Trigonometry and the Unit Circle

Opener

***Pre-Calculus 12*, pages 164–165**

Suggested Timing
30–45 min

Blackline Masters
BLM 4–1 Chapter 4 Prerequisite Skills
BLM U2–1 Unit 2 Project Checklist

Planning Notes

In this chapter, students develop an understanding of the connection between angles in standard position and the unit circle. They then extend this understanding and develop a tool (unit circle) and methodology to solve linear and quadratic trigonometric equations. Help students appreciate that angles in standard position are not always static; they can be considered to be dynamic, and when in motion they produce cyclic relationships.

Consider having a discussion of space, location, and way-finding, and how these concepts relate to geometry and trigonometry. We use many of the concepts from trigonometry informally when we navigate without the use of technology. In fact, some Inuit elders have expressed concern that their youth are losing the ability to navigate the land without such technologies as GPS. Discuss some of the pros and cons of using technology for way-finding.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter. Encourage students to use a summary method of their choice.

Unit Project

In the Project Corner at the end of section 4.3, students are introduced to historical facts about different aspects of angular measurement. They are asked to consider different units of measure for angles, the advantages and disadvantages of using these, and the careers and contexts in which they might be used. In the project wrap-up at the end of the unit, students may choose to extend this knowledge by researching the history, usage, and relationship of types of units of angular measurement.

Meeting Student Needs

- Consider having students complete the questions on **BLM 4–1 Chapter 4 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out to students **BLM U2–1 Unit 2 Project Checklist**, which provides a list of all the requirements for the Unit 2 Project.
- Provide students with a checklist containing the learning outcomes for this unit. Discuss specific terms. Develop a sense of understanding of what they need to learn by the end of the unit.
- Discuss the Key Terms. Students may wish to create a word wall for the chapter or create a list of key terms and definitions in their notebook. Consider having students work in pairs to discuss what they believe the terms mean. They can check their predictions as they work through the chapter.
- Students could research the use of trigonometry in the recording of music.
- Whenever possible, use animations and interactive activities, such as those listed in the chapter planning chart. There are many videos and interactive tools on the Internet that provide instruction about trigonometric concepts.

Gifted

Ask students to consider the statement from the chapter opener, "In order to analyse these repeating, cyclical patterns, you need to move from using ratios in triangles to using circular functions to approach trigonometry." Have them hypothesize what this means and why there is a need for such a change in approach.

Career Link

Discuss with students what they know about collision investigation:
- Why is collision investigation needed?
- Who would conduct this type of investigation?
- What type of information might be considered? What types of skills would be needed to work with this type of information?

Suggest that students go online to research careers related to collision investigation and the qualifications for these careers.

4.1 Angles and Angle Measure

***Pre-Calculus 12*, pages 166–179**

Suggested Timing
90–120 min

Materials
- masking tape
- sidewalk chalk
- string
- measuring tape
- grid paper (optional)
- compass (optional)
- protractor (optional)
- ruler

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 4–2 Section 4.1 Extra Practice
BLM 4–4 The Unit Circle

Mathematical Processes for Specific Outcomes
T1 Demonstrate an understanding of angles in standard position, expressed in degrees and radians.
✓ Connections (CN)
✓ Mental Math and Estimation (ME)
✓ Reasoning (R)
✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–4, 6–9, 11, 12, 14, 18
Typical	#1–3, 5–9, 11–14, one of 15 or 16, 17, 18, 21, C1–C5
Extension/Enrichment	#10, 13, 17, two of 19–23, 24–27, C4, C5

Planning Notes

Read the Focus on… bullets as a class. Ask
- What does it mean for angles to be in standard position? How will your understanding of this concept be helpful when sketching angles greater than 360°?
- What does the term *negative angle* mean?

As you read through the Focus on… bullets, ask students to identify terms with which they are unfamiliar, and to share their predictions on what the terms might mean. Ask students to share any other ideas that they have learned in previous courses that may relate to concepts involving angular measure listed in the Focus on… points.

Investigate Angle Measure

The Investigate helps students develop a concrete understanding of radian measure. If you do not have room in your class for students to draw circles with a radius of 1 m, consider doing the Investigate on a smaller, desktop scale.

Before beginning the investigation, ask students which element of a circle a radian might be related to. If students are having difficulty, ask
- How is the formula for the circumference a circle related to radius?
- What is the approximate value of 2π? How does this value relate to the number of radii you found that fit on the circumference of your circle?
- How many degrees are there in a circle? If $\angle AOB$ was exactly 60°, how many $\overset{\frown}{AB}$ would there be in the circumference? What does this tell you about the measure of $\angle AOB$ that you found?
- Use the circumference formula to find the value for r. How does this relate to the measure of a radian?
- How could you use what you have learned in this investigation to estimate the circumference of a circle if you know the radius? (It is a about six and a quarter r.)

Meeting Student Needs

- Before beginning the Investigate, discuss the outcome(s) and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.
- Consider creating a collage of pictures, or have the class create one that illustrates where angles can be found in nature and in construction. The collage could include the dorsal fin of a shark, which is triangular, the honeycombs in a beehive, etc. Ask students to suggest other places that angles exist naturally or in human constructions.
- When working on the Investigate, encourage at least two students of different sizes to work through steps 2 and 3 to illustrate that the relationship is the same. The answer is not dependent upon the unit of measure.
- If possible, keep one of the circles on the floor for future reference. You might even consider dividing it into fractional parts using different coloured painters' tape.

Answers

Investigate Angle Measure

3. a bit more than six

4. $\frac{\text{circumference}}{\text{radius}} = \frac{2\pi r}{r} = 2\pi \approx 6.28$

5. Less than 60°, since a circle has 360° and there are more than 6 radii on the circumference of a circle: $\frac{360°}{6+} < 60°$.

6. $\frac{360°}{2\pi} \approx 57.3°$

7. The number of radii that lie on the circumference remains the same. In the general case, since the ratio of circumference to radius is $\frac{2\pi r}{r}$ and the r's cancel out, the size of the radius does not affect the size of the ratio.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Encourage students to estimate the size of their angle before measuring. • For the Investigate, you may wish to have groups construct circles of different sizes, rather than having all groups use circles of the same radius. This would allow for a greater discussion of the last Reflect and Respond question. Alternatively, you could have groups rotate through stations, with each station having a circle of a different size. The groups could then reach their own conclusions.

Example 1

This example demonstrates a number of ways to convert degrees to radian measure and vice versa. Discuss the three given techniques with your class:

1. Unitary method, where $1° = \frac{\pi}{180}$ or 1 radian $= \frac{180°}{\pi}$.
2. Proportional method, where students set up a ratio such as $\frac{D}{R} = \frac{180}{\pi}$ or $\frac{R}{D} = \frac{\pi}{180}$.
3. Unit Analysis method, where students set up a product to eliminate the given measure, and then determine the other degree measure. To eliminate radians they use $x\ \cancel{\text{radian}}\left(\frac{180°}{\pi\ \cancel{\text{radian}}}\right)$. For degrees, they should use $x\cancel{°}\left(\frac{\pi}{180\cancel{°}}\right)$

Suggest that students try each of the given methods at least once when doing the Your Turn questions. Ask them to choose their favourite technique and share why they prefer this method.

To help students understand radian measure, have them draw a circle or provide one for them (consider handing out **BLM 4–4 The Unit Circle**). Ask students to consider that π ends at 180°, and have them label it on the circle. Then, ask students to divide π (180°) into six equal parts. Ask, "How much is one part?" $\left(\frac{1}{6}\pi \text{ or } \frac{\pi}{6}\right)$ Have students use one colour to draw the six lines and label them 0π, $\frac{1}{6}\pi$, $\frac{2}{6}\pi$, $\frac{3}{6}\pi$, $\frac{4}{6}\pi$, $\frac{5}{6}\pi$, and $\frac{6}{6}\pi$. The degree measures can also be labelled $\frac{\pi}{6} = \frac{180°}{6} = 30°$, etc.

Ask students to work through the same process dividing π first into four equal parts, and then three equal parts. Use a different colour for each set of lines and labels. Ask students to note any similarities and overlaps.

Have students extend this process into the second π (bottom half of the circle). When the circle is complete, students will have developed a tool to better understand radian and degree measurements, and the relationship between $\frac{\pi}{6} = 30°$, $\frac{\pi}{4} = 45°$, and $\frac{\pi}{3} = 60°$.

Example 2

Before exploring this example, encourage students to share their understanding of angles in standard position on the coordinate plane. Ensure that students understand the terms *initial arm*, *terminal arm*, *positive rotation*, *negative rotation*, and *coordinate plane quadrants*. Students will need to understand what *coterminal angles* and *reference angles* are and how to determine them.

