Trigonometric Functions and Graphs

Opener

***Pre-Calculus 12*, pages 220–221**

Suggested Timing
30–45 min

Blackline Masters
BLM 5–1 Chapter 5 Prerequisite Skills
BLM U2–1 Unit 2 Project Checklist

Planning Notes

Approach this chapter by exploring as many ways to represent a question as possible. Students should be able to graph the function, identify the important parts, and determine the equation given a graph or parts of the function. Using graphs and technology will help students visualize the differences and similarities of the sine, cosine, and tangent functions.

The first two sections of the chapter focus on the sine and cosine function. Students begin by discovering the effects of a and b on the function. Then, students move on to discover the effects of c and d on the function. Students should be proficient in interchanging the equation and the graph of a sine and cosine function. Students should understand the cosine and sine functions are the same, and only differ by a phase shift.

Section 5.3 deals with the tangent function. Rather than teaching it as a separate function, it should be approached as the ratio of sine to cosine. This function is different from the previous two functions primarily with respect to the period and asymptotes.

Section 5.4 allows students to apply the skills and knowledge they gained from the previous sections to solve real-world problems by graphing and solving trigonometric equations. If they have a true understanding of periodic functions they can choose which function should be used to solve each question. Students should also be comfortable applying the parts of a function to the context of any given situation.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter. Encourage students to use a summary method of their choice.

Meeting Student Needs

- Consider having students complete the questions on **BLM 5–1 Chapter 5 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out **BLM U2–1 Unit 2 Project Checklist**, which provides a list of all the requirements for the Unit 2 project.
- Provide students with a copy of the student learning outcomes for the unit.
- Discuss the Key Terms. Students may wish to create a word wall for the chapter or create a list of key terms and definitions in their notebooks.
- Prepare a handout illustrating the graph of a sinusoidal function. You can either label the key parts beforehand, or have students label them either as a class or in small groups.

Enrichment

The chapter begins with natural examples of sinusoidal and periodic patterns, such as the image of Painted Hills at sunset, John Day Fossil Beds National Monument Painted Hills Unit, Oregon, USA. Encourage students to keep a list of examples of such patterns they have seen in nature, on the news, and so on. Have them speculate on the cause of the patterns. For example, the gravitational force of the moon, along with the orbit of the Earth and the moon, produces the periodic pattern of the tides.

Gifted

Ask students to use the unit circle to explain how exact sine values were arrived at before the advent of computer technology. Note that, later, students will be asked to actually do the calculations.

Career Link

Discuss with students what they know about geology. Ask
- What do you know about geology?
- What do you think geologists study?
- With what kind of industries are geologists involved?
- How do you think geologists might use mathematics and, in particular, trigonometry?

Suggest that students go online to research careers in geology. Are there any aspects of the career that surprise them?

5.1 Graphing Sine and Cosine Functions

***Pre-Calculus 12*, pages 222–237**

Suggested Timing
135–180 min

Materials
- grid paper
- ruler
- graphing calculator
- protractor
- compass
- spreadsheet software (optional)

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 5–2 Section 5.1 Extra Practice
TM 5–1 How to Do Page 226 Example 2a) Using TI 83/84
TM 5–2 How to Do Page 226 Example 2a) Using TI-Nspire™

Specific Outcomes
T4 Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.
- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–10, 12, 14, 18
Typical	#1–11, 13, 14, one of 15–17, 18, 19, C1, C3
Extension/Enrichment	#6, 10, 20–24, C2, C5

Planning Notes

Begin this section by discussing the learning outcomes. As a class, read the opening paragraph and examples of cyclic phenomenon. Ask students
- Can you think of other events or quantities that are cyclic?
- Why would it be beneficial to be able to mathematically model cyclical events?

Tell students that in this section they will be learning about cyclical functions, the sine and cosine functions in particular.

Investigate the Sine and Cosine Functions

Prepare for the Investigate by having students recall the side lengths and angles of the special triangles. You may wish to have students work on the investigation in groups of two or three.

If students choose to use spreadsheet software to complete the table in step 1, ensure they create the graphs by hand.

In step 2, students may be somewhat familiar with the term *amplitude*, but they may not know exactly how to determine it. The term is defined later, in Example 1; so, you may need to explain to students that amplitude is half of the distance between the maximum and minimum values of a periodic function.

Meeting Student Needs

- Provide students with **Master 3 Centimetre Grid Paper**.
- Provide students with graphs of $y = \sin \theta$ and $y = \cos \theta$. Ask students to discuss the similarities and differences as a class, or in small groups.
- After students have completed the Investigate, help them link the discussions about $y = \sin \theta$ and $y = \cos \theta$ with the graphs of tides, Earth's orbit, and/or animal populations.

Common Errors

- Some students have difficulty determining in which quadrant an angle in radians lies.

R_x Reinforce that 2π rad is equivalent to 360°. Ask students what $\frac{1}{4}$, $\frac{1}{2}$, and $\frac{3}{4}$ of each of these angle measures represents. Ask them how to determine these values and how they can express the angle measure of quadrants I, II, III, and IV using inequalities.

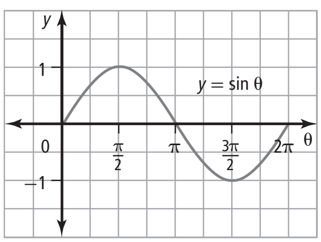
Answers

Investigate the Sine and Cosine Functions

1. Example:

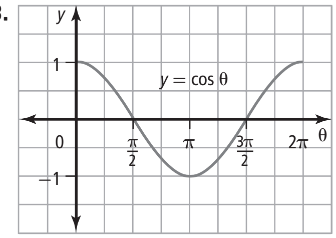
Angle, θ	$y = \sin \theta$	$y = \cos \theta$
0	0.00	1.00
$\frac{\pi}{6}$	$\frac{1}{2} = 0.50$	$\frac{\sqrt{3}}{2} \approx 0.87$
$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}} \approx 0.71$	$\frac{1}{\sqrt{2}} \approx 0.71$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$	$\frac{1}{2} = 0.50$
$\frac{\pi}{2}$	1.00	0.00
$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2} \approx 0.87$	$-\frac{1}{2} = -0.50$
$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}} \approx -0.71$	$-\frac{1}{\sqrt{2}} \approx -0.71$
$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2} \approx -0.87$	$\frac{1}{2} = 0.50$

2. a)



b) maximum value: 1, minimum value: -1, repeat interval: 2π, zeros: $x = 0, \pi, 2\pi$, y-intercept: 0, domain: $[0, 2\pi]$, range: $[-1, 1]$

3.



maximum value: 1, minimum value: -1, repeat interval: 2π, zeros: $x = \frac{\pi}{2}, \frac{3\pi}{2}$, y-intercept: 1, domain: $[0, 2\pi]$, range: $[-1, 1]$

4. a) Example: The graph will continue to curve up and down to the right between $y = 1$ and $y = -1$. The next cycle will end at $\theta = 4\pi$.

b) Example: The graph will continue to curve down and up to the left between $y = 1$ and $y = -1$. The next cycle will end at $\theta = -2\pi$.

5. Example: The graph will continue to curve up and down in both directions between $y = 1$ and $y = -1$. The next cycle to the left will end at $\theta = -2\pi$ and the next cycle to the right will end at $\theta = 4\pi$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigation. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • If students are having difficulty with #4, ask them to review how they answered the interval over which their graphs repeated in #2b). Check their understanding and ensure they responded to #2a) correctly. Ask students how this information could assist them in working through #4. Alternatively, students could be referred to a pre-drawn unit circle and asked to show where they would locate sin 390° and/or cos 420°. Ask them how many degrees would take them to a second complete trip around the unit circle and how this could help answer #4.

Example 1

Use this example to have a discussion about the mode function on a calculator. Discuss when students should use radians and when they should use degrees. You might help students recall how to convert between degrees and radians.

Ask students to describe in their own words what *amplitude* is. They should understand that the amplitude is the distance from the horizon to the maximum and minimum. Use this discussion to talk about the *horizon of the graph*. This discussion will lead nicely into Example 2, as well as provide a good foundation for section 5.2.

After students work on the Your Turn, have a class discussion about the similarities and differences between the sine curve and the cosine curve.

Example 2

Use this example to talk about what the amplitude affects. Ask whether amplitude affects the

- maximum and minimum
- period
- domain
- range
- x-intercepts
- y-intercept

This discussion could be used to examine the effects of all the parameters on the sine and cosine functions. Ask students

- Is the effect of a on sine the same as its effect on cosine?
- What is the difference between a positive value of a and a negative value of a?

