #17, ask how you can work backward to determine the polynomial. Is there more than one answer for each question? How can you check that the polynomial satisfies each set of conditions?

Meeting Student Needs

- Students could be grouped for the assignment. Each student could be given one question to complete and then work through the solution with the other members of the group. Mini whiteboards are an excellent tool for this. If whiteboards are not available, then supply students with poster paper. Encourage other students to write comments for the solutions on the poster paper and display in the classroom. Not all solutions need to be displayed, and both correct and incorrect solutions could be posted and discussed. Students will learn from looking at the "What Not To Do" solutions. Of course, only post solutions with student permission.
- For #8, model one solution on the whiteboard.
- For #11 and 12, students may wish to use algebra tiles.
- Provide **BLM 3–4 Section 3.2 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2, 3a), b), 4a)–c), 5a)–c), 6, 8a), b), 9, and 11. Students who have no problems with these questions can go on to the remaining questions.	• Questions #1 to 3 deal with long division. Students are prompted in #1 to identify restrictions and verify their work. Although not explicitly stated in #3, students should retain the same process and make the same connections without being asked. The same holds true for the synthetic division in #4 and 5. • Remind students that the remainder theorem requires them to set the divisor equal to zero and then substitute or evaluate the polynomial. Coach students how to set up #8, since the remainder is given. Ask them to verbalize the differences between #6 and 8. • Students may find it beneficial to draw and label a diagram for #11. Ask students who are having difficulty how they would solve this problem if the area was 8 and the length was 4. How can this be used to solve #11?
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• Discuss the answer to C1 as a class. Then, have students include the solution in their notebook or graphic organizer.

The Factor Theorem

3.3

***Pre-Calculus 12*, pages 126–135**

Suggested Timing
60–90 min

Materials
• graphing technology

Blackline Masters
BLM 3–5 Section 3.3 Extra Practice

Mathematical Processes for Specific Outcomes
RF11 Demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients).

☑ Communication (C)
☑ Connections (CN)
☑ Mental Math and Estimation (ME)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2a), b), e), 3a), b), e), 4, 5, 6a), b), 7–9, 11
Typical	#1, 2c), d), f), 3c), d), f), 4, 5, 6c)–e), 7, 9, 10, 13, C1–C3
Extension/Enrichment	12–16, C1–C3

Planning Notes

Before beginning this section, discuss the student learning outcomes for this section, or provide these in a handout. Discuss with students what they have learned in previous sections, especially in respect of factoring polynomials. Encourage students to develop a list of the terms and definitions found within this section.

Investigate Determining the Factors of a Polynomial

In Part A, how can you determine the quotient? What method do you use? If you know the remainder is zero when the polynomial is divided by $x + 1$, what does this mean about $x + 1$?

In Part B, how can you factor the cubic function? How can you determine if a possible factor is an actual factor? It is important that students see the connections between the zeros of the function and the factors. Make sure they have a good grasp of this before moving on to the examples.

Meeting Student Needs

- Students may start by determining if a given number is a factor of a larger number by dividing. If it is, they will discover that the remainder is zero. For example, is 23 a factor of 299? When 299 is divided by 23, the result is 13 with a remainder of zero.
- For #3, have students graph $P(x)$ using technology. What do they notice about the x-intercepts and the given factors?

Common Errors

- Students may make mistakes substituting values into the equation, especially when the substitution involves negative values. Example:

  $x^2 - x + 3$ divided by $x + 1$
  $P(-1) = -1^2 - 1 + 3$
  $P(-1) = -1 - 1 + 3$
  $P(-1) = 2$

R_x Encourage students to use brackets for all substitutions. This will avoid many mistakes, especially the double negatives. Example:

  $x^2 - x + 3$ divided by $x + 1$
  $P(-1) = (-1)^2 - (-1) + 3$
  $P(-1) = 1 + 1 + 3$
  $P(-1) = 5$

Answers

Investigate Determining the Factors of a Polynomial

1. a) 0
b) $x^2 + x - 6$; $x^3 + 2x^2 - 5x - 6 = (x + 1)(x^2 + x - 6) + 0$
c) $(x + 3)(x - 2)$
d) $(x + 3)(x - 2)(x + 1)$
e) The remainder is always 0.

2. The remainder is 0 when a polynomial is divided by one of its factors.

3. $(x - 1)$, $(x - 2)$, $(x + 3)$; Replacing x with a, it is easy to determine when the remainder is 0 and thus a factor.