Remind students that to add $\frac{8\pi}{3}$ and $2\pi n$, they must find a common denominator:

$$\frac{8\pi}{3} + 2\pi n = \frac{8\pi}{3} + \frac{6\pi n}{3}$$
$$= \frac{2\pi(4 + 3n)}{3}$$

Some students may require coaching on how to translate 2π into a radian measure with a denominator other than 1. A quick rule to show them is to multiply 2π by 1 in the form $\frac{\text{desired denominator}}{\text{desired denominator}}$: $2\pi\left(\frac{3}{3}\right) = \frac{6\pi}{3}$.

Example 3

Some students may need help with the general form. Discuss why n is multiplied by 360° in the expression $110° \pm (360°)n$, and why n must be a natural number.

Students may need to revisit the concept of a solution domain. Ask:

- How many angles are coterminal with angle 110°? What have we effectively done to the number of possible coterminal angles when we stipulate a domain for the solution?
- How many possible coterminal angles are there in a domain of $-720° \leq \theta < 720°$? What is the smallest possible angle that is coterminal with 110° in this domain? the largest?
- How can you use the general solution to find all coterminal angles greater than 110° and smaller than 720°? Can you use a similar method to find all coterminal angles greater than −720° and less than 110°?

Draw students' attention to the effectiveness of a table when solving these types of problems.

Example 4

Before moving on to Example 4, be sure that students understand the terms *subtend*, *arc*, and *central angle*. Students may also better understand the ratios and relationships outlined in the student resource if they develop the formula for themselves. Suggest that they construct a figure showing concentric circles, similar to the one on page 173. They might use one colour to draw $\overset{\frown}{AB}$ and label radius 1 and arc length x, and a second colour to draw $\overset{\frown}{CD}$ and label radius r and arc length a. They can then use these two colours when writing out the ratios. This will help relate the parts of the ratios to the diagram.

Have students answer one of the Your Turn questions using the proportion method and answer another using the arc length formula. Ask each student to share with you or another student which method they prefer and why.

Meeting Student Needs

- Suggest that students record their own summary of the Key Ideas, including examples, and then store the summary for each section in the same location.
- You may want to have students work in pairs. Have students attempt the question before discussing it as a class. They will have a greater interest in the solution after they try to develop their own answer.
- Suggest students complete the Your Turn questions, and compare their solutions with those of other students.
- Create dominoes (flash cards) to practise matching coterminal angles. Ensure that both degree and radian measures are used.
- Provide students with circles of various radii. Ensure that the centres are clearly marked. Instruct them to create a $\frac{1}{4}\pi$ central angle. Ask them to measure the radius and then use string to measure the length of the arc. They can then compare the length of the arc with the length of the radius multiplied by the central angle. Have them repeat this process in a circle with a different radius. Ask them what they notice.

Common Errors

- Students often confuse when to multiply by $\frac{\pi}{180}$ or $\frac{180}{\pi}$ when converting from one system of measure to another.

Rx Point out to students that a radian measure has a π value and a degree measure does not. To convert a radian measure to degrees, students must apply a mathematical operation that gets rid of the π value; that is, multiply by $\frac{180}{\pi}$ so that the π values cancel. Conversely, to write a radian measure, multiply by $\frac{\pi}{180}$ to introduce the π value.

- Students often improperly set up a proportion that compares ratios of radians to degrees, and π to 180°.

Rx Have students get in the habit of writing a proportional relationship similar to $\frac{R}{D} = \frac{\pi}{180}$, or $\frac{D}{R} = \frac{180}{\pi}$, with the first ratio having the unknown measure in its numerator and the known value in its denominator.

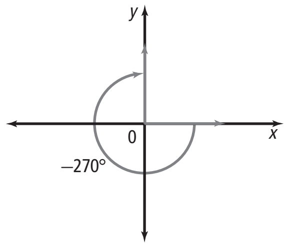
- Students may forget to use the correct format or generate answers outside of a given domain when solving for angular solutions written in the general case.

Rx Have students correct their work after each question, making sure that they use a format exactly the same as that shown in the answer provided. Remind them to make sure that their answers lie within the given domain if one is given in the question.

Answers

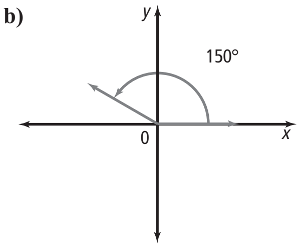
Example 1: Your Turn

a)



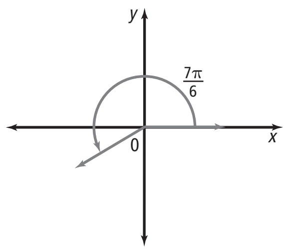
$\frac{-3\pi}{2}$; −4.71 radians

b)



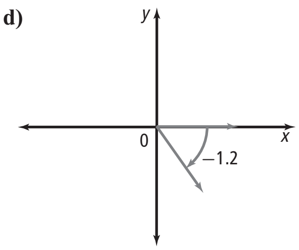
$\frac{5\pi}{6}$; 2.62 radians

c)



210°

d)



$\frac{-216°}{\pi}$; −68.75°

Example 2: Your Turn

Example:

a) 630° and −90°

b) $\frac{3\pi}{4}$ and $-\frac{13\pi}{4}$

c) 20° and −340°

Example 3: Your Turn

a) $220° \pm 360°n$, $n \in$ N; 220°, −140°

b) $290° \pm 360°n$, $n \in$ N; 290°, −70°

c) $\frac{\pi}{4} \pm 2\pi n$, $n \in$ N; $\frac{\pi}{4}$, $-\frac{7\pi}{4}$

Example 4: Your Turn

a) 11.4 cm

b) 2.6 mm

c) 2.6

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Encourage students to select one method with which they feel comfortable. • Review how a negative sign affects an angle measure with regard to direction of rotation.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Review that multiple rotations around the circle are multiples of 360°, both positive and negative.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Ensure students have a good understanding of the value of *n* in the general form before asking them to complete the Your Turn questions.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Encourage students to select one method with which they feel comfortable.

Check Your Understanding

Help students start #14b) by suggesting that they draw a circle and sketch a sector on the circle. Remind students of the formula for the area of a circle. Ask, "If you let the sector area be represented by A_s, can you write a ratio of sector area to circle area?" $\left(\frac{A_s}{\pi r^2}\right)$ Then discuss how students might write the ratio of the sector angle in radians. $\left(\frac{\frac{5\pi}{3}}{2\pi}\right)$ Then, help students set up a proportion to solve for the sector area: $\frac{A_{sector}}{A_{circle}} = \frac{\theta_{sector}}{\theta_{circle}}$.

For #17, you may need to remind students how to convert revolutions to degrees, degrees to radians, and revolutions to radians.

If students are having difficulty with #18, suggest that they consider the similarities and differences between Joran's and Jasmine's answer? Ask, "What is the difference between $n \in$ I and $k \in$ N? Keeping that in mind, why does Joran describe his answer as $78° + (360°)n$ and Jasmine as $78° \pm k(360°)$?"

For #19c) ask students how many gradians are in a right angle. How might this be related to the metric measurement?

For #21a), discuss with students that they will have to convert kilometres to metres and hours to minutes. Ask, "Can you use a unit analysis method to change km/h to m/min?" To complete this question, some students may also have to revisit the relationship of radius to diameter. If students are having difficulty, ask

- What is the circumference of the wheel expressed as an exact value?
- Given the speed of the wheel in m/min and the circumference in m, how can you determine the number of revolutions the wheel makes per minute?
- How can we use this information to determine how many radians per minute the wheel turns?

In #22, some students may need to revisit the relationship of diameter to circumference and how the circumference is related to the distance the wheel turns in one revolution. Once they determine how far the wheel turns in 15 revolutions, some students may require prompting to make the conversion from m/min to km/h. Suggest that students share their ideas with a partner.

Students may find it helpful to sketch a diagram of a circle in #26, showing a sector and a segment subtended by the same two points on the circle. Ask

- Can you find the area of the sector by comparing the ratio of sector area to circle area, and the ratio of sector angle to angle of a circle?
- What circular angle measure should you use, degrees or radians?
- Can you find the area of the triangle determined by the radii and the chord?

Guide students in how they can use a trigonometric ratio to find the height of the triangle ($h = r\sin\theta$), and to realize that the base of the triangle is r. They can then substitute into the formula for the area of a triangle, $A = \frac{1}{2}bh$.

Meeting Student Needs

- Provide **BLM 4–2 Section 4.1 Extra Practice** to students who would benefit from more practice.
- Students should complete the checklist of understanding provided in the opener. Doing so will guide them when deciding which questions they need to practise to gain a deeper understanding of the outcomes for this section.
- Students could create an example to represent each outcome listed in the Key Ideas.
- Students could research various applications of angular velocity to gain a better understanding of the concept.

Enrichment

Some students may be interested in designing a *GeoGebra* applet for #27. This applet would display degrees and radians of the angle formed when a user moves the minute hand around the face of a clock. Some students might take it even further and do the same for the hour hand.