Example 3

Talk about the effects of changing the value of b. Ask:

- How does the value of b affect the
  - maximum and minimum?
  - period?
  - domain?
  - range?
  - x-intercepts?
  - y-intercept?
- What is the difference between a positive value of b and a negative value of b?
- Is the effect of b on sine the same as its effect on cosine?

Make sure students understand that a negative value of b indicates that the function is reflected in the y-axis. The absolute value of b is used to determine the period because period is a length and must be positive.

Example 4

Some students may wonder if it matters whether they begin the graph of the function with the vertical or horizontal stretch. Suggest that half of the class do the Your Turn question doing the horizontal stretch first, and the other half do the vertical stretch first. This will illustrate to students that it does not matter which one they do first.

You could go over the sine and cosine curves from 0 to 2π radians, using Method 2. The following table may be useful in your explanation.

	0	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Sine	x-intercept	maximum	x-intercept	minimum	x-intercept
Cosine	maximum	x-intercept	minimum	x-intercept	maximum

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them save the summary for each section in the same location so they can revisit these concepts later.
- Students can add the terms from the Link the Ideas to the Word Wall or pocket dictionary they have started.
- You may wish to provide students with **TM 5–1 How to Do Page 226 Example 2a) Using TI 83/84** or **TM 5–2 How to Do Page 226 Example 2a) Using TI-Nspire™**.
- Develop the concept that the ratio of the circumference of a circle to its diameter is π by providing students with circles of various diameters. Make sure the centre is clearly marked on each circle. Students can then measure the diameter and circumference. What do they find when they divide the circumference by the diameter? (Students may have done this activity in an earlier grade, but doing this activity again would serve as good reinforcement.)
- For Example 1, students may wish to convert the exact values in the chart to decimal values to assist with graphing the points.
- Explain to students that a chord is a line segment joining two points on a circle. Ask them to illustrate how half-chords are related to the sine ratio.
- Create a chart listing the characteristics of the sine graph, allowing two blank columns for the cosine graph and, in a later section, the tangent graph. Suggest that students use brightly coloured paper to allow them to find this chart quickly.
- Create a poster for $y = a$ (trig function) $b(\theta - c) + d$. As you work through the section, develop the parameters individually:
  - amplitude $= |a|$
  - period $= \frac{2\pi}{|b|}$ (sine or cosine)
- Encourage students to remember that sine and cosine functions are the same except for a horizontal translation: sine passes through (0, 0) and cosine passes through (0, 1). Otherwise, the concepts developed in this section are the same for both functions.
- Consider teaching all concepts for sine on the whiteboard and have students work in pairs to develop the same concepts for cosine at their desks, either as partners or individually (allowing for student preference). As students determine the key points for cosine, add the information to the whiteboard table.

Enrichment

Have a class discussion about, or have students discuss in pairs, how the sine and cosine functions are the same and how they are different. Ask students to prepare a Venn diagram that highlights the similarities and differences.

Gifted

Before the age of computers, inexact trigonometric values of sine and cosine were calculated by interpolation between exact values. Challenge students to use the unit circle and the Pythagorean theorem to find the exact values of angle measures, where possible, and to record them in a table. Ask students to then use interpolation to expand their table. Have them compare their table values to values that they calculate with a calculator. Ask, "What could account for any differences?"

Common Errors

- Some students think that b is the period.

Rx Reinforce the notion that b is the number of times the function repeats in 360° or 2π radians.

- Some students may think that amplitude can have a negative value: $-a$.

Rx Reinforce that amplitude is a distance. A distance cannot be negative, so a function cannot have a negative amplitude. Since amplitude is a positive value, it is equal to $|a|$.

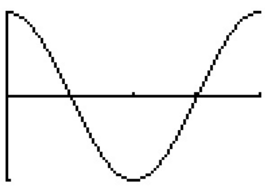
- Students get confused with the value of b. If $y = f(bx)$, they sometimes think that there is a horizontal stretch by a factor of b, rather than $\frac{1}{b}$.

Rx Use the examples of $f(x)$, $f(2x)$, and $f\left(\frac{x}{2}\right)$ to explain why horizontal stretches are different. Stress that $f(2x)$ does not mean that you are multiplying x by 2, but that you are taking half of the original value of x, which is why there is a horizontal stretch by a factor of $\frac{1}{2}$.

Answers

Example1: Your Turn

a)

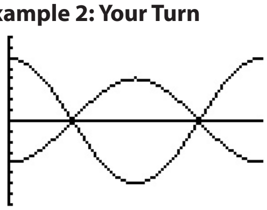


Use window settings:
x: [0, 360, 90]
y: [−1, 1, 1]

- The curve is periodic.
- The curve is continuous.
- The domain is $\{\theta \mid \theta \in R\}$.
- The range is $\{y \mid -1 \le y \le 1, y \in R\}$.
- The maximum value is +1.
- The minimum value is −1.
- The amplitude of the curve is 1.
- The period is 2π or 360°.
- The y-intercept is 1.
- The x-intercepts, in radians, are ... $\frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$, … , or $\frac{(2n-1)\pi}{2}$, where $n \in I$. In degrees, they are ... −270°, −90°, 90°, 270°, ... , or $(180n - 90°)$.

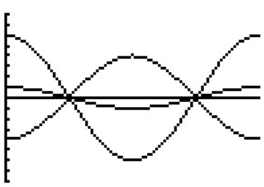
Example 2: Your Turn

a)



b) The amplitude of $y = 6 \cos x$ is 6 The amplitude of $y = -4 \cos x$ is 4.

c)

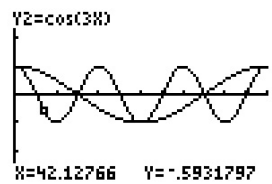


Function	Period	Amp.	Specified Domain	Range
$y = \cos x$	2π	1	$\{x \mid 0 \le x \le 2\pi, x \in R\}$	$\{y \mid -1 \le y \le 1, y \in R\}$
$y = 6 \cos x$	2π	6	$\{x \mid 0 \le x \le 2\pi, x \in R\}$	$\{y \mid -6 \le y \le 6, y \in R\}$
$y = -4 \cos x$	2π	4	$\{x \mid 0 \le x \le 2\pi, x \in R\}$	$\{y \mid -4 \le y \le 4, y \in R\}$

d) The amplitude of $y = 1.5 \cos x$ is 1.5.

Example 3: Your Turn

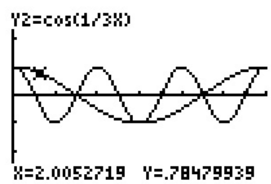
a)



Use window settings:
x: [0, 360, 30]
y: [−3, 3, 1]

The period is 120°. It has the same amplitude, domain, and range as $y = \cos x$, but the period is three times smaller.

b)



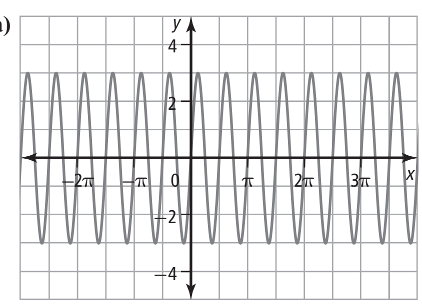
Use window settings:
x: $[0, 6\pi, \pi/2]$
y: [−3, 3, 1]

The period is 6π. It has the same amplitude, domain, and range as $y = \cos x$, but the period is three times greater.

c) The period is $\frac{2\pi}{3}$.

Example 4: Your Turn

a)



b) amplitude: 3; period: $\frac{\pi}{2}$; maximum: 3; minimum: −3; x-intercepts: $\frac{\pi}{4}n$, $n \in I$; y-intercept: (0, 0); domain: $\{x \mid x \in R\}$; range: $\{y \mid -3 \le y \le 3, y \in R\}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Revisit the table and the characteristics of the sine function before students start on the cosine function in Your Turn. • Suggest that students create a table and draw graphs similar to the ones in the example. • Some students will be able to identify the characteristics from the table, while others may need guidance. • Have students label the length of the period and the amplitude on their graph. • Remind students that amplitude measures distance, so it cannot be negative.
Example 2 Have students do the Your Turn related to Example 2.	• Ask students to identify what the amplitude is for $y = \cos x$. What is the value of a? • Have students refer to the graph of cosine they made in Example 1 Your Turn. Then, suggest that they label the amplitudes and the lengths of periods for each of the functions in the question on which they are working.
Example 3 Have students do the Your Turn related to Example 3.	• Remind students to graph $y = \cos x$ first, and then compare the graph of that function with $y = \cos 3x$ and $y = \cos \frac{1}{3}x$. • Students should revisit transformations if they wish to approach the questions from this perspective. Revisiting this concept will also assist with the next example. • If students are having difficulty recognizing the difference between the functions in parts a) and b), suggest that they graph all three functions on the same set of axes.
Example 4 Have students do the Your Turn related to Example 4.	• Remind students that they can either create their graph using transformations or using key points, whichever they find easier. Allow students to choose their approach. • Many students will benefit from visuals of a graph when answering part b). Some students may find it easier to express the values in degrees, rather than radian measures.