4. A divisor is a factor when the remainder is 0. Thus, $P(a) = 0$.

5. The zeros or x-intercepts are the a values that are used in the remainder theorem.

6. a) Example: Guess at a factor. Then, use division to determine the others.
b) $(x - 1)(x + 2)(x + 1)$

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• It is important for students to do #2, since it sets the foundation for this section. Discuss the responses as a class and ensure students realize that a factor will give a remainder of zero. • You may wish to have students create a chart for #5 to help organize their responses. This is important information to include in their notebook or graphic organizer.

Example 1

Ask students why it is useful to determine the factors of a polynomial. Why do you think these binomials were chosen? What are the possible zeros of the polynomial, given the possible factors? What do these zeros have in common with the constant term? Ask students how the degree of the function is related to the number of zeros. Ask them how the zeros of a function and its x-intercepts are related.

Example 2

Once you know the values to test, is there a way to test them using graphing technology? Can the table function help? How could you use the trace function? Would it be useful to determine the zeros of the function using technology? How are the zeros and factors of a function related? Is there another way to check that you factored the expression correctly?

Example 3

Is there a way to narrow down the factors to try? Can you determine which factor sets the polynomial to zero in another way? Which method do you prefer to use to factor the cubic function? If there are two $x - 2$ factors, how does this affect the number of x-intercepts? Based on the factored form, how many x-intercepts does this polynomial have? What are they?

Example 4

Have students explain why it is necessary to factor the polynomial to determine the dimensions. How does the factored form of the polynomial relate to the formula for volume? Using graphing technology to determine the zeros of the function makes factoring quick and easy. What happens if the zeros are not whole numbers? How would you factor then? If one of the roots was $\frac{3}{5}$, what would the factor be?

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Organize students into small groups. Assign each group a polynomial to factor completely. Within each group, students can use the integral zero theorem and the factor theorem to find the factors of the given polynomial. Each student should then choose a different factor to begin. Use synthetic division and further factor theorems to determine all the factors. Compare answers with the other students in the group.
- For Example 3, Method 2, emphasize that grouping only works with polynomials with four terms.
- For Example 4, Method 1, work through the use of a spreadsheet on the computers available. How are the values in the second column calculated?

Common Errors

- Students may forget the initial factors when factoring a higher degree polynomial, or forget to record double roots.

R_x If the polynomial can be fully factored, the number of factors should equal the degree of the polynomial.

- Students may determine the zeros but list them as the factors, forgetting to change the signs.

R_x Remind students that $P(x) = 0$ tests potential zeros, not potential factors. To determine the factors, they must state the factor and then set the equation equal to zero.

Answers

Example 1: Your Turn

$(x - 2)(x + 3)(x - 5)$

Example 2: Your Turn

a) $(x + 5)(x - 2)(x + 1)$

b) Example: The zeros are numbers that make each factor 0.

Example 3: Your Turn

$(x - 3)^2(x + 2)(x + 1)$

Example 4: Your Turn

The dimensions are $(x + 3)$, $(x + 2)$, and $(x + 2)$.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Remind students that a factor written $(x - a)$ is evaluated as $P(a)$. • Have students compare their answers with those of a partner.
Example 2 Have students do the Your Turn related to Example 2.	• Remind students that possible integral zeros are ±. • Suggest that students always begin with the easiest factors, such as ±1and ±2.
Example 3 Have students do the Your Turn related to Example 3.	• Suggest to students that they may use a combination of methods to determine the factors of polynomials of higher degree. Graphing may be a quick way to isolate some of the zeros. • Encourage students to approach the question with the method that they are most comfortable with. Challenge them to try more than one way. • Some students may need coaching and a review of factoring by grouping.
Example 4 Have students do the Your Turn related to Example 4.	• Encourage students to try to solve questions using more than one method. • If students are using a calculator, remind them that a table can be read for the initial zeros as well.

Check Your Understanding

For #1, check that students are correctly stating the corresponding binomial factor. You could ask them to do the reverse to reinforce the concept: Given a factor, how can you determine the zero?

In #2, ask students what value they will substitute into the polynomial. What methods could they use to determine if $x - 1$ is a factor?

In #5 and 6, have students explain how to check that the polynomials are factored correctly. How can determining the x-intercepts of each polynomial help you factor?

For #11, how does knowing one dimension help you determine the other dimensions? Could you determine them without the given dimension? Within the context of the question, are there any restrictions on the variable?

In #12, is it possible to place restrictions on the variables to ensure there are no weak pieces?

For #14, have students consider the following: If the polynomial is divisible by $x - 1$, what is the zero of the function? What can you do with the zero of the function? What do you know about the coefficients if their sum is zero?