Gifted

Ask students to consider the unitary, proportion, and unit analysis methods. Suggest that they create a table that outlines the advantages and disadvantages of each of these methods, and to provide an example of where each might be used.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–4, 6–9, 11, 12, 14 and 18. Students who have no problems with these questions can go on to the remaining questions.	• Allow students to use the circles they constructed on the floor to complete the questions. Alternatively, they could draw a smaller version of the circle in their notebook, labelling the direction of the angles with a "+ arrow" and a "– arrow." Help them see the relationship between the Investigate and the questions. • You may wish to have some of the coaching tips from the Examples posted on the wall for students who need extra support in the conversions for #2–4. • Coach students in verbalizing the meaning of angles in standard position and how to tell whether an angle reflects more than one revolution. Ensure students have a clear understanding of these concepts before working on #8, 9, and 11. • Encourage students to draw and label a diagram for #14.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• C1 ensures that students understand the purpose behind the Investigate. Have them refer back to their responses to the investigation if they are having difficulties with this question. Ask them to verbalize the similarities between the Investigate and this question. • Encourage students to use diagrams for C2. Students should clearly understand this question before proceeding. Their communication should be clear, and it should firmly establish that they understand the differences. • You may wish to revisit the term *reference angle* before students begin C3.

The Unit Circle

4.2

***Pre-Calculus 12*, pages 180–190**

Suggested Timing
90–120 min

Materials
- can or other cylinder
- scissors
- tape
- grid paper
- compass
- straight edge
- protractor
- ruler

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 4–3 Section 4.2 Extra Practice
BLM 4–4 The Unit Circle

Mathematical Processes for Specific Outcomes
T2 Develop and apply the equation of the unit circle.
☑ Connections (CN)
☑ Reasoning (R)
☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2, 3a)–d), 4–6, 9,11, 13
Typical	#1c), d), 2, 3c)–f), 4–9, 13, C1–C3
Extension/Enrichment	one of 15–17, 18–20, C2, C4

Planning Notes

This section develops students' understanding of relative measures of exact angles as drawn on the coordinate plane. While developing this understanding, students produce a model of the unit circle, a useful tool in solving trigonometric equations.

The section begins with an introduction to the difference between linear and circular gauges. If possible, bring a linear and a circular tire gauge to class. Invite students to use the two gauges to measure the tire pressure of a bicycle or automobile tire. Ask, "What are the advantages and disadvantages of each type of gauge?" As a class, discuss what other types of circular items or cyclical events students can think of (e.g., analogue clocks, ocean tides, etc.).

Investigate Circular Number Lines

The Investigate should help students develop an intuitive understanding of the relative sizes of exact radian measures when placed on the coordinate plane.

For step 3, suggest that students fold the strip of paper in half three times to create eight equal subdivisions and label as fractional values of π. They could use a similar folding strategy in step 7.

In step 5, you might suggest that students make a mark on the circle to represent each of the divisions of the number line. You might also ask students to share how they found the centre of the circle (e.g., use a compass and a straight edge to draw the perpendicular bisectors of any two non-parallel chords of the circle. The point of intersection of the two bisectors is the centre of the circle.) Once the centre of the circle is determined, have students draw a horizontal line to represent the x-axis and a vertical line to represent the y-axis.

In step 6, ask the class how they could generalize their findings.

In step 10, instruct students to construct a perpendicular line from the point P to the positive x-axis. Ask, "What type of triangle is this? (isosceles) What does this tell you about the measures of the angles?"

Meeting Student Needs

- Discuss the outcome(s) and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found in this section.
- Before beginning the Investigate, ask students what patterns they notice in the circle developed in the previous section. For example, guide them to see that the 60° angle is reflected through the x-axis and y-axis.
- In the Investigate, students could sketch the two reference triangles on a sheet of paper with the right angle in the bottom, right corner. They could label the lengths of the sides and then reduce the sides proportionately so that the hypotenuse has a value of 1. For example, in the 30°-60°-90° triangle, the opposite sides, respectively, are 1, $\sqrt{3}$, and 2.
  By dividing by 2, they will now have $\frac{1}{2}$, $\frac{\sqrt{3}}{2}$, and 1. Suggest that they repeat this process for the 45°-45°-90° triangle.
- Some student may need to be reminded how to simplify points. For example, $\left(\frac{4\pi}{12} = \frac{\pi}{3}\right)$.

Common Errors

- When creating the right triangle, some students draw a line horizontally to the y-axis, instead of vertically to the x-axis.

Rx Suggest students think of construction workers constructing the triangles. When the angle is in quadrant I or II, a worker climbs the terminal arm and drops a line to coworkers on the ground (x-axis). When the terminal arm is in quadrant III or IV, the worker scales down the arm, and coworkers drop a line down to the worker on the arm.

- Student often forget to reduce fractions to lowest terms.

Rx When discussing the Investigate with the class, ask students to share their answers in reduced form. Meet with any students who experienced difficulty reducing exact-value fractions correctly.

- Student sometimes list positive angles for clockwise motion.

Rx This can occur when students place their number line upside down or backward. Discuss beforehand what the implications of misaligning the number line might be. Then circulate between groups to make sure that number lines are taped to the cylinder correctly.

Answers

Investigate Circular Number Lines

2. 2π

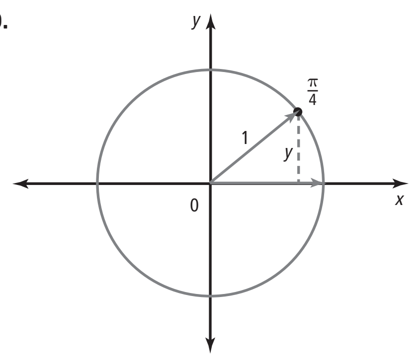
3. π; $\frac{1}{2}\pi$

8. The assumption is that the radius is 1.

9. a) Yes. Since a circle has a radian angle measure of 2π, $\frac{1}{4}$ of 2π is $\frac{2\pi}{4} = \frac{\pi}{2}$.

b) Yes. Since there are 8 subdivisions, each subdivision has a measure of $\frac{2\pi}{8} = \frac{\pi}{4}$.

10.



Draw a vertical line to the x-axis from the point on the circle at $\frac{\pi}{4}$. This forms a right angle triangle with a hypotenuse of 1 and a vertex angle of $\frac{\pi}{4}$, or 45°. Since this triangle has two equal angles of 45°, the opposite sides (base and height) are equal. Using the Pythagorean relationship,

$$a^2 + b^2 = c^2$$
$$a^2 + a^2 = (1)^2$$
$$2a^2 = 1$$
$$a^2 = \frac{1}{2}$$
$$a = \pm\sqrt{\frac{1}{2}}$$
$$a = \pm\frac{1}{\sqrt{2}}$$

Since the point $P\left(\frac{\pi}{4}\right)$ is in quadrant I, the x- and y-values are positive, so $P\left(\frac{\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

The only difference between the points midway between axes is the sign of one or both of the coordinates:

- $P\left(\frac{3\pi}{4}\right)$ is in quadrant II, so the x-value is negative and the y-value is positive: $P\left(\frac{3\pi}{4}\right) = \left(\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.
- $P\left(\frac{5\pi}{4}\right)$ is in quadrant III, so the x-value and y-value are both negative: $P\left(\frac{5\pi}{4}\right) = \left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.
- $P\left(\frac{7\pi}{4}\right)$ is in quadrant IV, so the x-value is positive and the y-value is negative: $P\left(\frac{7\pi}{4}\right) = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • The response to #8 sets up the remainder of the questions, so ensure that students' thinking is correct for this question before having them continue to #9 and 10. • Students may need coaching for the ratios of the sides of a 45°-45°-90° triangle before linking it to the coordinates of $\frac{\pi}{4}$. As a class, ensure that students have correctly identified that the number values of the coordinates in each quadrant are the same, but the signs vary. This understanding is important for students to understand the unit circle and the patterning that is involved.

Example 1

If students are struggling with this example or the Your Turn questions, ask

- What general formula did you discover for the equation of any circle centred at the origin?
- Which variable represents the radius of the circle? What value will you use to replace r when the radius is 2?

Example 2

Before students work through this Example, ensure that they are clear about the general equation for any circle centred on the origin.

If students need guidance for part a), ask them to identify which variable is given, which is missing, and how they can solve for the missing variable. Some students may need to reactivate their knowledge of solving equations involving square roots, and the fact that there are two possible answers when taking the square root of a real number.

It may help some students to represent these concepts visually. Have students sketch a unit circle. (Consider handing out **BLM 4–4 The Unit Circle**.) Ask

- How many points lie on the unit circle where the x-coordinate is $+\frac{2}{3}$?
- Given a unit circle on the coordinate plane, in which quadrants do the points lie?

For part b), have a similar discussion about the number of points on the unit circle that have a y-coordinate of $-\frac{1}{\sqrt{2}}$. Suggest students use their sketch of the unit circle to answer. Then ask how many points on the unit circle have a y-coordinate of $-\frac{1}{\sqrt{2}}$ and lie in quadrant III. Ask, "Is the x-coordinate positive or negative in quadrant III?"