Check Your Understanding

For #4, students can check their answer using the formula or by simply recognizing that a = amplitude.

Students can check #5 using the formula.

Encourage students to complete #6 without their graphing calculator.

For #7, you may have to remind students that the vertical stretch is $|a|$ about the x-axis.

For #8, students should remember the horizontal stretch is $\frac{1}{|b|}$ about the y-axis.

For #10, ask students how they can determine if they are looking at a sine or a cosine graph.

For #14, have students consider how their answers would change if they were told that the graphs were of cosine waves.

For #16, which is a Mini Lab, have students work in groups of two or three. In step 2, groups will have different measurements for their opposite and hypotenuse because they will have different circle sizes. Regardless of the circle size, the sine ratio should be the same for all groups. Ask students why the y-axis is labelled from -1 to $+1$. In step 4, students are asked to complete the chart using the symmetry of the sine curve. If students are having difficulty, have students recall the signs of sine in the four quadrants. Sine is positive in quadrants I and II (0° to 180°), and negative in quadrants III and IV (180° to 360°).

You may want to discuss #19 as a class because part c) leads nicely into section 5.2.

For #21, ask students to discuss the methods they chose and why.

Students see an application of period in #22.

Do C1 as a class. Use the view screen to illustrate the question.

Trigonometric tables might help in C5. Encourage students to substitute three or four values into the expression to prove that the value it returns is constant.

Meeting Student Needs

- Provide **BLM 5–2 Section 5.1 Extra Practice** to students who would benefit from more practice.
- Before students begin the Check Your Understanding section, revisit the terms learned in this section.
- If students completed the table of information while the concepts were developed, they can omit #3.
- For #11, outline the expectations as to the amount of detail required for the sketch. Usually the five key points (or in this case, the ten key points) are required.

- For #13, you might invite a student to demonstrate a second and third harmonic on a keyboard or you may choose to find examples on the Internet to assist students to develop a better understanding of harmonics.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–10, 12, 14, 18. Students who have no problems with these questions can go on to the remaining questions.	• In #1 to 3, the basics of the sine and cosine functions are revisited. It is important that students are confident with these graphs before moving on to questions that vary the parameters. It may be beneficial to have some students create a class poster reflecting sine and cosine, which could then be posted in the classroom as a visual reference. • You may need to prompt and coach students regarding which parameters are linked to amplitude and period. Ask – How does the *sign* of *a* affect the graph? – How does the *value* of *a* affect the stretch? Similar prompts may be needed for the period. • Ask students to start a page graph and chart of information for each of the trigonometric functions. They could build on this (or some other type of graphic organizer) as they study each parameter. A four door Foldable would be an excellent way for students to model their thinking. • For #6, use the graphs students made in #1 and 2 or the visuals on the wall, and have students verbalize what changes have occurred in each of the graphs. • For #9, ask students what parameter they have studied will help to answer the question. For each question, ask them to identify the value for *a* and *b*, and then have them verbalize how they can use *b* to find the period. This will assist them for #14 and 15 that present real world data and ask students to interpret it.
Assessment *as* Learning	
Create Connections Have all students complete C1. Students not experiencing any difficulties can move on to the remaining questions.	• It may benefit some students to work in partners for C1, a Mini Lab, so they can compare and discuss their thinking. This Mini Lab asks students to compare the relationship between the unit circle and the sine and cosine graphs, using technology. This relationship is important for students to understand. If possible, complete the lab as a class and display the results for all students. • Students who have no problems with this mini lab should be encouraged to try the remaining questions.

5.2 Transformations of Sinusoidal Functions

***Pre-Calculus 12*, pages 238–255**

Suggested Timing

90–135 min

Materials

- grid paper
- graphing technology
- coloured pencils

Blackline Masters

BLM 5–3 Section 5.2 Extra Practice

Specific Outcomes

T4 Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.

- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Technology (T)
- ✓ Visualization (V)

RF2 Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

RF3 Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

RF4 Apply translations and stretches to the graphs and equations of functions.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–6, 11–13, 15–17, 21
Typical	#1–7, 12, 14–17, one of 18–22, 26, C1, C2, C4
Extension/Enrichment	#5, 12, 14, 19, 23, 25–28, C1, C3, C4

Planning Notes

Begin this section by discussing the learning outcomes for the section or providing them on a handout. Ask students

- Have you ever heard the word *phase*? What does this word mean to you?
- Have you ever heard the word *displacement*? What does this word mean to you?
- From your understanding of these two words, what do you think *phase shift* means? How about *vertical displacement*?

Tell students that in this section, they are going to explore these terms in the context of sinusoidal functions. Ask them to predict what the terms might mean in this context. Tell them that the motion of the hands of a clock, the movement of a planet around the Sun, and the motion of the tip of a fan blade are all examples of circular, repetitive motion, and that each of these can be modelled using sinusoidal functions. However, transformations of functions are necessary for the graphs of sinusoidal functions to be applied to real-world situations.

Investigate Transformations of Sinusoidal Functions

The investigation provides the opportunity for students to discover physically what effects the parameters c and d have in a transformation.

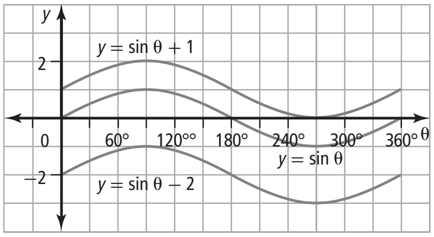
Encourage students to use their knowledge of transformations to predict what the graph of each function will look like. Ensure that students can distinguish each sinusoidal function they sketch, perhaps using coloured pencils.

Meeting Student Needs

- For #3, allow students to justify their predictions by substituting values or by graphing.
- Students may benefit from working in pairs and taking turns sketching the graphs.
- Some students may need to be reminded to sketch the functions only over the given domains.

Answers

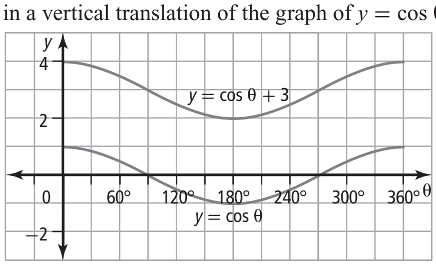
Investigate Transformations of Sinusoidal Functions

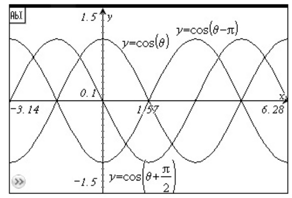
1.


2. Compared to the graph of $y = \sin\theta$, the graph of $y = \sin\theta + 1$ is translated 1 unit up and the graph of $y = \sin\theta - 2$ is translated 2 units down.

3. Example: The graph of $y = \sin\theta + 3$ will look like the graph of $y = \sin\theta$ translated 3 units up and the graph of $y = \sin\theta - 4$ will be the graph of $y = \sin\theta$ translated 4 units down.

4. a) When $d > 0$ in the function $y = \sin\theta + d$, the graph of $y = \sin\theta$ is translated d units up.
b) When $d < 0$ in the function $y = \sin\theta + d$, the graph of $y = \sin\theta$ is translated $|d|$ units down.

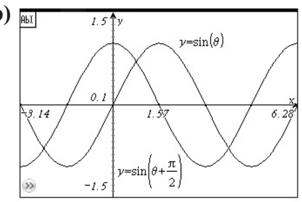
5. a) Example: The parameter d in the function $y = \cos\theta + d$ will result in a vertical translation of the graph of $y = \cos\theta$.
b)


6.


