Trigonometric Identities

Opener

***Pre-Calculus 12*, pages 288–289**

Suggested Timing
45–60 min

Blackline Masters
BLM 6–1 Chapter 6 Prerequisite Skills
BLM U2–1 Unit 2 Project Checklist

Planning Notes

Prior to beginning this chapter, you may wish to review with students how to graph trigonometric functions in degrees and in radians, how to choose appropriate window settings, how to identify the key features of a graph, and how to determine the domain and range of a function.

The first two sections of this chapter focus on trigonometric identities. Students learn and compare a variety of ways to verify an identity. In section 6.3, students learn how to prove an identity. They also learn how proving an identity differs from verifying the identity. Finally, in section 6.4, students use trigonometric identities to solve trigonometric equations with the variable in both degrees and in radians.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter. Encourage students to use a summary method of their choice.

Unit Project

In this chapter, students revisit the Unit 2 Project introduced in Chapter 4. In section 6.2, they use trigonometric identities to explore Mach numbers. Finally, at the end of this chapter, students use the skills they developed in Chapters 4 to 6 to complete the Unit 2 Project Wrap-Up.

Meeting Student Needs

- Consider having students complete the questions on **BLM 6–1 Chapter 6 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out **BLM U2–1 Unit 2 Project Checklist**, which provides a list of all the requirements for the Unit 2 Project.
- Review the key terms and have students review related terms and their definitions. Ensure students understand the terms before proceeding.
- Provide students with a checklist containing the learning outcomes for this unit. Discuss specific terms. Develop a sense of understanding of what students need to learn by the end of the unit.
- You may wish to have students research the meaning of the terms *identity* and *trigonometric identity*.
- Create a laminated list of all trigonometric identities that will be used in this chapter. Provide headings for the various equations. Provide the list to students at the beginning of the chapter.

Enrichment

Suppose the equation $d = \frac{2v^2 \cos \theta \sin \theta}{g}$ models the distance, d, travelled by a javelin after it is thrown. Encourage students to estimate the best angle for the javelin to be thrown at to maximize distance.

Gifted

Physical ergonomics involves the way in which people use technology to improve their work and lives. Challenge students to observe and measure the distances, angles, and proportions involved in a worker sitting at a computer work station. Have students identify the variables that could influence the comfort level of a particular worker. How might the mathematics affect the comfort level?

Career Link

Discuss with students what they know about the field of kinesiology and the role of an athletic therapist. Ask
- What do you know about kinesiology and athletic therapy?
- What education do you think is required to work in this field?
- What other fields are related to athletic therapy?
- How might mathematics, trigonometry in particular, be used in the study of kinesiology?

Suggest that students go online to research careers in athletic therapy. Are there any aspects of the career that surprise them?

6.1 Reciprocal, Quotient, and Pythagorean Identities

***Pre-Calculus 12*, pages 290–298**

Suggested Timing
60–90 min

Materials
- grid paper
- ruler
- graphing technology

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 6–2 Section 6.1 Extra Practice

Mathematical Processes for Specific Outcomes
T6 Prove trigonometric identities, using:
- reciprocal identities
- quotient identities
- Pythagorean identities

☑ Reasoning (R)
☑ Technology (T)
☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 3, 5, 7, 8, 10, 11
Typical	#1, 4, 6–8, 10, 12, 13, C1–C3
Extension/Enrichment	#14–17, C2, C3

Planning Notes

Discuss the learning outcomes for this section. As a class, read the opening paragraph about digital music players. Students will learn more about trigonometry and music in section 6.4.

Tell students that in this section they will apply the reciprocal and quotient identities they learned in previous chapters, and they will learn the Pythagorean identities.

Meeting Student Needs

- Students may need to study the identities in this section before they work on the Investigate.
- Have students determine the exact values for the trigonometric ratios for special angles and their multiples (in both degrees and radians), display them in a graphic organizer, and highlight the values that are undefined. Students can use this as a reference when working through questions involving non-permissible values.
- Display the three reciprocal identities, the quotient identities, and the three Pythagorean identities on a poster.

Investigate Comparing Two Trigonometric Expressions

For step 1, ask students to think about what window settings would be appropriate. Have them decide on window settings before they graph each function.

For step 3, ask students how the tangent ratio is related to the sine and cosine ratios. Are there any non-permissible values for $y = \tan x$? Will $y = \cos x \tan x$ have any non-permissible values?

For step 6, have students graph the equations over the domain $-720° \leq x < 720°$. Students should carefully study the table of values as well as the graphs.

Common Errors

- Some students may not recognize the non-permissible values for $y = \cos x \tan x$.