Example 3

To assist students with part a), work as a class to identify the multiples of $\frac{\pi}{3}$ between 0 and 2π radians. You could also discuss this concept using the idea of "cutting" pi into equal parts: $\frac{1}{3}\pi, \frac{2}{3}\pi, \frac{3}{3}\pi, \frac{4}{3}\pi, \frac{5}{3}\pi, \frac{6}{3}\pi$. Do this for $\frac{1}{4}$ and $\frac{1}{6}$ as well. Ask, "In which quadrants does each of the angles lie?" Consider creating a table during this discussion.

Suggest that students sketch the right triangle. Then help students relate this triangle to the unit circle, asking them to label the side lengths with the hypotenuse equal to 1. Ask

- What is the coordinate of the point of intersection of the terminal arm and the unit circle for a central angle of $\frac{\pi}{3}$ radians?
- What are the coordinates of the intersection between the terminal arm and the unit circle for all multiples of $\frac{\pi}{3}$ from 0 to 2π rads?
- Connect the angles of rotation and endpoints on the unit circle using the format $P(\theta) = (x, y)$.

Consider having students discuss part c) as a class or in small groups.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Ask students to attempt the Examples in pairs before discussing them as a class.
- Students can refer to the circle created in section 4.1, which contains the multiples of the 30°, 45°, and 60° angles in the four quadrants. They can use the reference triangles created in the section 4.2 Investigate to label the points on the unit circle.
- Providing triangles on transparency paper allows students to manipulate the transformations for all four quadrants. The length of the sides will be visible through the transparency paper.
- For Example 3, some students may need help to recall the side ratios for a 30°-60°-90° triangle. Help them create these ratios and ask them to write the trigonometric ratios for sin θ, cos θ, and tan θ. Repeat this exercise with a 45°-45°-90° triangle.
- In a corner of each quadrant, students may wish to draw a reminder of the signs for the x-coordinate and y-coordinate: quadrant I (+, +), quadrant II (−, +), quadrant III (−, −), quadrant IV (+, −).
- If the floor circle created in section 4.1 is still available, students may wish to use masking tape to label the points on the circumference of the circle. Include the coordinates of the signs as listed in the previous point.

Common Errors

- Some students forget that when using the Pythagorean relationship, they must square the radius to find the length of the sides for the final answer.

R_x Suggest that students write out the formula $a^2 + b^2 = c^2$ and then the subsequent numerical substitution into this equation).

- Students often use degree measures instead of radian measures when finding the arc length of a sector.

R_x Discuss why radian measures represent real number values for angles but degree measures do not. Ask students to try solving for arc length using both degree and radian measures for the central angle, and to discuss why the answers differ.

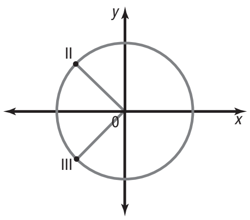
Answers

Example 1: Your Turn

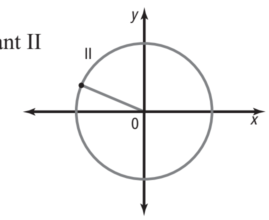
$x^2 + y^2 = 36$

Example 2: Your Turn

a) $\left(-\frac{5}{8}, \frac{\sqrt{39}}{8}\right)$ and $\left(-5/8, -\frac{\sqrt{39}}{8}\right)$; quadrants II and III

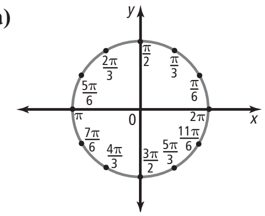


b) $\left(-\frac{12}{13}, \frac{5}{13}\right)$; quadrant II

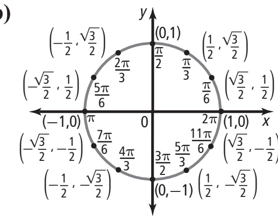


Example 3: Your Turn

a)



b)



c) The coordinates of the points reflected across the axes are the same except for the signs.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Students should be familiar with the Pythagorean relationship. If they are having difficulty, suggest that they draw the diagram of their circle first and label the radius. It is important that they understand that any point on the unit circle will fit the form of $x^2 + y^2 = 1$.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs, or compare their work with a partner. • Have students use a unit circle drawing as a visual reference.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • The development of the coordinates of the unit circle is sometimes difficult for students to follow. However, the pattern established by the coordinates is easier for them to see and remember. It may help students to connect the coordinate points with lines that will form a rectangle. This approach makes it easier to see the relationship between the coordinates, and to see that the sign changes correspond to the quadrant that the points are in.

Check Your Understanding

Suggest that students use the unit circle they constructed in Example 3 as a visual reference to answer #4 to 6 and later questions in the question sets.

It is best in #8, the Mini Lab, if students complete steps 1, 2, and 3 for one set of points at a time. Then, for step 4, they can summarize what they have discovered in the other parts.

In #9, students may need guidance to find the coordinates for point B.

For #10a), you may want to revisit with students the values of cos 0°, cos 0, cos 90°, and cos $\frac{\pi}{2}$. Ask, "What is the range of values for cos θ from 0 to $\frac{\pi}{2}$?"

For #13, students may wish to draw three unit circles. The first contains the points for -2π to 0π, the second contains the points for 0π to 2π, and the third contains the points for 2π to 4π radians. The point $P(\theta) = \left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ should be plotted on each circle, and then the angle θ can be determined.

You might point out to students that #13c) is similar to #8. You could ask students what generalization they discovered for angles on the unit circle that are a one-quarter rotation apart.

For #16c), be sure that students are familiar with the decimal values of the points listed on the unit circle. Students need to determine between which two points a rotation has a decimal value of 5 radians.

Before working on #17, be sure that students are comfortable with the equation of a unit circle and how to work with this equation. Ask

- What is the equation of a unit circle?
- Given the equation $y = -3x$, how can you determine the x-value of the point of intersection between the line and the unit circle?
- How can you determine the y-value of the point of intersection?

Discuss how students might use a trigonometric ratio in part b) to determine the angle of rotation to one of the points in part a).

Suggest that students sketch a diagram for #18. Students may even consider doing part b) before part a). They should then be able to compare the triangle produced by the point and the x-axis with the triangle produced by the point of intersection on the unit circle and the x-axis. By producing a proportion of a ratio of radii to a ratio of x-values or y-values, they should be able to find one of the coordinates on the unit circle and extend that to the second point.

For #19, suggest that students draw a unit circle centred at the origin and construct terminal arms in each of the four quadrants. They can then list general points of intersection of the terminal arms and unit circle in terms of x and y. Using their knowledge of sine and cosine ratios, have students determine the value for sin θ and cos θ in each of the quadrants. Discuss what happens to the sine and cosine ratios as angles increase from 0° to 360°.

Before working on C2, discuss with students what they know about an isosceles triangle, the comparative length of its arms, and its internal angles.

Before working on C4, revisit the concept of the areas of a square and a circle. You may want to discuss what the areas of the square and circle are in terms of the radius. Ask, "What is the ratio of area of scrap paper to area of the square? Why is this ratio important in determining the percent of paper cut off?"

Meeting Student Needs

- Provide **BLM 4–3 Section 4.2 Extra Practice** to students who would benefit from more practice.
- Students should complete the checklist of understanding provided in the opener. This will guide them when deciding which questions they need to practice to gain a deeper understanding of the outcomes for this section.

Enrichment

Have students show visually the relationship between the radius value of the unit circle and the key distances around that circle. How could they use the visual to explain the value of the unit circle and radian measure?

Gifted

Ask student to explain why more than one quadrant of a unit circle is necessary to explain some natural phenomena.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2, 3a)–d), 4–6, 9, 11, and 13. Students who have no problems with these questions can go on to the remaining questions.	• In #1, students use $x^2 + y^2 = r^2$ for circles with radii other than 1. Discuss with students that the origin does not transform the centre, so the first half of the equation remains unchanged. Showing students how the centre is substituted in will help them to answer C3 for any circle. It might benefit them to graph a circle with centre at (2, 2) and $r = 1$. • In #2 and 3, students use their understanding of $x^2 + y^2 = 1$ to determine whether points are on the unit circle or to find a missing coordinate. Ensure that students understand why the quadrants have been indicated in #3. You may wish to complete #3a) as a class, and then use #3f) as an assigned question. • Opposite operations are presented in #4 and 5. Have students verbalize how they are opposites. If they are still having difficulty, use the coordinates $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ as a sample to show them that this point belongs to 30°. • You may wish to have students work with a partner for #9 and have them compare their responses. • If students have difficulty with #13, go through the question using the coordinates of $\left(\frac{-1}{2}, \frac{-\sqrt{3}}{2}\right)$. Help them to make links to the coordinates in the question with each step.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C3.	• Visual learners will benefit from placing C1 into their notebook. This question provides a visual link to the patterns the coordinates and radian values create on the unit circle. • When students work on C3, they can link their understanding from #1 in the Practice section to their previous work on transformations.