7. Compared to the graph of $y = \cos\theta$, the graph of $y = \cos\left(\theta + \frac{\pi}{2}\right)$ is translated $\frac{\pi}{2}$ units to the left and the graph of $y = \cos(\theta - \pi)$ is translated π units to the right.

8. Example: The graph of $y = \cos\left(\theta - \frac{\pi}{2}\right)$ will look like the graph of $y = \cos\theta$ translated $\frac{\pi}{2}$ units to the right and the graph of $y = \cos\left(\theta + \frac{3\pi}{2}\right)$ will be translated $\frac{3\pi}{2}$ units to the left.

9. a) When $c > 0$ in the function $y = \cos(\theta - c)$, the graph of $y = \cos\theta$ is translated c units to the right.
b) When $c < 0$ in the function $y = \cos(\theta - c)$, the graph of $y = \cos\theta$ is translated $|c|$ units to the left.

10. a) Example: The parameter c in the function $y = \sin(\theta - c)$ will result in a horizontal translation of the graph of $y = \sin\theta$.
b)


Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigation. Encourage them to generalize and reach a conclusion about their findings.	• If students are having difficulty with either parameter, suggest they graph more familiar functions, such as $y = x^2$, $y = x^2 + 2$, and $y = x^2 - 3$, and then describe how they differ. Ask students to generalize what they believe would be similar for a sine or cosine function. Use a similar set of parabolic functions for the other parameter, if necessary.

Example 1

Use this Example to talk about what the values of d and c affect. Ask students

- Does d affect
  – the maximum and minimum?
  – the period?
  – the domain?
  – the range?
  – the x-intercepts?
  – the y-intercepts?
- What is the difference between a positive value of d and a negative value of d?
- Does d have the same effect on sine as it has on cosine?

Then, ask students the same questions about the value of c.

Example 2

Ensure that students can visualize that the sinusoidal axis is the mid-line of the curve. Students are often confused why the phase shift is to the left, rather than to the right. Have a discussion about why this is. You might want to discuss with students that this question could also be done using key points.

Example 3

An alternative way to use the key points to graph the function is as follows:

- Since the period is π, find the x-coordinate of the five points that split the first cycle into quarters: $0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}$, and π.
- Since there is a phase shift of $\frac{\pi}{3}$ to the right, add $\frac{\pi}{3}$ to the above x-coordinates: $\frac{\pi}{3}, \frac{7\pi}{12}, \frac{5\pi}{6}, \frac{13\pi}{12}$, and $\frac{4\pi}{3}$.
- To find the y-coordinates of the five points, combine the amplitude, 3, with the vertical displacement, 2: $\left(\frac{\pi}{3}, 2\right); \left(\frac{7\pi}{12}, 5\right); \left(\frac{5\pi}{6}, 2\right); \left(\frac{13\pi}{12}, -1\right)$, and $\left(\frac{4\pi}{3}, 2\right)$

This alternative approach eliminates students having to memorize the inequality $0 \leq b(x - c) \leq 2\pi$, and to use the method from section 5.1.

Example 4

Ask students, "Is there another way to find the value of d?" (The value of d is the average of the maximum and minimum points.)

Two methods of finding the value of b are presented. Discuss with students when to use each method. For Method 1 to be effective, each interval length of 2π must have an integral number of cycles.

To find the value of c for the sine curve, students can look along the sinusoidal axis and find the point to the right of the y-axis.

To find the value of c for the cosine curve, students can look along the sinusoidal axis and find the point to the right of the y-axis, where cosine begins at a maximum and decreases.

Example 5

Discuss with students that they need to use their knowledge of all the parameters to determine a suitable window setting on their calculator. Reinforce why $\frac{\pi}{6}$ is the value of b, rather than $\frac{1}{6}$. Invite students to share their answers to the green question in part a). Explain that the data for most applications of trigonometric functions involve real numbers, such as for time. Radians are appropriate to represent these measurements. The use of degrees should be reserved for applications involving angles.

When determining the number of hours the ocean liner can dock, ask students, "Why do you substitute $y = 13$ and not $x = 13$?"

To determine the berth depth students can also use the table function on their graphing calculator.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location so that they can revisit these concepts later.
- Revisit the equation $y = a$ (trig function) $b(\theta - c) + d$. Discuss the effect varying a and b had on the graph, as students learned in the previous section.
- In Example 1, develop the understanding of the effect d has on the graph. (The curve moves up d units when d is positive, or down $|d|$ units when d is negative.)
- In Example 2, develop the understanding of the effect c has on the graph. It is important that students understand that the equation indicates the opposite of c.
- Summarize the effects of each of the parameters:
  – a affects the amplitude.
  – b affects the period.
  – c affects the phase shift (left or right).
  – d affects the vertical displacement (up or down).
  Add this information to the poster you started in the previous section.
- Some students may need to recall the order of performing transformations when graphing. Remind them to apply stretches and reflections before translations.
- Break students into groups of three or four. Each group could be given a set of four or fve equations. Use different equations for each group. The equations should emphasize all four factors affecting the standard sine and cosine graphs. Ask students to graph the set of equations they have been given using the method(s) of their choice.
- Have students study Examples 1 to 4, and then write a brief summary of the information they learned from the examples. Encourage students to look at a few of the graphs they created, and ask them to determine the amplitude, period, phase shift, and vertical displacement. Then, ask them to reverse the process, creating the equation from the graph. Ask, "Which two parts require the most thought?"
- Work through Example 5 as a class, encouraging discussion from all students willing to participate. This will allow others to listen and learn.

Enrichment

Ask students why computers might have different values for the sine of the same angle, and why most calculators have the same values.

Gifted

The height of a tsunami is a function of physical characteristics, such as ocean depth and shoreline shape. Ask students to predict how a particular wave's mathematical model might account for those characteristics, as the wave approaches land. Ask, "How might modelling a tsunami mathematically be of benefit?"

Common Errors

- Students often make mistakes with the phase shift. For example, for $y = \cos\left(\theta + \frac{\pi}{2}\right)$, students may think that because you are adding $\frac{\pi}{2}$, there is a shift to the right.

Rx Help students recognize that adding $\frac{\pi}{2}$ results in a shift to the left by asking, "If you add $\frac{\pi}{2}$ to θ, what would you have to do to return to the original point?"

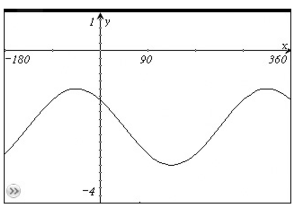
- Some students forget to factor out the b value.

Rx Show students the example $y = 3 \sin\left(2x - \frac{2\pi}{3}\right)$. If the 2 is not factored, it appears that the phase shift is $\left(\frac{2\pi}{3}\right)$. By factoring out the 2, the equation becomes $y = 3 \sin 2\left(x - \frac{\pi}{3}\right)$, which makes it apparent that the phase shift is $\left(\frac{\pi}{3}\right)$.

Answers

Example 1: Your Turn

a)

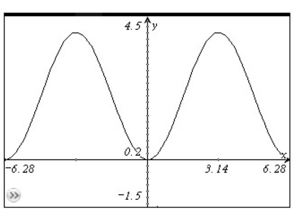


b) domain: $(-\infty, \infty)$, range: $[-3, -1]$

c) The graph of the function $y = \cos(x + 45°) - 2$ has been translated 2 units down and 45° to the left.

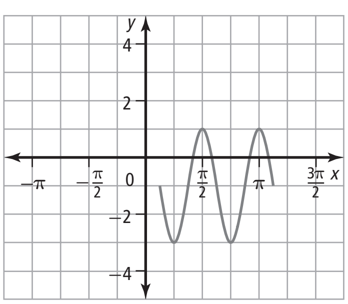
Example 2: Your Turn

a)



b) The graph of $y = 2 \sin\left(\theta - \frac{\pi}{2}\right) + 2$ has an amplitude of 2 and a phase shift of $\frac{\pi}{2}$ units to the right. The graph of the function also has a vertical displacement of 2 units up.