In #16, are there terms missing from any of the polynomials? How do you account for missing terms when you use synthetic division? Do you have to record the missing terms as zero? What is the result if you do not?

Meeting Student Needs

- Create ten learning stations using questions from this section. Students can rotate through the learning stations in pairs, discussing the possible solutions.
- When finished the questions, students can be given an "exit slip" asking them to list, on their own, the Key Ideas from this section.
- Students can complete a compare and contrast chart for the remainder theorem and the factor theorem.
- To build on #9, students could measure the length of the classroom and then determine the width and height in relation to the length. Calculate the polynomial that would represent the volume of the classroom. For example, if the length is 10 m, the width is 7 m, and the height is 3 m, the dimensions would be x, $x - 3$, and $x - 7$, respectively. So, the polynomial would be $V(x) = x^3 - 100x^2 + 21x$. This will develop an understanding of how these types of equations are developed.
- Provide **BLM 3–5 Section 3.3 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2a), b), e), 3a), b), e), 4, 5, 6a), b), 7–9, and 11. Students who have no problems with these questions can go on to the remaining questions.	• For #1, you may wish to prompt students by starting with a factor and reminding them how to obtain the zero ($x - 2 = 0$ becomes $P(2) = 0$). • Questions #2 and 3 are the opposite of #1. Therefore, students should have no difficulty in determining the zero. If students are having difficulty, you may wish to use the example above. Remind students that a zero makes the equation equal to zero. This should also assist with #7. • If students are having difficulty with #4, complete the first question as a class while students verbalize the potential integral zeros. Then, using the steps from the previous questions, identify one of the zeros. Remind students that quadratics can often be factored. This should help to determine the dimensions in #11. • Remind students to look for common factors as a part of their factoring. This will assist with #8 and 9.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• C1 provides an opportunity for students to explain their thinking when using zeros on a graph. Ensure students are able to completely explain how they can obtain the integral zero and why this is a zero before they go on to C2. • Encourage students to explain as many ways as they can to identify possible factors. They may wish to draw on their discussion in C1. • If students are confused by C3, have them identify each process listed separately. Ask them to describe how that process helps to determine a zero. Have them look for commonalities or points at which one method finishes and another begins. You may wish to do an example question with them so they can visualize each of the processes at work.

Equations and Graphs of Polynomial Functions

3.4

***Pre-Calculus 12*, pages 136–152**

Suggested Timing
90–120 min

Materials
• graphing technology

Blackline Masters
BLM 3–6 Section 3.4 Extra Practice

Mathematical Processes for Specific Outcomes
RF12 Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5).

✓ Communication (C)
✓ Connections (CN)
✓ Technology (T)
✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–5, 7, 8, 10–12
Typical	#3, 4, 6–10, two of 13, 15–18, C1, C2
Extension/Enrichment	#6, 19–23, C1–C4

Planning Notes

Discuss the outcome and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section. Review the zero principle with students before beginning the Investigate.

Investigate Sketching the Graph of a Polynomial Function

Once you have factored a polynomial, how can you determine the zeros? If the product of two or more factors is zero, which factor is zero? Since you do not know which factor is zero, what must you do? Once you know the number of x-intercepts, how can you determine the number of intervals the graph has been divided into? What effect does the sign of the leading coefficient have on the end behaviour of the arms? Does the degree of the function also tell us something about the end behaviour of the arms?

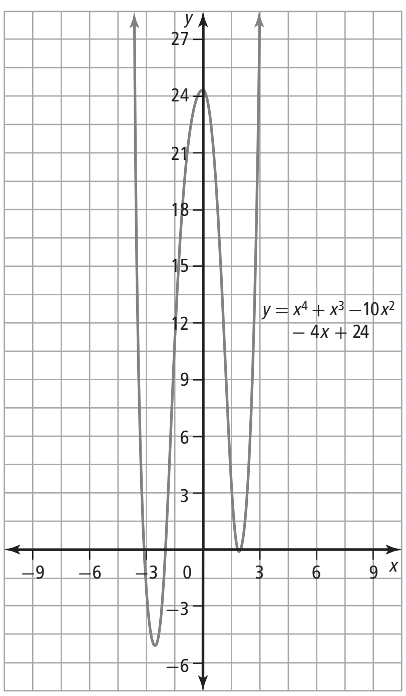
Meeting Student Needs

- Prepare sketches of the polynomial functions used in the investigation. Display them in the classroom for students to view.

Answers

Investigate Sketching the Graph of a Polynomial Function

1. a)



b) $x = -3, -2, 2$
c) $(x + 3)(x + 2)(x - 2)^2$

2. a) $x = -3, -2, 2$
b) The roots of the equation are the x-intercepts.