4.3 Trigonometric Ratios

***Pre-Calculus 12*, pages 191–205**

Suggested Timing
90–120 min

Materials
- grid paper
- compass
- protractor
- ruler
- scientific calculator
- straight edge

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 4–4 The Unit Circle
BLM 4–5 Section 4.3 Extra Practice

Mathematical Processes for Specific Outcomes
T3 Solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.
- ☑ Mental Math and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☑ Technology (T)
- ☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1a)–f), 2a)–f), 3–12, 14
Typical	#1a)–f), 2a)–f), 3d)–f), 4–9, 10a), d), 11, 13, 15, 16, 18, 19, 22, C1–C3
Extension/Enrichment	#13, 17, 19–23, C2–C4

Planning Notes

This section develops a connection between the unit circle, angles in standard position, and the six trigonometric ratios. Students construct a unit circle model containing multiples of 30° and 45° angles $\left(\frac{\pi}{6} \text{ and } \frac{\pi}{4}\right)$ and the exact value coordinates for these points of intersection for the domain $0° \le \theta < 360°$ $(0 \le \theta < 2\pi)$. The unit circle supports students' understanding but does not replace problem solving skills. Students use their understanding to solve trigonometric equations and problems involving all six trigonometric ratios.

Tell students, "In grade 10 you studied the trigonometric ratios for cosine, sine, and tangent in right triangles. In grade 11 your study of trigonometric ratios was extended to include angles beyond the range 0° to 90°. Now you will relate trigonometric ratios to central angles and points on the unit circle. This leads to revised definitions of the trigonometric ratios."

Investigate Trigonometric Ratios and the Unit Circle

Before beginning the Investigate, break the class into groups of two or three students. Each group needs a compass to construct accurate drawings. You may also consider distributing **Master 3 Centimetre Grid Paper**.

When students are working on step 1, ask them whether points P and Q are inside, on, or outside the unit circle. Have them also describe the position of these points relative to A and B.

For steps 2 and 3, some students may need to revisit the trigonometric ratios. Ask them to discuss the effect on each ratio when the hypotenuse equals 1. Students can use this knowledge to respond to step 5 as well.

In step 3, some students may need a hint to identify a line segment for tan θ. Refer to the larger triangle, QBO. In △QBO,

$$\tan \theta = \frac{QB}{OB}$$
$$= \frac{QB}{1}$$

If students are having difficulty with step 8, guide them to recall that $\tan \theta = \frac{\text{opp}}{\text{adj}}, \frac{y}{x}$, or $\frac{\sin \theta}{\cos \theta}$. (You may need to help some students understand why $y = \sin \theta$, and $x = \cos \theta$.) Ask students when the tangent ratio is 0 and when it is undefined. Ask them to consider when these conditions occur on the unit circle. Then have them consider angle θ when these conditions occur.

Meeting Student Needs

- Discuss the outcome and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.
- Provide students with a handout listing the six trigonometric functions and ratios.

- Provide students with a handout of the unit circle containing coordinates of the points corresponding to multiples of 30°, 45°, and 60°. Alternatively, have students develop this tool as described below for Example 2.
- Some students may need help to realize that any point on the unit circle can be expressed in the form (x, y). Have students share how they represent any unknown point on the coordinate plane. Then ask how this relates to any point on the unit circle.

Answers

Investigate Trigonometric Ratios and the Unit Circle

1. P lies on the circle at the intersection of the terminal arm and the unit circle. Q lies outside the unit circle directly above point B.

2. $\sin \theta = y$ or $\overline{AP}$

3. $\cos \theta = x$ or $\overline{OA}$; $\tan \theta = \frac{y}{x}$ or $\frac{AP}{OA}$

4. $P(\theta) = (\cos \theta, \sin \theta)$

5. Given that $\triangle AOP$ is a right triangle and the circle is a unit circle,

$\cos \theta = \frac{\text{adj}}{\text{hyp}}$, where adj $= x$ and hyp $= 1$

$\cos \theta = \frac{x}{1}$

$\cos \theta = x$

6. $\cos^2 \theta + \sin^2 \theta = 1$; Yes, because all points on a unit circle are defined by the equation $x^2 + y^2 = 1$. Since $x = \cos \theta$ and $y = \sin \theta$, $\cos^2 \theta + \sin^2 \theta = 1$ defines all points of intersection between the terminal arm of an angle and the unit circle.

7. The maximum value of both $\cos \theta$ and $\sin \theta$ is 1 and the minimum value is -1. So, $-1 \le \cos \theta \le 1$ and $-1 \le \sin \theta \le 1$.

8. Since $\tan \theta = \frac{y}{x}$, $\tan \theta = 0$ when $y = 0$. The minimum value of $\tan \theta$ is 0. As the angle gets larger, from 0° to 90°, the size of y increases from 0 to 1 and the value of x decreases from 1 to 0. Therefore, as the angle increases, the value of $\tan \theta$ increases from 0 to $\frac{1}{0}$, which is undefined. Therefore, $0 \le \tan \theta <$ undefined.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs. • Discuss the approach to completing #5–7 as a class to ensure that students have a good understanding of cosine and sine. Have students complete #8 independently and compare their answer with a partner.

Example 1

Have students complete the example before discussing it with the class. Before students begin, you may want to review $\cos \theta$, $\sin \theta$, and $\tan \theta$. Ask them to use the definitions in the student resource to set up the ratios for $\sec \theta$, $\csc \theta$, and $\cot \theta$.

Example 2

Before working on Example 2, help students build a unit circle model, connecting special angles and their trigonometric ratios for $\sin \theta$, $\cos \theta$, and $\tan \theta$. Have students draw a unit circle centred at the origin. (Consider providing **BLM 4–4 The Unit Circle**). They can create labels on the inside of the circle for the angles as multiples of 30°, 45°, and 60° (including radian measure) and labelling the coordinates of the points for all multiples of $\frac{\pi}{6}\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$, $\frac{\pi}{4}\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$, and $\frac{\pi}{3}\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$. Students can complete Example 2 using their unit circles and then discuss their results with the class. You may also want to revisit the concept of reference angles before beginning this example.

Example 3

Before doing Example 3, revisit with students how to find trigonometric ratios using their calculator for both degrees and radians. Remind them that the calculator is only a tool, and they should understand what lies behind the numbers. Suggest that they sketch diagrams and use their knowledge of reference angles and sign conventions to estimate the values before entering them into their calculator.

Example 4

You may want to go over part a) with students. Doing so will help students avoid the error of finding only one answer for each ratio. Direct their attention to the domain and ask why it suggests that there will be two solutions. Then discuss with them how they can determine which quadrants the angles are in by looking at the sign of the given ratio. Students should not use rounded values in sequential calculations and only round after the final answer has been determined.

Students should work through the rest of the example on their own or with a partner. They can then discuss the results as a class. Suggest that they not move onto the Your Turn questions until they complete the example correctly. Meet with students who have difficulty and discuss any problems they are experiencing.

Example 5

Have students attempt this example before discussing it with the class. If students are having difficulty, ask:

- Can you sketch point A on the coordinate plane? In what quadrant does the point lie?
- When you draw a line from the point vertically to the x-axis, what is the value of the opposite side? the adjacent side?
- How can you find the value of the hypotenuse of the triangle?
- Can you write all six trigonometric ratios given the values of the sides of the right triangle?

Have students complete the Your Turn question and discuss it with the class.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Suggest that students look for videos on the Internet that show ways to remember different trigonometric concepts. For example, there are videos that show how students can use the palm of their hand to develop the values of the trigonometric ratios. Suggest that students try thinking of their own strategies for remembering what they learn.
- For Example 4, some students will benefit from a visual. For each angle, draw a sketch of the four quadrants. Label the quadrants using "A Smart Trig Class." Then, draw the two reference triangles in the correct quadrants according to the sign of the function. Students can then use their calculator to find the reference angle and the principal angle(s) for each question. For example, if $\tan \theta = -1.354$, sketch a reference triangle in quadrants II and IV because the tangent function is negative there.
- For Example 5, a sketch of the plotted point and required right triangle will assist students to find the missing side using the Pythagorean relationship.

Enrichment

Ask students to speculate about the reciprocal trigonometric ratios. When might they be used and why?

Gifted

Ask students to draw a representation of the rise and fall of an object as it rides on the waves of the ocean. Ask them to compare this movement to the value of sine ratios without a restricted domain. How are they similar? How are they different?

Common Errors

- Some students use the wrong form of angular measure in their answers.

R$_x$ Make sure that students get in the habit of writing down the given domain for each question and recognizing the type of angular measure used in the domain.

- When using a calculator, students often forget to change the angle mode.