Example 3: Your Turn



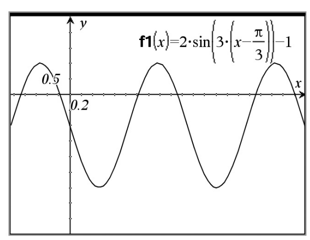
vertical displacement: 1 unit down; amplitude: 2; period: $\frac{\pi}{2}$; phase shift: π units to the left; domain: $\{x \mid x \in R\}$; range: $\{y \mid -3 \le y \le 1, y \in R\}$

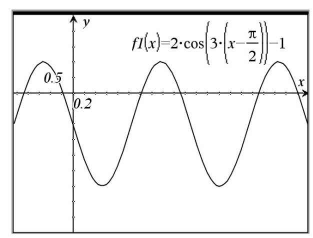
Example 4: Your Turn

a) Example: $y = 2 \sin 3\left(x - \frac{\pi}{3}\right) - 1$

b) Example: $y = 2 \cos 3\left(x - \frac{\pi}{2}\right) - 1$

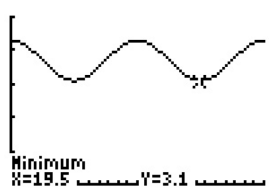
c)





Example 5: Your Turn

a)



b) period = 13 h

c) 7.91 h

d) At 7 h, the water level is at 3.1 m. Since the period is 13 h, this water depth will recur at $(7 + 13n)$ h, $n \in I$.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Revisit with students how the parameters *c* and *d* affect a graph before they sketch their graph. • Before students answer part c), review with them how *c* and *d* affect domain and range. • Guide students to the understanding that *d* is a centre line. You may wish to show them that if $a = 1$, then they can calculate the centre line using $\frac{\text{maximum} + \text{minimum}}{2}$.
Example 2 Have students do the Your Turn related to Example 2.	• If students are having difficulty, ask them to describe the effects of parameters *a*, *c*, and *d* in any transformation. • Have students orally identify the parameters given in the problem and then predict how these parameters might affect the graph, before students sketch their graphs. Clarify any misconceptions. • Ensure students can verbalize the effect of parameter *c* and why it appears to shift the graph in the direction opposite to the sign in the equation.
Example 3 Have students do the Your Turn related to Example 3.	• Students have little difficulty listing the values of the parameters *a*, *b*, *c*, and *d* when they are written in the form $y = a \sin b(x - c) + d$. Remind them that they may need to factor out *b* if this has not already been done. • Remind students that the stretch factor is determined by $\frac{1}{\lvert b \rvert}$.
Example 4 Have students do the Your Turn related to Example 4.	• Before students begin, ask them how the graph they were given is different from the one in the example. • Encourage students to copy the graph and identify the period, draw in the centre line, and so on. • Since *c* is the most difficult parameter for students to determine, remind them that it is the first *x* value to the right of the *y*-axis.
Example 5 Have students do the Your Turn related to Example 5.	• Encourage students to copy the graph and identify the period, draw in the centre line, and so on. • Remind students that the minimum depth means that they are looking for the area above this horizontal line.

Check Your Understanding

For #1 and 2, you may need to remind students to factor any number in front of *x* that is inside the brackets. For example, this needs to be done in #1e) and #2f).

Discuss with students how they can approach #3 and 5 without using a graphing calculator.

Students are required to apply the formula for *a* and *d* given the range of a function in #13.

Consider using #17 and/or 18 to show students how the sine graph can be obtained by applying horizontal translations to the cosine graph.

Students must apply their knowledge of *a* and *c* values given the domain of a function in #19.

As a class, discuss how to determine the window settings for #23.

Students may find #26 challenging. It deals with the phase shift required to make the sine graph similar to the cosine graph. Consider having students work with a partner.

For #27, students need a strong understanding of all the parameters and how they can be determined, given the maximum or minimum points of the graph.

Meeting Student Needs

- Provide **BLM 5–3 Section 5.2 Extra Practice** to students who would benefit from more practice.
- Review all terms and student learning outcomes. Have students complete a checklist indicating strengths and weaknesses.
- For #4 ensure that students compare each equation to the standard form. Note that *b* must be factored in parts c), d), and e).
- You may wish to post several questions individually on poster paper. Then, situate these at "stations," around the room, allowing students to work through the stations. Each student could complete the question in his/her notebook and then write comments on the poster paper for others to read. They could make a suggestion or just a comment about the question. The movement from question to question allows students to have a short break, which will assist them to concentrate better on the next question.

- Students may wish to support their learning using a graphing calculator. When writing equations for given graphs, they can enter the equation into the graphing calculator to determine if it looks like the given graph. Adjustments can be made until the graph on the calculator matches the graph in the textbook.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students #1–6, 11–13, 15–17, and 21. Students who have no problems with these questions can go on to the remaining questions.	• Prompt students to verbalize what the parameters *c* and *d* do to a function before they begin #1 and 2. Remind them that for a phase shift, the move is opposite to the sign in the brackets. For example, $(x - c)$ is a move to the right. • If students are having difficulty finding the range from the parameters, encourage them to use technology to graph the equations and then work from the visual. Coach them through one question using the visual, and then one using the values of the parameters. Students should be able to use both strategies. • For #4, ask students to verbalize the difference between parts b) and d). Coach them to remember to remove *b* as a factor in questions like part d). Ask them how they can use this approach to find the range, and so forth. Use a similar approach with #5. • For #6 and 7, encourage students to make a list of the four parameters on their page, and then list the corresponding value for each before beginning to write the equation. Revisit with students the general forms, such as $y = a \sin b(x - c) + d$. Post these general forms as visuals in the classroom. • Using the original graphs of sine and cosine as a reference, discuss with students how the graphs have changed in #15 and 16. Have students identify the parameters by drawing arrows and symbols similar to those used in section 5.1. • Encourage students to use technology for #21. Prompt them to identify the similarities in the parameters before graphing.
Assessment *as* Learning	
Create Connections Have all students complete C1. Students not experiencing any difficulties can move on to the remaining questions.	• The response that students write for C1 should be included in their notes section for future reference. Encourage students to be explicit in the description of each parameter and its effects. Suggest that they generate their own example function, and then that they use it to describe the effects of the parameters, both specifically and in general.

5.3 The Tangent Function

***Pre-Calculus 12*, pages 256–265**

Suggested Timing
60–90 min

Materials
- grid paper
- ruler
- protractor
- compass
- graphing technology
- spreadsheet software (optional)
- coloured pencils (optional)

Blackline Masters
Master 4 Unit Circle on Grid
BLM 5–4 Section 5.3 Extra Practice

Specific Outcomes
T4 Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.
☑ Connections (CN)
☑ Problem Solving (PS)
☑ Technology (T)
☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–5, 7, 8, 10
Typical	#1–8, 10, 12, C1, C2
Extension/Enrichment	#7, 8, 10, C2

Planning Notes

Discuss or handout the student learning outcomes for this section. Suggest that students add any unfamiliar terms to the Word Wall or individual dictionaries. Also, as students work through this section, they will add to the chart of characteristics they began in section 5.1. They should have access to their chart and be prepared to add the tangent graph characteristics.

Investigate the Tangent Function

You may wish to have students work through this Investigate in small groups and then share their answers. It is important that students understand the terminology and follow the steps carefully. For step 2, terminal arms in quadrants I and III are extended to the positive y-axis, whereas terminals arms in quadrants II and IV are extended to the negative y-axis. Students may benefit from recalling the sign of the tangent ratio with terminal arms in each quadrant.

Meeting Student Needs

- Students may wish to organize their ordered pairs for part A using a spreadsheet.
- Invite students to write a paragraph explaining the results of the Investigation.
- Some students may benefit from using **Master 4 Unit Circle on Grid** and following the steps in the Investigate.

Enrichment

Ask students to explain why pi is so important to the unit circle and the trigonometric ratios. Ask them to use examples in their explanation.

Gifted

Challenge students to explore the world of musical notes and their relationship to periodic functions. How does the physics of sound equate to the mathematics of the sound waves that a note represents? You may wish to invite a First Nations, Métis, or Inuit traditional musician to participate and broaden the learning experience for all students.

Ask students to investigate how a graph of a tangent function could be transformed using appropriate terminology.

Common Errors

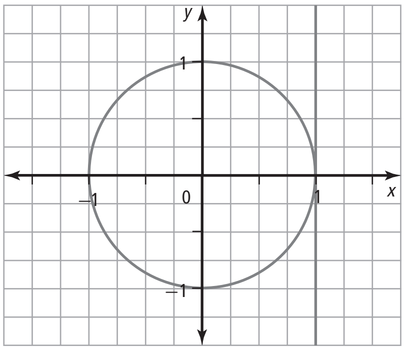
- Some students may forget that tangent has non-permissible values.