3. The zeros, x-intercepts, and roots of the polynomial are the same.

4.

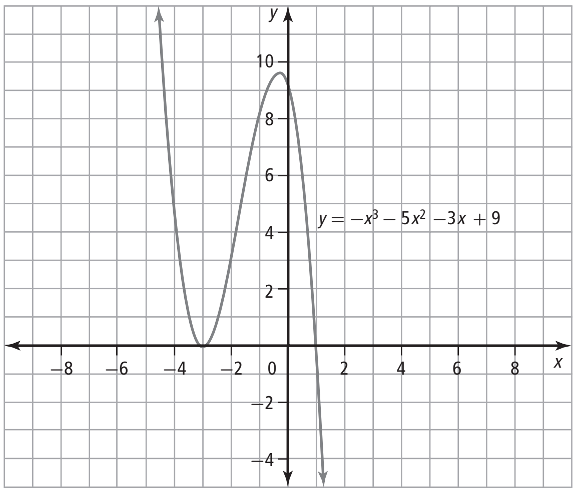
Interval	$x < -3$	$-3 < x < -2$	$-2 < x < 2$	$x > 2$
Sign of $f(x)$	Positive	Negative	Positive	Positive

5. a) It changes sign.
b) When there is a double zero the function does not change sign.

6. The degree is 3; the leading coefficient is −1; the zeros are 1, −3; the y-intercept is 9. The function is positive over $x < -3$ and $-3 < x < 1$; the function is negative over $x > 1$.

Answers

7.



8. Example: Start with the intercepts. Place them on the Cartesian plane. Then, using the leading coefficient and the degree, it is easy to tell if the graph opens upward or downward. Using this knowledge, connect the dots, leaving space for local maximum or minimum values, which occur within the curve.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Discuss the answer to #2 as a class before having students respond to #3. If they did not understand or make a link in #2, they will have difficulty drawing the relationship in #3. • Students may benefit from substituting actual values into the function as it gets close to the *x*-axis. You may also have students use the trace function of their graphing technology and watch as the values change. This will assist with #5. • Question #8 is an excellent summative question. Have students write their response for you to review, or have them share their response with the class for the benefit of classmates who may be unclear about how to obtain the information needed for the sketch.

Example 1

Ask students about the multiplicity of roots. What does the sum of all the multiplicities yield? How does the multiplicity of roots help us graph a function? What happens when the multiplicity of a root is 1? 2? 3? Is it possible to determine the largest possible degree? How are the leading coefficient and the end behaviour of the arms similar to slope of a line segment?

Example 2

Why is it easier to graph a function that is already fully factored? What are the zeros of the function? What is the behaviour of each zero at the x-axis? What is the impact of a positive or negative leading coefficient? Is it easier to determine x-intercepts from a polynomial that has been factored? Is it easier to determine y-intercepts from a polynomial that has been factored? Why is it always a good idea to factor a common factor first? How does factoring a common factor affect the potential zeros of a function?

Example 3

Students may struggle with filling out the table for specific points. They might need coaching to help them make the connection between the parameters and the effects on individual points. Ask them how they think the points given in the first column, $y = x^3$, are determined.

For the second column, $y = (4x)^3$, ask students what x would have to be to keep the same value from the first column. In other words, if x has an initial value of -2, what value would x have to be so that $4x = -2$?

For the third column, $y = -2(4x)^3$, ask students if the value of x has changed from the second column. (It has not.) What has changed? The function in multiplied by -2, so what value must be multiplied by -2?

Have students look at the x value in the fourth column, $y = -2(4(x - 1))^3 + 3$. Since 1 is being subtracted from the x value, what is the value of x so that it remains the same? In other words, if the initial value of x is -0.5, what would x need to be so that $x - 1 = -0.5$? In the fourth column, the function is increased by 3. What value should be increased by 3? For the Your Turn question, can you graph the function without using a table of values? What methods could you use?

Example 4

Why are the factors $5 - x$ and not $x - 5$? What would a factor of $x - 5$ represent? What would a factor of $5 - x$ represent? In Method 1, why do you determine the point of intersection? Why can you not determine the x-intercept of each function? In Method 2, why do you equate the equation to zero?