R$_x$ Tell students that they should either draw a figure or create a quick mental image of the angle and mentally estimate the value or measure they expect. If the result produced by the calculator varies greatly from the expectation, they may want to check their calculator mode first.

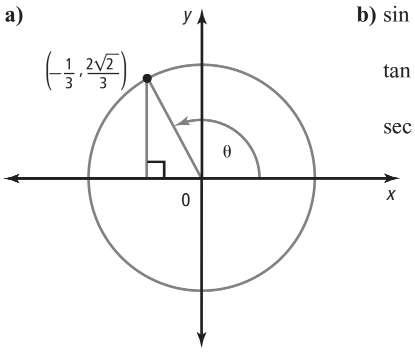
- Students often confuse which of the trigonometric ratios and their reciprocals go together. They frequently state that cosecant is the reciprocal ratio of cosine, and that secant is the opposite ration of sine.

R$_x$ A quick rule to give students having difficulty is to remind them that the lead letters in each word cannot match (the *cc* do not go together and the *ss* do not do together). Alternatively, help students recognize that there is only one "co" term in each pair: Sine/**Co**secant and **Co**sine/Secant.

Answers

Example 1: Your Turn

a)



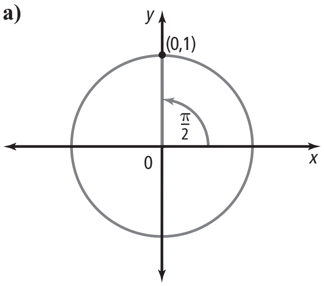
b) $\sin\theta = \frac{2\sqrt{2}}{3}$, $\cos\theta = -\frac{1}{3}$,

$\tan\theta = -2\sqrt{2}$, $\csc\theta = \frac{3\sqrt{2}}{4}$,

$\sec\theta = -3$, $\cot\theta = -\frac{\sqrt{2}}{4}$

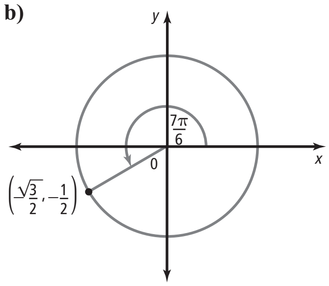
Example 2: Your Turn

a)



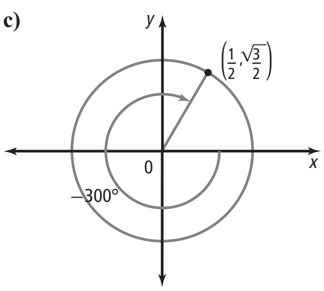
$\tan\frac{\pi}{2} = \text{undefined}$

b)



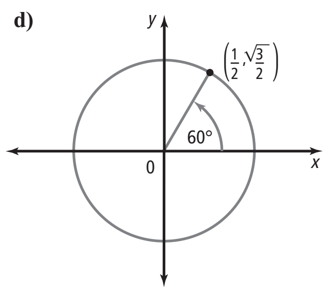
$\csc\frac{7\pi}{6} = -2$

c)



$\sin(-300) = \frac{\sqrt{3}}{2}$

d)



$\sec(60°) = 2$

Example 3: Your Turn

a) $\sin 1.92 \approx 0.9396$; $\sin\theta$ in quadrant II is positive

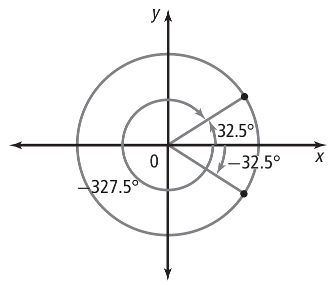
b) $\tan(-500°) \approx 0.8391$; $\tan\theta$ in quadrant III is positive

c) $\sec 85.4° \approx 12.4690$; $\sec\theta$ in quadrant I is positive

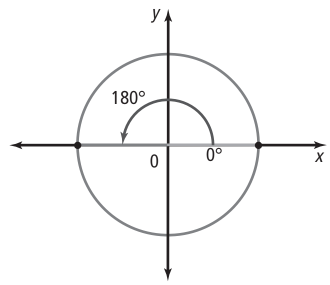
d) $\cot 3 \approx -7.0153$; $\cot\theta$ in quadrant II is negative

Example 4: Your Turn

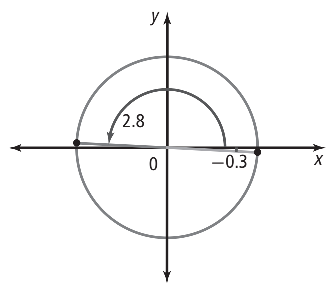
a) $\theta = -327.5°$, $-32.5°$ and $32.5°$



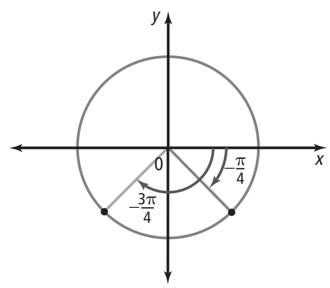
b) $\theta = 0°$ and $180°$



c) $\theta = -0.3$ and 2.8



d) $\theta = -\frac{\pi}{4}$ and $-\frac{3\pi}{4}$



Example 5: Your Turn

Sides are $x = -5$, $y = -12$, and $r = 13$

$\cos\theta = \frac{x}{r} = -\frac{5}{13}$; $\sec\theta = \frac{1}{\cos\theta} = -\frac{13}{5}$; $\sin\theta = \frac{y}{r} = \frac{-12}{13}$;

$\csc\theta = \frac{1}{\sin\theta} = -\frac{13}{12}$, $\tan\theta = \frac{y}{x} = \frac{12}{5}$; $\cot\theta = \frac{1}{\tan\theta} = \frac{5}{12}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Coach students to verbalize what quadrant the point will lie in. • Remind students that $x = \cos\theta$ and $y = \sin\theta$. This should assist in obtaining four of the six ratios. • You may need to review the process of rationalizing the denominator. For example, students should understand that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.
Example 2 Have students do the Your Turn related to Example 2.	• Have students draw in the rectangles for the 30°, 150°, 210°, and 330° measures, as well as for 60°, 120°, 240°, and 300°. This provides a visual that students could use to answer this and later questions in the chapter. • A square could connect 45°, 135°, 225°, and 315°.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Have students identify which quadrant the angle is in. Then, have them state the sign the answer should have before finding the measure with their calculator. • Remind students to change modes on their calculator as needed.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Revisit the reciprocal functions with students. • Have students identify what quadrant their value is in. Remind them that some may have more than one possible location. It is important to draw their attention to the domains given so that students recognize the number of rotations and/or direction of rotation involved. Some students may benefit from a quick review of coterminal angles.
Example 5 Have students do the Your Turn related to Example 5.	• You may wish to have students work in pairs. • Some students would benefit from using the coordinates to draw the triangle on a coordinate grid. Remind them it is a right triangle so they are able to find the missing side.

Check Your Understanding

For #2 to 4, 6, and 7, suggest that students make a sketch of the angle and use their knowledge of reference angles and sign conventions to determine the solutions.

For #5, students will have to sketch the point on the coordinate plane and then draw a right triangle. They can then use their knowledge of trigonometric ratios to determine the reference angle and to find the angle in standard position. Ask

- What quadrant does the point lie in?
- Which trigonometric ratio can you use?
- If the reference angle is in the first quadrant, what is the value of the angle in standard position? How do you determine the angle in standard position if the reference angle is in the second, third, and fourth quadrants?

Students can use a similar strategy in #8 as they did in #5. Suggest that they consider what the possible quadrants are for the point $\left(\frac{3}{5}, y\right)$. Ask, "What is the sign of y when the point is in the fourth quadrant?" Then guide them to use their knowledge of the side lengths to find tan θ and csc θ. Some students may need to be guided or see a worked example for this question. You may also tell students that they can use unit circle coordinates or the special reference triangles to substitute values and then simplify.

For #10, encourage students to draw direction arrows at the vertex to indicate the movement of the terminal arms within the given domain. Doing this should help them understand the possible solutions expected in the question. Ask, "What tool can you use to find the exact measure of an angle?"

Students must use reference angles and knowledge of sign conventions to solve #11. Point out to students that this question asks them to create a diagram, but that they can use this strategy for most problems. Suggest that students begin by considering the domain, the trigonometric ratio, and the sign to determine the possible quadrants of the angle.

Students should understand that #12 does not involve the unit circle. The strategy for solving these questions, however, is still based on a right triangle and understanding the trigonometric ratios. Suggest students draw a diagram showing the reference triangle for the given angle. Then, guide them in how they might use the Pythagorean relationship to determine the third side, and then write the other trigonometric ratios.

For #14, some students may need guidance to understand how to interpret 13.61 rotations and how to deal with the 0.61 rotation. Ask

- How many degrees are there in one full rotation?
- How many degrees are there in 0.61 of a rotation?
- What is the reference angle for a 220° angle?