R_x Ask students
- Can you divide by zero? Why not?
- How is dividing by zero related to non-permissible values?
- Considering that $\tan \theta = \frac{\sin \theta}{\cos \theta}$, what can you say about the value of $\cos \theta$ in the tangent function? (it cannot be 0) How is this related to non-permissible values for tangent?

Tell students that division by zero results in an asymptote for the function.

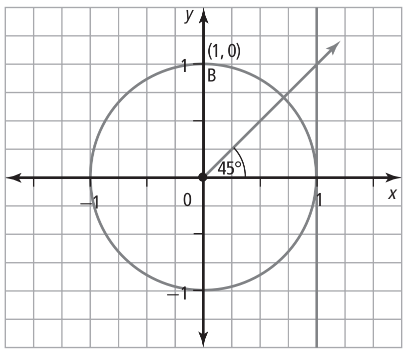
Investigate the Tangent Function

1.



2. a) (1, 0)

b)

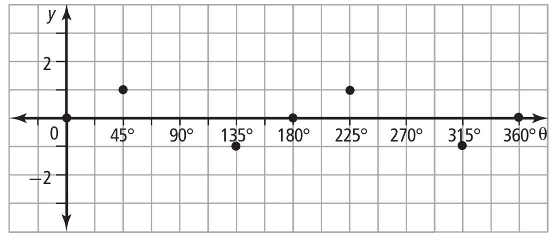


The extension of the terminal arm intersects the tangent line at (1, 1).

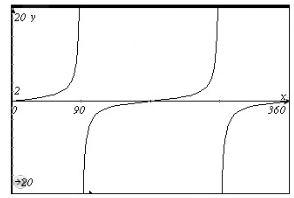
c) They never intersect.

d)

Angle Measure	0°	45°	90°	135°	180°	225°	270°	315°	360°
y-coordinate on Tangent Line	0	1	does not exist	−1	0	1	does not exist	−1	0



3.



4. There is an asymptote at $\theta = 90°$ and $\theta = 270°$.

5. The period of the tangent function is 180°.

6. The tangent function has no maximum or minimum values. Therefore, there is no amplitude for the tangent function.

7. $\sin \theta = y$, $\cos \theta = x$, $\tan \theta = \frac{y}{x}$

8. $\tan \theta = \frac{AB}{AO}$

9. $\tan 60° = \sqrt{3}$; $\sqrt{3}$ units

10. $\sqrt{3}$

11. The slope is equal to the tangent of the angle.

12. When the angle is 0°, there is no rise so the slope is 0. Therefore, $\tan 0° = 0$. When the angle is 90°, there is no run so the slope is undefined. Therefore, tan 90° is undefined.

13. The values of tangent approach infinity as θ approaches 90°. tan 90° is undefined.

14. Example: At 90°, the terminal is vertical and since the tangent of the line passing through (1, 0) is also vertical, the two never intersect. They are parallel lines.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigation. Encourage them to generalize and reach a conclusion about their findings.	• Ensure that students have a good understanding of the answer to #4 and 5 before they move on to the rest of the questions. • Have a class discussion to help students determine that $\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin x}{\cos x}$. • For the second Reflect and Respond, it may assist students to draw tangent lines and the corresponding right triangle. • Help students determine the tangent at 90°. Use technology with the division of zero, if needed.

Example 1

Ask students to explain why there is a break in the graph when $\theta = \frac{\pi}{2}$ and $\frac{3\pi}{2}$. Help students recall that asymptotes are imaginary lines that a function approaches, but never touches or crosses. Help students connect this to the break in the graph of the tangent function.

Emphasize that the first asymptote is at $\frac{\pi}{2}$ and, since the period for $y = \tan \theta$ is π, there will be another asymptote every π radians. As an extension of this discussion, explain the set notation for the domain of the tangent and how it reflects the asymptote. Students often struggle with writing the domain in this way. (You might also ask students to explain why tangent does not have a maximum or minimum value. How is this related to amplitude?)

Example 2

Rather than giving students the window settings to graph this function, discuss as a class how to come up with the settings. Students often struggle with this. Ask students to explain what the x- and y-values are within the context of the question.

Encourage students to draw a diagram of the situation and ensure they label the angle, θ, from the plane to the vertical, not to the horizontal. Some students may wonder how *distance* can be negative. Ask students to describe what negative values of distance and negative angles represent in this situation.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Students should identify two ways that the tangent function differs from the sine or cosine function. (No amplitude and period = 1π.)
- Invite students to remember another concept that involved asymptotes. (Rational functions) Ask
  - What is the importance of an asymptote?
  - Why are asymptotes part of the tangent function?
  - What will cause the asymptote lines to move from $\frac{-\pi}{2}$ and $\frac{\pi}{2}$?
- Encourage students to check Internet sites for the development of the graph of a tangent function. Another visual aid will assist some students with the understanding of the concept of the tangent line.

Common Errors

- Students may think the period of the tangent function is 2π as it was for sine and cosine.

R_x Reinforce for students that the period is the distance between the asymptotes, which is π radians.

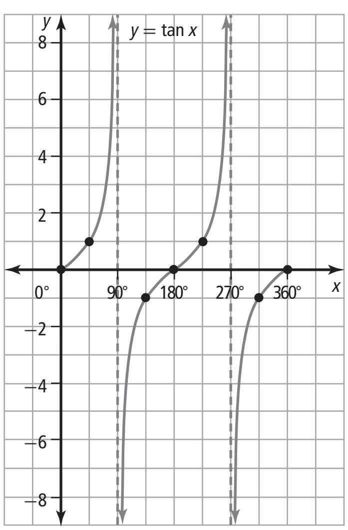
- Some students think that a is the amplitude of the tangent function. For example, they may think that $y = 3 \tan \theta$ has an amplitude of 3.

R_x Ask students what the maximum and minimum values of any tangent function are. The arms of the graph extend infinitely in both directions; therefore, there is no maximum or minimum value. If there is no maximum or minimum value, there can be no amplitude.

Answers

Example 1: Your Turn

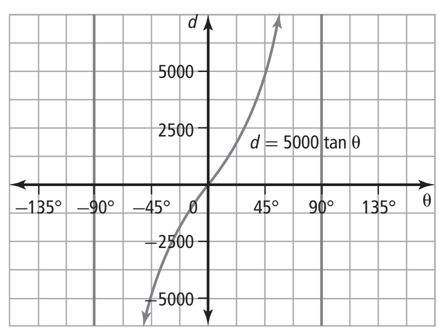
a)



Example:
- The curve breaks at θ = 90° and θ = 270°, where the function is undefined.
- tan θ = 0 when θ = 0°, θ = 180°, and θ = 360°.
- tan θ = 1 when θ = 45° and θ = 225°.
- tan θ = −1 when θ = 135° and θ = 315°.
- The domain is {θ | θ ≠ 90° + 180°n, θ ∈ R, n ∈ I}.

Example 2: Your Turn

a) $d = 5000 \tan \theta$ where d is the horizontal distance from the observer to the plane, and θ is the angle from the vertical to the plane, in degrees.



b) The tangent function has an asymptote at θ = 90°. This value is undefined since the plane is never directly beside the observer. As the plane flies toward the observer, from the right, the horizontal distance decreases. Since tan 0° = 0, the horizontal distance is 0 m when the plane is directly overhead and the angle to the plane from the vertical is 0°. On the left side of the observer, the values are negative. The angle approaches −90° as the tangent function decreases to −∞.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Students should be directed to use the graph from Example 1. • Before students attempt the Your Turn, have them describe how the radian measurements will equate to degrees. • Ask students whether the characteristics would be any different if the references are expressed in degrees.
Example 2 Have students do the Your Turn related to Example 2.	• The Your Turn problem can be patterned after the example. • Encourage students to draw the triangle, as well as the graph, and to indicate the asymptotes. • As students complete the question, have them verbalize what the asymptotes mean.

Check Your Understanding

Students are required to find the in-between points on the graph in #2. This is an excellent question as it requires students to accurately read the graph. It also reinforces and tests their knowledge of radians because they have to identify points that are not already labelled.

For #4, ask students if there are other ways to answer this question. Can they use the unit circle to find the value?

For #13, revisit with students how to graph a linear equation to accurately complete this question.