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Prepare a "bookmark" for students to reference while working through this section. The bookmark could contain the Key Ideas of polynomial functions. For example, the leading coefficient determines the end behavior of a graph: If the coefficient is positive, the graph rises to right and finishes in quadrant 1. If the coefficient is negative, the graph falls to right and finishes in quadrant 4. Include other topics such as degree, zeroes, multiplicity behaviors, and y-intercept.
- Prepare graphs illustrating various degrees, multiplicities of the zeros, positive leading coefficients, and negative leading coefficients. Include the equation below each graph. Students can look at the graphs and discuss the parts of the equations that determine the behaviours illustrated on the graph. Students should create a chart listing equation, leading coefficient, degree, x-intercepts, multiplicities, beginning quadrant, end quadrant, y-intercept.
- Students could work in groups to create the graphs in Example 3. Colour code the graphs.
- Students may wish to create a bookmark containing their solution to part a) of Example 3. It may be useful to refer to this example as they work through other questions.

Common Errors

- Students may confuse the effect of the a parameter.

R_x Ask students how far the points are from the x-axis. For example, points on the graph of $y = 3x^3$ are three times farther from the x-axis than points on the graph of $y = x^3$, so the graph of $y = 3x^3$ is tall and narrow. Points on the graph of $y = x^3$ are one third the distance from the x-axis compared to points on the graph of $y = \frac{1}{3}x^3$, so the graph of $y = \frac{1}{3}x^3$ is short and wide.

- Students may confuse the effect of the b parameter.

R_x You could use a table of values to explain this. Think about the point $x = 2$. For $f(2x)$, x has to be 1 to end up with $f(2)$. Since x has to be 1, it is one half the original value. Therefore each value of x in $f(2x)$ is half the distance from the y-axis compared to the values of x in $f(x)$.

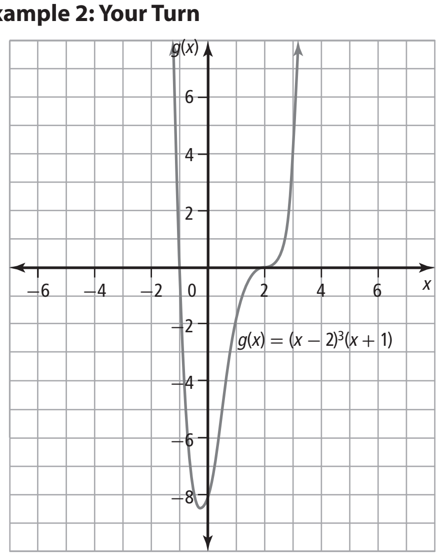
Answers

Example 1: Your Turn

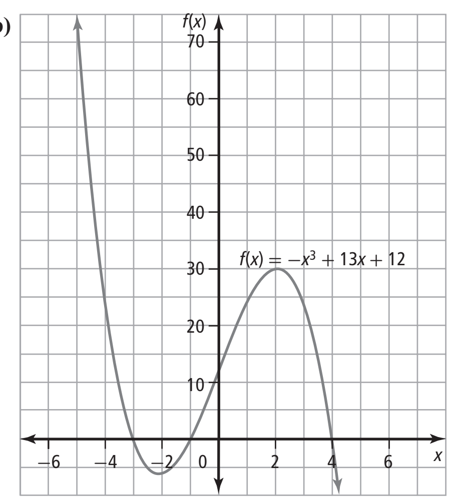
The least possible degree is 3; the sign of the leading coefficient is negative; the x-intercepts are -2 and 3; the factors are $-(x + 2)^2$ and $(x - 3)$. The function is positive is for $x < -2$ and $-2 < x < 3$; the interval when the function is negative is for $x > 3$.

Example 2: Your Turn

a)



b)



Example 3: Your Turn

Reflect the graph of $y = x^3$ in the x-axis. Then, apply a vertical stretch by a factor of 4 and a horizontal stretch by a factor of $\frac{1}{2}$. Finally, translate 5 units down and 2 units left.

Example 4: Your Turn

a) $-210 = x(x + 1)(x + 2)$ **b)** $-7, -6, -5$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Coach students who are uncertain of the multiplicity of an x-intercept. Remind them that when the graph changes direction and stays the same sign, rather than passing through an x-intercept, the x-intercepts have even multiplicities. • Some students may find setting up a table helps them to organize their thinking and to determine the shape of the graph.
Example 2 Have students do the Your Turn related to Example 2.	• Encourage students to set up their solution using a table much the same as that shown in the example. This should assist them in organizing the information for a sketch.
Example 3 Have students do the Your Turn related to Example 3.	• If students are having difficulty, suggest they set up a table and use the same points from the example that are on $y = x^3$. They can then go through and apply the transformations one parameter at a time.
Example 4 Have students do the Your Turn related to Example 4.	• Coach students to determine the meaning of consecutive integers. Giving them concrete numbers such as 7, 8, and 9 may help them develop their polynomial function. If you let $x = 7$, ask them how they get to 8 (add 1 or $x + 1$). Ensure that they use x as their reference for the third number. Ask them how they get to 9 from 7 (add 2 or $x + 2$). • Prompt students to identify when they would use the quadratic formula. • Encourage them to solve the problem in more than one way and compare their solutions between methods and with those of another student.