Then guide students to find the trigonometric ratios for the reference angle, 40°.

Be sure that students recognize that they are dealing with radians in #17 and that they set the mode of their calculator accordingly. Discuss with students how the sign for cos θ differs from sin θ in quadrants II, III, and IV. Ask how they could use this information to make their prediction in part c).

For #18b), students must first find how long it will take the wheel to make one rotation. They can then determine how many rotations the wheel will make in one minute, and which quadrant it will be in at that time. Ask

- What are the values of x and y after the wheel rotates for 1 minute?

- Which value, x or y, on point P will determine how far the piston has travelled horizontally?
- How far will the piston move when point P is on the y-axis?
- How far will point P have travelled in radians for the remaining part of a rotation?

For #22, suggest that students use their model of the unit circle to assist them with parts a) and b). For part c), ask

- If you represent the new systems angle with α and the old conventional angles with θ, what do you get when you add, subtract, multiply, and divide each of these angles together?
- Do you see any relationship between these two sets of angles?
- Does your relationship work for degree measure? This question will help them with part d).

For #23, students can use $\sec \theta = \frac{1}{\cos \theta}$ in $\triangle OBQ$ to show that $\sec \theta = OQ$. This can be proved using similar triangles $\triangle OAP$ and $\triangle OBQ$.

Meeting Student Needs

- The Project Corner provides information about the origin of degrees and pi. Encourage students to research answers to the questions in the last bullet and to share their findings with the class.
- Provide **BLM 4–5 Section 4.3 Extra Practice** to students who would benefit from more practice.
- Students should complete the checklist of understanding provided in the opener. This will guide them when deciding which questions they need to practise to gain a deeper understanding of the outcomes for this section.
- Students should practise creating the unit circle, complete with 16 angles and the coordinates of the points on the circumference of the circle. Students can check each other's work. Once they are certain that the circles are correct, they can use the circle to help answer the questions in the Practise section.
- Students may wish to create a laminated, poster-size unit circle containing the angles and the coordinates of each point on the unit circle. A moveable terminal arm could be attached using a clip. Students could then rotate the terminal arm as indicated in a question and read the answer from the unit circle.
- For #4, a quick sketch of the given angle may assist students to find the reference angle and the sign of the answer. For the example, they could sketch 110° and find the reference angle using $180° - \theta = 70°$. They will see that the angle terminates in quadrant II where sine and its reciprocal is positive.
- Direct students to sketch a diagram whenever possible and to refer to it when working through the problems.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1a)–f), 2a)–f), 3–12, and 14. Students who have no problems with these questions can go on to the remaining questions.	• You may wish to have students work in pairs to draw an entire unit circle with 0° and multiples of the angles 30°, 45°, 60°, 90° labelled. They could also include the radian measures. This would be a useful reference tool for #1–6. • Guide students to recognize that a square could connect 45°, 135°, 225°, and 315°. • Ensure that students have a good understanding of how to determine the values for the coordinates if they do not have a reference circle. • Have students verbalize the difference between #7a) and b), and between c) and d). • Remind students that the equation $(\cos \theta)^2 + (\sin \theta)^2 = 1$ could be used to complete #8. • Ensure that students watch the restrictions on the domains for #10–12. You may wish to have them identify what quadrant they are working with before they begin each question. Doing so will assist them in determining the sign of the answer(s).
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2.	• Some students might find C1 easier to complete if they organize the values in a table. This way they can observe the changes in the sine values. • Remind students what a hexagon is and ask them to verbalize what the term *regular* implies when drawing polygons. This should provide them a lead for C2.

4.4 Introduction to Trigonometric Equations

***Pre-Calculus 12*, pages 206–214**

Suggested Timing

90–120 min

Materials

- grid paper
- compass
- protractor
- ruler
- scientific calculator

Blackline Masters

Master 3 Centimetre Grid Paper
BLM 4–4 The Unit Circle
BLM 4–6 Section 4.4 Extra Practice

Mathematical Processes for Specific Outcomes

T5 Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–4, 6–12, 15–17
Typical	#1–4, 6, 7, 9, 10, 12, 14-18, C1, C2, C4
Extension/Enrichment	#18–20, C1–C4

Planning Notes

This topic connects all the previous concepts and skills covered in this chapter. Students use their knowledge of solving linear and quadratic equations to develop a strategy for solving linear and quadratic trigonometric equations. They also learn to recognize the difference between finding a limited number of solutions within a given domain and expressing solutions in the general case. They then apply this understanding.

Throughout the section, help students appreciate the value and purpose of solving trigonometric equations. Begin by discussing with the class situations or applications that may involve cyclic patterns. Then, as a class, read over and discuss the Focus on … bullets. Discuss and define any terms with which students are unfamiliar.

Investigate Trigonometric Equations

The Investigate helps students recognize the similarities and differences between solving algebraic linear and quadratic equations and solving linear and quadratic trigonometric equations. Before the investigation, you may want to help students revisit skills in solving linear equations and solving quadratic equations by graphing, factoring, or using the quadratic formula. Also, reactivate their understanding of the difference between exact and approximate value solutions.

Discuss with students how the equation in the Investigate is similar to the equations they have solved in the past. Ask them how it is different? If some students are struggling, ask how they would solve the equation if they were to replace $\cos \theta$ with x. Then, ask what the difference is between solving the equation for θ and for $\cos \theta$. You may have to remind students to consider how many possible solutions they would expect to find in the given domain.

In step 4, ask students to consider how solving these two equations was similar and different. If students stop after reaching $\cos \theta = 0$ and $\cos \theta = 1$, discuss with them why their solution is incomplete. Make sure they understand that they are solving for the angle.

Meeting Student Needs

- Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found in this section.
- To see how light refracts in water, students could look at an object partially submerged in an aquarium or a sink. They can then use a glass or metal rod to touch the object. Ask them what they notice. Some students may want to find further information about the refraction of light in water.
- Encourage students to change the given equations to similar equations they know how to solve. For example, in step 1, ask them to explain how they would solve $x + \frac{1}{2} = 0$
- Some students are confused by the form of a trigonometric equation. Suggest that they set the trigonometric ratio equal to a variable, such as m, and then solve for m. They can then replace m with the original trigonometric ratio, and solve for the angle in each “mini equation.”

Answers

Investigate Trigonometric Equations

1. $\theta = \frac{2\pi}{3}$ and $\frac{4\pi}{3}$; they are the same

2. $\theta = 120°$ and $240°$

3. $\theta \approx 1.91$ and 4.37

4.

Quadratic Equation	Trigonometric Equation
$x^2 - x = 0$ $x(x - 1) = 0$ $x = 0$ and $x = 1$	$\cos^2 \theta - \cos \theta = 0$ $\cos \theta(\cos \theta - 1) = 0$ $\cos \theta = 0$ and $\cos \theta = 1$ $\theta = 0, \frac{\pi}{2}, \frac{3\pi}{2}$

5. Solving for a linear trigonometric ratio is the same as solving a linear equation for a variable. But once the ratio is determined, the trigonometric ratio and its sign are used to solve for the angle measure that makes the equation true.

6. The answer should be expressed using the same units as those listed in the given domain.

7. The strategy for solving quadratic trigonometric equations for the trigonometric ratio is exactly the same as solving quadratic equations for the variable. The difference is that in quadratic equations, once the variable has been determined and checked for correctness, the operation is finished. When solving quadratic trigonometric equations, after the trigonometric ratios have been determined, the angle that satisfies the given trigonometric equation needs to be found.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in partners for this activity. • Consider completely solving the quadratic on the board while asking students to verbalize the process. Ensure the term *common factor* is used, and that the process of setting each factor equal to zero is shown. Ask students if there are any parallels that could be applied to the trigonometric equation. • Ensure students understand that the values they have learned for the unit circle are often repeated. As a result, depending on the domain given, they may have multiple answers. For example, $\cos x = 0$ could mean $\frac{\pi}{2}$ or $\frac{3\pi}{2}$, depending on the domain.

Example 1

Before doing this Example, remind students that they are familiar with solving equations and that it is only the context that has changed. Suggest that students replace the trigonometric ratio with a variable, such as x. After solving the equation for x, they can replace x with the trigonometric ratio. Remind students that they are solving for the angle, so they need to determine the angle that satisfies the equation (e.g., in part a), the value for θ that satisfies the equation $\sin \theta = -\frac{1}{2}$).
In part b), ask students why $\csc x = 2$ equals $\sin x = \frac{1}{2}$. What trigonometric equation would they have solved for if the equation was $\sec x = 2$?

Example 2

If students are struggling with this example, ensure that they are comfortable with factoring. Work through a similar example using a quadratic equation. Ask how your sample is similar to the question in Example 1.