Meeting Student Needs

- Provide **BLM 5–4 Section 5.3 Extra Practice** to students who would benefit from more practice.
- Students should reflect on the student learning outcomes for this section and focus on the outcomes that they do not understand.
- Students can be paired and numbered 1 and 2. They can have a "Partner Talk," where 1s tell 2s something they learned in the section, and then 2s tell 1s something they learned from the section, and so forth. The challenge is to see who has the last word.
- All students should complete #8 and #10. The use of technology will be helpful. Students could have the graph of $y = \tan x$ for $0 \leq x \leq 2\pi$ on their graphing calculator while making the other calculations. The visual display will assist with the understanding of the concept.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–5, 7, 8, 10. Students who have no problems with these questions can go on to the remaining questions.	• The solutions to #1 will reflect whether students understood the investigation at the beginning of the section, as well as Example 1. Remind them that $\tan x = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin x}{\cos x}$. Have students identify how this formula relates to the ordered pairs. Understanding of this question will impact their work for #9. • For #2, students are asked to use the tangent graph. Review with students how to read a graph with an asymptote. Ensure that they understand why there is an asymptote. This concept will be important for students to deduce the information required in #8. • For #3, have students discuss their response as a class. Address any misunderstandings regarding what determines amplitude. Some students believe any curve in a graph can determine amplitude. • Prompt students when writing the equation for #10. Coach them as to how the tangent ratio can apply to the question when determining the equation.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2. Students not experiencing any difficulties can move on to the remaining questions.	• Consider combining C2 onto one large graph and encourage students to use different colours to draw each function. An 11 × 17 sheet of paper with a grid already placed on it would provide students with sufficient space to draw and would make visual comparisons easier.

5.4 Equations and Graphs of Trigonometric Functions

***Pre-Calculus 12*, pages 266–281**

Suggested Timing

90–135 min

Materials

- marker
- ruler
- compass
- stopwatch
- centimetre grid paper
- graphing calculator
- coloured pencils (optional)

Blackline Masters

BLM 5–5 Circle With 8 cm Radius on Grid

BLM 5–6 Section 5.4 Extra Practice

Specific Outcomes

T4 Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.

- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Technology (T)
- ✓ Visualization (V)

T5 Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 5–11, 15–17
Typical	#5, 6, 8, 12–15, 17 or 18, 20, C2
Extension/Enrichment	#19, 22, 23, C2, C3

Planning Notes

Before beginning this section, discuss the student learning outcomes for this section, or provide these in a handout. Discuss with students what they have learned in previous sections. Ask:

- How are the functions related?
- Do you have a sense of how these functions might be used?
- Do you have a sense of how graphing a trigonometric function may help to solve real-world problems?

Revisit the situations described in the openers for the different sections and ask students if they have a better sense of how the mathematical concepts they have learned are related to the situations described. Then, tell them that in this section they are going to explore how solving trigonometric equations by graphing can help solve real-world problems.

Investigate Trigonometric Equations

Suggest that students use the tip of a marker or even coloured pencils to move around the circle and identify the timed intervals to measure. Guide students to aim for a 20 s travel time around the circle. This will result in numbers that are ideal to work with and will allow students to be able to compare answers with other groups.

To determine the first coordinate of the point found in #6b), students will need to refer to their table to predict the time that the distance is the same as in part a).

Meeting Student Needs

- Some students may benefit from using **BLM 5–5 Circle With 8 cm Radius on Grid** for the Investigate.
- Allow students who confuse clockwise and counterclockwise to complete the Investigate moving clockwise around the circle as long as they are consistent.
- In #5, some students may choose to write the equation using a cosine curve, rather than a sine curve. Allow these students to work through the remaining steps using cosine.

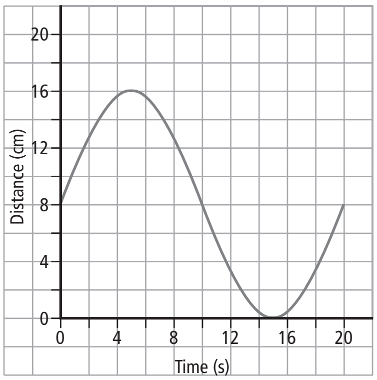
Answers

Investigate Trigonometric Equations

3. Example:

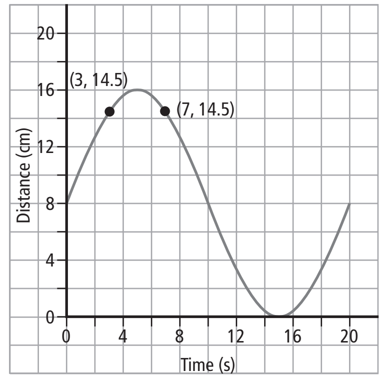
Time (s)	0	2	4	6	8	10	12	14	16	18	20
Distance (cm)	8	12.7	15.6	15.6	12.7	8	3.3	0.4	0.4	3.3	8

4.



5. $y = 8 \sin \frac{\pi}{10}t + 8$

6. a), b)



c) Example: The points (3, 14.5) and (7, 14.5) satisfy the equation $y = 8 \sin \frac{\pi}{10}t + 8$.

d) Example: The measured and calculated distances are essentially the same.

8. Example: The pattern around the circle creates a sine curve when graphed as a function of time.

9. Example: The radius of the circle is 8 cm and the amplitude of the sine curve is 8 cm. The curve is translated 8 cm up because the measurements start at the 3:00 position, which is a distance of 8 cm from the tangent line. It took 20 s to travel around the circle and the period of the sine curve is 20 s. The distance from the tangent line increased to a maximum, then decreased to a minimum and returned to the starting position; this follows the sine curve.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigation. Encourage them to generalize and reach a conclusion about their findings.	• Orally discuss the response to #8 as a class. You may wish to display a sample graph from a group to discuss how to create the equation. Have students from other groups share and compare their equations. • Encourage students to answer #9 using terminology related to transformations and trigonometry. • Discuss the results as a class to ensure students can take data from a chart and use it to generate an equation. Help students link the key points on the graph to the situation.

Example 1

The notation $\cos^2 x$ may be new to students. Explain that this represents the square of the cosine ratio. Encourage students to solve the trigonometric equations using both graphical and algebraic methods.

Example 2

You may wish to remind students to adjust the mode of their calculator to radians. Make sure students understand that the *general* solution is not over a specific interval and that they will need to use the period of the function in their answer. Students may need to recall how to determine the period of a trigonometric function.

When working through the Your Turn, students may require guidance to help them determine the equation for the second possible value of x. Ask:

- In which quadrants is the sine ratio positive?
- What is the reference angle in each of these quadrants?
- What are the measures of the angles in standard position, in degrees? in radians?
- How are these angles related?

Example 3

Have a discussion about the terms *interpolate* and *extrapolate* with students, to ensure that they know the difference and that they know the situations that call for each of them. Make sure students have their graphing calculator set to radians. Have a discussion about window settings, and how to come up with these. Students often struggle with setting these. Ask students if they are able to figure out the period from the equation.

For the Your Turn, ask students if they are able to calculate the period using two different methods. Which do they prefer?

Example 4

Have a calendar available to students for this question.

Ask students if finding the values for part e) is an example of interpolation or extrapolation. There are many ways to find these values:

- the Trace function on the calculator
- the Table function on the graphing calculator
- substitution

Encourage students to try all three methods, and ask them which they prefer.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Interpolation and extrapolation are taught in grade 9. However, be sure to revisit these concepts to ensure that students understand them.
- Students will benefit from reading the Key Ideas prior to working through the examples.
- For Example 3, students need to understand how to interpret the graph. Discuss the importance of the window settings, and discuss the meaning of each tick on the graph. For part a) the period can also be found by taking $\frac{(2\pi)}{(120\pi)}$ as discussed in earlier sections.
- Ensure students graph Example 4 by hand, but allow them to check it using a graphing calculator. Ask, "What are the characteristics of the graph?"
- Students can work in pairs or small groups to study the examples. Ask them to state the similarities in the problems. Ask
  - What were the steps followed in each example?
  - What strategy can you use for future problems?

  Have students write the steps involved.

Common Errors

- Some students neglect to set their calculator to degrees or radians as appropriate for a particular problem.

R_x Encourage students to always consider whether their answers are reasonable, and when they are not, suggest that students remember to check the mode of their calculator as part of their verification.

Answers

Example 1: Your Turn

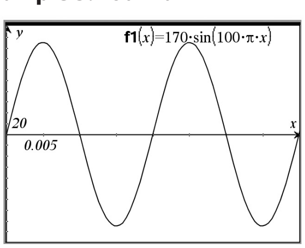
The solutions for the interval $0° \leq x \leq 360°$ are $x = 60°$, $120°$, $240°$, and $300°$.