Check Your Understanding

In #1 and #2, ask students to explain the connection between an algebraic and graphical solution. In solving these equations, what information can they state about the polynomials?

Ask students how they can check whether their answers in #3 are correct.

For #5, ask students which parameters are easiest to identify on the graph. Which parameter do students prefer to begin with?

Before students sketch the graphs in #8, have them identify the important parts of each graph that need to be identified and labelled.

In #9, is it possible to sketch the graphs of the functions in their given form without using technology? Before students factor each expression, have them write down all the information they know about each graph.

To determine the domain and range in #11, students need to determine restrictions or limitations on the variables. Are there any such limitations on x or y?

In #12, ask students if the question can be solved algebraically. Why must the equation be set to zero? Can you create a system of equations that could help you solve this question? What would the point of intersection tell you? Which coordinate are you interested in determining at the point of intersection, the x-coordinate or the y-coordinate?

To help students with #15, encourage them to build an expression in steps. What can be done to any odd number to make it even? Write an expression for an even number. Then, what can be done to any even number to make it odd? Add to the expression for even numbers to develop a new expression for odd numbers. How far apart are odd numbers from each other? Have students adjust their expression for an odd integer to show the odd integer that follows it. Finally, how far is the third odd integer from the first? Adjust the expression for an odd integer to show the odd integer that follows the second one.

Completing #16 may help students work through #19 and see that if the first integer is x, the next three consecutive integers are $x + 1$, $x + 2$, and $x + 3$.

In #18, make sure students understand that x does not represent the length of the side. The expression $x^2 - 12$ represents the length.

In #21, how can you determine the roots of the equation? How can you use those roots to determine the new roots? How can these roots help you determine the equation?

For #23, explain to students that buoyancy is an upward force exerted by a fluid, that opposes an object's weight. The difference in the pressure between the top and the bottom of a column of fluid is equivalent to the weight of the fluid that would otherwise occupy the column. For a floating object, only the submerged volume displaces fluid. The mass, m, of an object floating in a liquid with density ρ_{fluid} can be calculated using $m = \rho_{\text{fluid}}V_{\text{disp}}$, where V_{disp} is the volume of fluid displaced by the object.

The solution to #23 requires the formula for the volume of a spherical cap/segment. Using the variables from the diagram, the formula is $V_{cap} = \frac{\pi x}{6}(3a^2 + x^2)$. Students will need to use the Pythagorean theorem to substitute an expression for a^2. The solution follows:

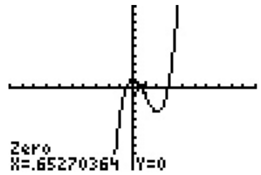
Substitute the expression for a^2 into $V_{cap} = \frac{\pi x}{6}(3a^2 + x^2)$.

$$V_{cap} = \frac{\pi x}{6}(3a^2 + x^2)$$
$$V_{cap} = \frac{\pi x}{6}(3(2x - x^2) + x^2)$$
$$V_{cap} = \frac{\pi x}{6}(6x - 2x^2)$$
$$V_{cap} = \frac{\pi x^2}{3}(3 - x)$$
$$V_{cap} = V_{disp} = \frac{\pi x^2}{3}(3 - x)$$

Then, substitute $V_{cap} = V_{disp} = \frac{\pi x^2}{3}(3 - x)$ into $\rho_{fluid}V_{disp} = \rho_{buoy}V_{buoy}$ and solve for x.

$$\rho_{fluid}V_{disp} = \rho_{buoy}V_{buoy}$$
$$\rho_{fluid}\left(\frac{\pi x^2}{3}(3 - x)\right) = \frac{1}{4}\rho_{fluid}\left(\frac{4\pi}{3}\right)$$
$$\pi x^2(3 - x) = \pi$$
$$3x^2 - x^3 = 1$$
$$x^3 - 3x^2 + 1 = 0$$

Students can graph the corresponding function and determine the value of the zero that makes sense in this case.



So, $x \approx 0.65$. The buoy sinks to a depth of approximately 0.65 m.