Ensure that students set up their ratio correctly and that they consider the domain when determining their solution. You may also want to revisit the strategy for checking answers. If students are still not getting the correct solution, ask

- Did you use the given domain to determine if the solution should be expressed in degrees or radians?
- Did you sketch a diagram to determine the quadrants of the angles, given the values of the trigonometric ratio in your solution?
- Did you correctly determine the value of the reference angles and use a positive value for the reference angle ratio?

Encourage any students who are having difficulty with the notation of trigonometric equations to replace the trigonometric ratio with a variable. They can then solve the quadratic equation for the variable, replace the variable with the original trigonometric ratio, and then solve for the angle that satisfies the given equation. This strategy is often helpful for students who are having difficulty factoring trigonometric equations.

Example 3

Discuss with students that there may be more than one way to solve an equation. Ask

- How could you solve the equation for $\sin x$ by using a square root?
- Can this quadratic expression be factored?
- Do any of the solutions for $\sin x$ indicate that there is an exact value answer?

For part b), suggest that students write the solutions for the domain $0 \leq x < 2\pi$ as a general solution. Then ask

- How many general solutions did you find for this equation?

- How far apart are the two solutions you found?
- Given that the two solutions you found are π units apart, is there a way to write a general solution that combines the two?

Meeting Student Needs

- Consider posting examples of interval notation in the classroom. Ask students to write, in words, the meaning of each example you have posted.
- For Example 1, some students will find it helpful to first solve $x^2 - 5x + 4 = 0$. They may also wish to solve $\tan^2 \theta - 5 \tan \theta + 4 = 0$ using substitution.
- You may wish to provide more examples for the class to solve together prior to students doing individual work.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Students may forget to solve for the correct number of solutions.

R_x Remind students to get into the habit of determining the number of solutions before they solve equations. They need to use their knowledge of trigonometric sign conventions and rotations of angles around the unit circle to determine the number of solutions for all angles except those that may have non-permissible answers.

- Some students may not use the correct format to write the general solution of an equation.

R_x Remind students that they need to describe the original terminal angle in the first positive rotation and then add either $360n$ or $2\pi n$ to that angle. Suggest that they draw an angle and then trace an arrow with their pencil showing the effect of adding $360n$ or $2\pi n$. Also discuss the restrictions on the value of n.

Answers

Example 1: Your Turn

a) $\theta = -2\pi, 0, 2\pi$

b) $x = 120°$ and $240°$

Example 2: Your Turn

$\theta = 180°$

Example 3: Your Turn

a) $x = 0°$ and $180°$

b) $x = 0° + 360°n, n \in I$, and $180° + 360°n, n \in I$; or $\theta = 0° + 180°n, n \in I$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Students will need to recall the side lengths and angles of the special triangles. • Students may need to be reminded to consider the solution only in the specified domain.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Students may find it easier to initially change the cos θ in the equation to x and solve a basic quadratic first. Once they have done a few questions they will not need to make this substitution. • Remind them that rules of basic factoring apply here, as would the quadratic equation if needed.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • It is important to help students link the concept of coterminal angles with the term *general solution*. • Remind students to watch the limits on the domain, if any exist.

Check Your Understanding

For #6, discuss interval notation, where square brackets indicate "greater than and equal to" or "less than and equal to." Ask students to describe which equality symbol is associated with the opening and closing square brackets. Also discuss what type of inequality is described by a round bracket. Ask, "What does the order of values inside the brackets indicate?"

For #9, ask students to look at the equation and consider whether it is a linear or quadratic trigonometric equation. Ask, "How many possible solutions do you get if you divide both sides of this equation by sin θ?" Have students consider whether a single solution fits with their understanding of solving a quadratic equation. Alert students that dividing both sides in a situation like this is a common error. Ask them to describe the correct strategy.

Suggest that students use their unit circle to determine the minimum and maximum values for $\sin \theta$ for #11. Have them consider whether the value 2 lies within this range of values. Ask, "Will $\sin \theta = 2$ have a solution?" (Note that #17 is similar to this question.)

For #15a), some students may need guidance to recognize that since variable y, the number of air conditioners, is expressed as thousands, its value is 8.3. You may also have to lead some students through a discussion of how this problem represents a cyclical problem, and how it therefore relates to the unit circle and angular measure. Ask

- To solve for t, what type of angular measure should you use, degrees or radians?
- What radian measure satisfies the equation $\sin \theta = 1$?
- Given that $1 = \sin\left(\frac{\pi}{6}(t - 3)\right)$ in this question, what is $\left(\frac{\pi}{6}(t - 3)\right)$ equal to?

For #15b), ask students how they can determine the minimum and maximum values of the function graphically and algebraically. Note that #19 and 20 are similar to #15, and a similar line of questioning could be used to guide students having difficulty.

Meeting Student Needs

- Provide **BLM 4–6 Section 4.4 Extra Practice** to students who would benefit from more practice.
- Students should complete the checklist of understanding provided in the opener. Doing so will guide them when deciding which questions they need to practise to gain a deeper understanding of the outcomes for this section.
- For #1, students can rewrite the domain in interval notation.
- For #14, invite a physics teacher to present information about Snell's Law and how it is used. Examples can be presented of its applications in various careers.
- Students can work in pairs, exchanging and evaluating solutions, including identifying errors if possible.

Enrichment

Have students select one of the procedures developed in section 4.4 and create a flowchart that shows the process.

Gifted

Ask students to explore the connections between the procedures used to factor numbers, algebraic expressions, and trigonometric expressions. What makes the process the same? What makes them different?

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–4, 6–12, and 15–17. Students who have no problems with these questions can go on to the remaining questions.	• Allow students to refer to their drawing of the unit circle as an aid for #1–3. Remind them to watch the limits on the domain. Encourage students to show their work in solving all trigonometric equations in the same way they would for quadratics. This is especially true for #3, 7, and 8. • For students having difficulty with #9, present and discuss the following algebraic solutions before students try the question on their own: $x^2 = x$ $x^2 = x$ $\frac{x^2}{x} = \frac{x}{x}$ $x^2 - x = 0$ $x = 1$ $x(x - 1) = 0$ $x = 0$ and $x = 1$ Ask students to identify which solution is correct. Ensure that before they return to #9 they understand that dividing both sides eliminated one of the solutions. • If students are having difficulty, coach them through #10 before letting them try #11.
Assessment *as* Learning	
Create Connections Have all students complete C1, C2, and C4.	• Suggest that students refer back to earlier sections in the chapter to find ways to prove whether the point in C2 is on the unit circle. Remind them of the inverse function. • You may have to coach students to make a link between non-permissible values and undefined values to assist them in answering C3. • You may consider doing C1 to C4 as a mini project to summarize students' understanding of the skills and concepts covered in this section of the chapter. This would also be an opportunity for students to do peer assessments of each other's work, and it could open up discussion among students of the concepts covered in this chapter.

4 Chapter 4 Review and Practice Test

***Pre-Calculus 12*, pages 215–219**

Suggested Timing
60–135 min each

Materials
- grid paper
- compass
- protractor
- ruler
- scientific calculator

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 4–2 Section 4.1 Extra Practice
BLM 4–3 Section 4.2 Extra Practice
BLM 4–4 The Unit Circle
BLM 4–5 Section 4.3 Extra Practice
BLM 4–6 Section 4.4 Extra Practice
BLM 4–7 Chapter 4 Study Guide
BLM 4-8 Chapter 4 Test

Planning Notes

Have students who are not confident discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource. Consider distributing copies of **Master 3 Centimetre Grid Paper** and **BLM 4—4 The Unit Circle** to encourage students to draw diagrams to help them with the questions.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test. You may wish to provide students with **BLM 4—7 Chapter 4 Study Guide**.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1–12.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 4—2 Section 4.1 Extra Practice**, **BLM 4—3 Section 4.2 Extra Practice**, **BLM 4—5 Section 4.3 Extra Practice**, and/ **BLM 4—6 Section 4.4 Extra Practice**.
- Students should work through the complete list of outcomes provided at the beginning of the unit. Which outcomes and indicators do they know? Which outcomes and indicators require extra study and practice? Encourage students to partner with a student who needs similar practice. What other resources are available to assist them with the learning?
- Students are encouraged to write the practice test and then assess their own answers. Use the results to determine which areas require attention prior to the unit assessment.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 4 Review The Chapter 4 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #1–5, 7, 9–17, 19a), b), 20a), b), 21	• Encourage students to use **BLM 4–7 Chapter 4 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 4 Study Guide Have students use their responses on the practice test and work they completed earlier in the chapter to identify skills or concepts they may need to reinforce.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. Students should be able to answer confidently the suggested minimum questions. Encourage students to try additional questions beyond the minimum. • Consider allowing students to use any summative charts, concept maps, or graphic organizers in completing the practice test.
Assessment *of* Learning	
Chapter 4 Test After students complete the practice test, you may wish to use **BLM 4–8 Chapter 4 Test** as a summative assessment.	• You may wish to have students refer to **BLM 4–7 Chapter 7 Study Guide** and identify areas they need reinforcement in before beginning the chapter test.