Example 2: Your Turn

The general solutions for the trigonometric equation $10 = 6 \sin \frac{\pi}{4} x + 8$ are approximately $x \approx 0.43 \pm 8n$ radians and $x \approx 3.57 \pm 8n$ radians, where n is an integer.

Example 3: Your Turn

a)



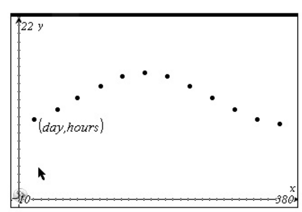
The vertical axis represents the number of volts. Each tick mark represents 20 V. The horizontal axis represents time. Each tick mark represents 0.005 s.

b) The period is 0.02 s.

c) The voltage reaches 110 V 100 times in the first second.

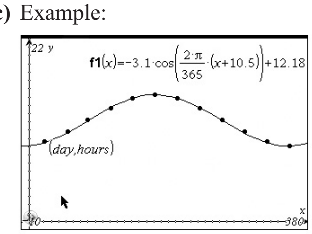
Example 4: Your Turn

a)

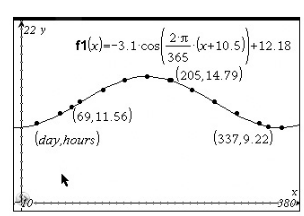


b) Example: $h(t) = -3.1 \cos\left(\frac{2\pi}{365}(t + 10.5)\right) + 12.18$

c) Example:



d) Examples:
i) ≈ 11.56 h **ii)** ≈ 14.79 h
iii) ≈ 9.22 h



c) Example: Iqaluit gets more hours of daylight each day during the summer and Windsor gets more hours of daylight each day during the winter.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to suggest that students who are using graphing as their method compare their graph with another student's graph for accuracy before determining the solutions. • The process followed in the example should be very helpful for students who are having difficulty as the questions are very similar. If needed, as a class, ask students to identify the similar characteristics between the two equations. • Listen for student responses, such as "I got it" or "It's actually right, I don't believe it!" Comments of this nature generally indicate that the student has patterned the question solely after the example and may not have a good understanding of what they have just completed. In this case, provide students with an additional question to solve. • Remind students of the importance of placing the $\pm$ in front of the root when solving algebraically.
Example 2 Have students do the Your Turn related to Example 2.	• Have students work in pairs for this question. • Students who are having difficulty should be encouraged to use Method 1 or Method 2. • Ensure students understand when to look for *x*-intercepts and when to look for points of intersection, and how they are different. • Students who complete the question quickly should be encouraged to verify their solutions using an alternative method.
Example 3 Have students do the Your Turn related to Example 3.	• Suggest that students compare their graph in part a) with that of another student to ensure that they have appropriate settings and a correct visual before completing the problem. • The process followed in Example 1 should be very helpful for students who are having difficulty as the questions are very similar. If needed, as a class, ask students to identify the similar characteristics between the two equations. Ask which parameters would be similar in both. This discussion should be sufficient to enable students to complete the question. • Have students label the length of the period on their graph.
Example 4 Have students do the Your Turn related to Example 4.	• Have students work in pairs for this question. • Coach students on how to enter their information to generate a scatterplot. • When answering part b), encourage students to use trial and error with different sinusoidal functions. Doing so provides valuable graphics. This should not only assist students to make the correct match, but also help them see each function's results. • Before estimating or calculating the number of hours of daylight, help students recognize that they need to change the date to reflect what day out of a monthly calendar is being tested. For example, March 1 becomes 3.03.

Check Your Understanding

For #6, students will need to state the domain and range of the actual function, and keep in mind any implicit constraints or restrictions given the context of the question.

For #7, mention that the hum of a fly is the sound of its wings flapping, which is about 600 Hz. Males, females, and different types of flies flap their wings at different frequencies, producing various tones. For example, a house fly hums in the key of F.

Challenge students to complete #9 without a graphing calculator.

Encourage students to solve #10 in two ways: using substitution and with a graph.

Discuss how to determine the window settings in #14. Compare the graphs within the context of the question.

For #16, ask students what part of the graph the average yearly temperature would be.

Ask students if #18 involves interpolation or extrapolation.

For #21, students are asked to choose which function to use to represent the situation. You might have them discuss this as partners and then justify their choice.

For #23, students are required to change the function based on a change in the period, which makes this an interesting question.

Meeting Student Needs

- For the Project Corner, you may wish to point out that AM and FM stand for Amplitude Modulation and Frequency Modulation. Some students may not be familiar with the word modulation. Explain to students that modulate means to vary or to change.
- Provide **BLM 5–6 Section 5.4 Extra Practice** to students who would benefit from more practice.
- Students can work in pairs, choosing to do eight to ten questions. Once students have worked through the questions, invite a discussion about the content of the questions. Students can also share methods used for success, as well as sharing problems that they might have encountered.

Enrichment

Encourage students to create a way to show what they have learned about trigonometric functions'

- history
- graphical representation
- place in the real world
- use by pre-computer mathematicians and engineers: how did they use these functions with accuracy?

Gifted

Ask students to research the trigonometric identities. Challenge them to prove the identity, $\sin^2 \theta + \cos^2 \theta = 1$.

Let students know that they will explore trigonometric identities in great detail in Chapter 6.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 5–11, 15–17. Students who have no problems with these questions can go on to the remaining questions.	• For #6, remind students that range is also linked to maximum and minimum values. Ask students to verbalize what formula they know to calculate these values. Then, ask them to identify which parameters they would use in the first equation to find these. • Students may wish to graph or plot the points for #8a) and b) before answering. • For #9 to 11, students should be able to answer the questions using the parameters from the equation. Some students however, may find it easier to graph the equation first to see the visual before starting.
Assessment *as* Learning	
Create Connections Have all students complete C2. Students not experiencing any difficulties can move on to the remaining questions.	• A summative approach to student learning in this section is presented in question C2. This question asks students to identify and comment on the effects of the parameters in the given equation.

Chapter 5 Review and Practice Test

5

***Pre-Calculus 12*, pages 282–287**

Suggested Timing
90–135 min each

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 5–2 Section 5.1 Extra Practice
BLM 5–3 Section 5.2 Extra Practice
BLM 5–4 Section 5.3 Extra Practice
BLM 5–6 Section 5.4 Extra Practice
BLM 5–7 Chapter 5 Study Guide
BLM 5–8 Chapter 5 Test

Planning Notes

Have students who are not confident discuss strategies with you or a classmate. Encourage students to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions with which they need no help, a little help, and a lot of help. You may wish to provide students with **BLM 5–7 Chapter 5 Study Guide**, which links the achievement indicators to the questions on the Chapter 5 Practice Test in the form of self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

Students can use their responses to help them prepare for the practice test. The practice test can be assigned as an in-class or take home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: 1–8, 10–12, 15, 17.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 5–2 Section 5.1 Extra Practice**, **BLM 5–3 Section 5.2 Extra Practice**, **BLM 5–4 Section 5.3 Extra Practice**, or **BLM 5–6 Section 5.4 Extra Practice**.
- Students could create a collage of information summarizing the Key Ideas for the chapter. They may include Key Terms and definitions, graphs, or other important information. Making the collage will serve as a review of the Key Ideas. To ensure understanding, you may then ask each student one or two questions about the information displayed in the collage.
- Students could choose a song and rewrite one or two verses and the chorus, using lyrics that describe the Key Ideas for the chapter.
- You may wish to provide students with **Master 3 Centimetre Grid Paper**.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 5 Review The Chapter 5 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #1–5, 6a), d), 7a), d), 8–12, 14–15, 17, 19–21	• Ask students to revisit any section from the review that they are having difficulty with, prior to working on the practice test. • Before looking at the Practice Test, suggest that students review **BLM 5–7 Chapter 5 Study Guide** to identify areas they feel they in which they need some extra practice before starting the Practice Test.
Chapter 5 Study Guide This blackline master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas for which they require further assistance. The minimum questions suggested are questions that students should be able to confidently answer. Encourage students to try additional questions. • Consider allowing students to use any summative charts, concept maps, or graphic organizers when completing their practice test.
Assessment *of* Learning	
Chapter 5 Test After students complete the practice test, you may wish to use **BLM 5–8 Chapter 5 Test** as a summative assessment.	• Before the test, coach students in areas in which they are having difficulties. • Suggest that students refer back to **BLM 5–7 Chapter 5 Study Guide** and identify those areas in which they may still need reinforcement.