Meeting Student Needs

- You may wish to prepare a chart for students to use for #7.
- You may wish to prompt students to refer to the headings of the chart in #7 to assist them in creating the graphs in #8 and #9.
- The same type of chart may be used for #10.
- You may wish to create an exit slip containing the three Create Connections questions. Students could complete the exit slip before leaving class or the questions could be given as an assessment the next day.
- Provide **BLM 3–6 Section 3.4 Extra Practice** to students who would benefit from more practice.
- Students may be interested to learn more about some French festivals that have ice sculptures as an activity, Refer to the Weblink on page 59.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–5 and 10–12. Students who have no problems with these questions can go on to the remaining questions.	• Encourage student to complete #1 and 2 by inspection. • Some students may find it easier to create a table for #3, 4, 7 and 11. Encourage them to review the example tables for information about a function needed to sketch a graph. • Students who are able to work through #10 have a good understanding of the concepts in this lesson. • Suggest that students verify their solution for #12 using a different method.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2.	• You may wish to have students complete C2 before C1. Have them share their responses with the class. They can then use these responses to answer C1.

3 Chapter 3 Review and Practice Test

***Pre-Calculus 12*, pages 153–156**

Suggested Timing
90–135 min

Materials
• graphing technology

Blackline Masters
BLM 3–3 Section 3.1 Extra Practice
BLM 3–4 Section 3.2 Extra Practice
BLM 3–5 Section 3.3 Extra Practice
BLM 3–6 Section 3.4 Extra Practice
BLM 3–7 Chapter 3 Study Guide
BLM 3–8 Chapter 3 Test

Planning Notes

Have students who are not confident discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test. You may wish to provide students with **BLM 3–7 Chapter 3 Study Guide**, which links the achievement indicators to the questions on the Chapter 3 Practice Test in the form of self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1–5, 6a), b), 7 b), c), 8, 10.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 3–3 Section 3.1 Extra Practice**, **BLM 3–4 Section 3.2 Extra Practice**, **BLM 3–5 Section 3.3 Extra Practice**, and **BLM 3–6 Section 3.4 Extra Practice**.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 3 Review The Chapter 3 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #1–8, 10, 12–14	• Have students revisit any section from the review that they are having difficulty with prior to working on the practice test. • Encourage students to use **BLM 3–7 Chapter 3 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 3 Study Guide This master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. The minimum questions suggested are questions that students should be able to answer confidently. Encourage students to try additional questions beyond the minimum. • Consider allowing students to use any summative charts, concept maps, or graphic organizers in completing the practice test.
Assessment *of* Learning	
Chapter 3 Test After students complete the practice test, you may wish to use **BLM 3–8 Chapter 3 Test** as a summative assessment.	• Before the test, coach students in areas in which they are having difficulty. • You may wish to have students refer to **BLM 3–7 Chapter 3 Study Guide** and identify areas they need reinforcement in before beginning the chapter test.

UNIT 1

Unit 1 Project Wrap-Up

***Pre-Calculus 12*, page 157**

Suggested Timing

60 min

Blackline Masters

Master 1 Holistic Project Rubric
Master 2 Ana-Holistic Project Rubric
BLM U1–1 Unit 1 Project Checklist

Mathematical Processes

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

General Outcome

Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes

RF2 Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

RF3 Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

RF4 Apply translations and stretches to the graphs and equations of functions.

RF5 Demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the:
- x-axis
- y-axis
- line $y = x$.

RF12 Graph and analyze polynomial functions (limited to polynomial functions of degree ≤ 5).

RF13 Graph and analyze radical functions (limited to functions involving one radical).

Planning Notes

The Unit 1 project, The Art of Mathematics, helps students draw a connection between the mathematical functions they study in this unit and real-world equations of functions found in art, nature, and manufactured objects.

Ensure students are aware of the Unit 1 project information provided on page 3, the Project Corners on pages 43, 89, and 117, and the Unit 1 Project Wrap-Up on page 157.

Have students use **BLM U1–1 Unit 1 Project Checklist** to make sure that all parts of their project have been completed. As a class, brainstorm different ways students can present their work.

You may wish to work with the class to create a specific rubric for the project using either **Master 1 Holistic Project Rubric** or **Master 2 Ana-Holistic Project Rubric** as a template. Review the general holistic points within the 1–5 scoring levels. Discuss with students how they might achieve each level in the Unit 1 project. A completed rubric in each style for this project is available on the *McGraw-Hill Ryerson Pre-Calculus 12* web site. Note that these are just samples; your class rubric may have more detail.

Ask questions such as the following:
- What are the big ideas in the unit? (For example, one big idea is that vertical and horizontal stretches affect the graphs of functions and their related equations that can be predicted, graphed, and analysed.)
- Which of the big ideas are involved in the project?
- What part of the project could you complete or get partially correct to indicate that you have a basic understanding of what was learned in Chapters 1 to 3? (For example, should you get a pass mark if you can show how to apply translations and stretches to the graphs and equations of functions?)
- What would be on a level 1 project? What might you start on correctly? What could be considered a significant start?
- What would be expected for a level 5 project? What should it include? Try to help students realize that a level 5 project may have a minor error or omission that does not affect the final result.

- Knowing the expectations of levels 1, 3, and 5 projects, what would be expected for a level 4? Help students to understand that this is still an honours level and therefore the work should be reflective of this. However, even an honours project may have a minor error or omission. Discuss the difference between a major conceptual error and a minor miscalculation or omission. Understanding this point will help clarify for students the expectations and differences between a pass and above average result and may encourage some students to work toward the highest level. Repeat the process for level 2.

Use the rubric to ensure that students understand the criteria for an acceptable level, as well as what would warrant either an unacceptable or an honours grading.

Assessment	Supporting Learning
Assessment *of* Learning	
Unit 1 Project This unit project gives students an opportunity to apply and demonstrate their knowledge of the following: • the effects of vertical and horizontal stretches and translations on graphs of functions and their related equations • the effects of reflections on the graphs of functions and their related equations • applying translations and stretches to the graphs and equations of functions • graphing and analysing polynomial functions (limited to polynomial functions of degree ≤ 5) • graphing and analysing radical functions (limited to functions involving one radical) Work with students to develop assessment criteria for this project. **Master 1 Holistic Project Rubric** and **Master 2 Ana-Holistic Project Rubric** provide descriptors that will assist you in assessing students' work on the Unit 1 Project.	• You may wish to have students use **BLM U1–1 Unit 1 Project Checklist**, which provides a list of the required components for the Unit 1 Project. • Reviewing the Project Corner boxes at the end of Sections 1.3, 2.2, and 3.1 will assist students. • Make sure students recognize what is expected for the minimum requirements for an acceptable project as well as the difference between level 5 and level 4. • Clarify the expectations and the scoring with students using any of **Master 1 Project Rubric, Master 2 Ana-Holistic Project Rubric**, or the rubric you develop as a class. Review the scoring rubric at the beginning of the project, as well as intermittently throughout the project, to refresh student understanding of the project assessment.

UNIT 1

Cumulative Review and Test

***Pre-Calculus 12*, pages 158–161**

Suggested Timing
90–135 min

Materials
- grid paper
- graphing technology

Blackline Masters
Master 3 Centimetre Grid Paper
BLM U1–2 Unit 1 Test

Planning Notes

Have students work independently to complete the review, and then compare their solutions with those of a classmate. Alternatively, you may wish to assign the cumulative review to reinforce the concepts, skills, and processes learned so far. If students encounter difficulties, provide an opportunity for them to share strategies with other students. Encourage them to refer to their notes, and then to the specific section in the student resource. Once they have determined a suitable strategy, have students add it to their notes. Consider having students make a list of questions they found difficult. They can then use the list to help them prepare for the unit test.

Meeting Student Needs

- Have students review the checklist containing the learning outcomes for Unit 1. Students who require more practice on a particular topic may refer to the extra practice BLM for the relevant chapter and section.
- Encourage students to review their own summary of the key ideas, including examples, presented in each of Chapters 1, 2, and 3.
- Suggest that students review any learning aids they prepared, such as a compare and contrast chart for the remainder theorem and factor theorem in Section 3.3.
- You may wish to provide students with **Master 3 Centimetre Grid Paper**.

Assessment	Supporting Learning
Assessment *for* Learning	
Cumulative Review, Chapters 1–3 The cumulative review provides an opportunity for students to assess themselves by completing selected questions pertaining to each chapter and checking their answers against the answers in the back of the student resource.	• Have students review their notes from each chapter to identify topics they had problems with, and do the questions related to those topics. Have students do at least one question that tests skills from each chapter. • Have students revisit any chapter section they are having difficulty with. • You may wish to have students review the study guide blackline masters from Chapters 1, 2, and 3 as well as practice tests and chapter tests they completed to help identify any skill areas that still require reinforcement.
Assessment *of* Learning	
Unit 1 Test After students complete the cumulative review, you may wish to use the unit test on pages 160 and 161 as a summative assessment.	• Consider allowing students to use their graphic organizers. • You may wish to have students complete **BLM U1–2 Unit 1 Test**, which provides a sample unit test. You may wish to use it as written or adapt it to meet the needs of your students. The answers to this unit test can be found on **BLM 3–9 Chapter 3 BLM Answers**.