R_x Ensure students understand that, although the graphs of $y = \sin x$ and $y = \cos x \tan x$ appear to be identical, the function $y = \cos x \tan x$ is not defined when $\cos x = 0$. This is because $\tan x$ (which is the ratio $\frac{\sin x}{\cos x}$) is not defined when $\cos x = 0$.

Answers

Investigate Comparing Two Trigonometric Expressions

1. The graphs of $y = \sin x$ and $y = \cos x \tan x$ are the same.

2.

x	$y = \sin x$	$y = \cos x \tan x$	x	$y = \sin x$	$y = \cos x \tan x$
$-360°$	0	0	$0°$	0	0
$-300°$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$	$60°$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{2}$
$-270°$	1	N/A	$90°$	1	N/A
$-210°$	0.5	0.5	$150°$	0.5	0.5
$-180°$	0	0	$180°$	0	0
$-150°$	-0.5	-0.5	$210°$	-0.5	-0.5
$-120°$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$240°$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$-90°$	-1	N/A	$270°$	-1	N/A
$-60°$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$	$300°$	$-\frac{\sqrt{3}}{2}$	$-\frac{\sqrt{3}}{2}$
$-30°$	-0.5	-0.5	$330°$	-0.5	-0.5
			$360°$	0	0

3. $\cos x \tan x = \sin x$; $x \neq \pm 90°, \pm 270°$

4. a) Yes, since $\cos x \tan x$ can be simplified to $\sin x$.

$$\begin{aligned}\cos x \tan x &= \cos x \left(\frac{\sin x}{\cos x}\right)\\ &= \sin x\end{aligned}$$

b) From the graph, it appears the expressions are equal for all values of x. From the table, we see that when $x = \pm 270°$ and $\pm 90°$, $\cos x \tan x$ is undefined but $\sin x = 1$.

5. $x \in \mathrm{R}$, $x \neq 90° + 180n$; $n \in \mathrm{I}$

6. Yes; for example, $\cos x \tan x \neq \sin x$ when $x = \pm 450°, \pm 630°$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigation. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs. • Encourage students to discuss what makes the two equations appear the same and what makes them appear different. • You may wish to discuss students' responses to #4b) and #6 as a class to ensure a clear understanding.

Example 1

In this example, students learn how to verify a potential identity numerically and graphically. Ensure students consider all the factors when determining non-permissible values. Have students explain why you have to state the non-permissible values for $\tan \theta$ even though it is in the numerator.

When verifying a potential identity by graphing, ask
- What are you looking for?
- How will you graph the equation?
- What would the graph look like if the equation were not an identity?
- Do the graphs have to be identical for all points in the domain $-360° < x \leq 360°$?

Discuss the fact that the numerical example is not any type of proof. It simply shows that the equation works for one value. This is an important concept.

Example 2

Use this example to talk about how identities can be used to simplify trigonometric expressions. Again, ensure students consider each factor when determining non-permissible values.

Example 3

Use this example to introduce the Pythagorean identity. This identity connects concepts students are already familiar with—the Pythagorean theorem, the basic trigonometric ratios, and the unit circle. Students do not need to memorize all the different forms of the Pythagorean identity. Ask students to explain how to derive the other forms of the Pythagorean identity given the basic form $\cos^2 \theta + \sin^2 \theta = 1$.

Meeting Student Needs

- Some students may benefit from seeing more steps when identities are used to simplify expressions.
- You may wish to use colour-coded cards to illustrate how to simplify a trigonometric expression.
  For example: $\sin \theta$, $\frac{1}{\csc \theta}$, $\cos \theta \tan \theta$, and other combinations that represent $\sin \theta$ could all be written on red cards. When simplifying a trigonometric expression, use the cards instead of markers.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Enrichment

Have students consider the graphical method of verifying identities. What are some advantages and disadvantages of the graphical method compared to algebraic methods?

Gifted

Challenge students to explain the process, step by step, from initial Pythagorean thinking to the three forms of the Pythagorean identity.

Common Errors

- Students may have difficulty knowing where to start when working with trigonometric identities.

R_x Encourage students to begin by rewriting the identity in terms of sine and cosine.

Answers

Example 1: Your Turn

a) $x \neq 180n$; $n \in I$

b) For $x = 45°$: $\cot 45° = 1$; $\frac{\cos 45°}{\sin 45°} = 1$

For $x = \frac{\pi}{6}$: $\cot \frac{\pi}{6} \approx 1.732$; $\frac{\cos \frac{\pi}{6}}{\sin \frac{\pi}{6}} = 1$

c) The graphs appear to be identical, so $\cot x = \frac{\cos x}{\sin x}$ could be an identity.

Example 2: Your Turn

a) $x \neq n\pi$; $x \neq \frac{\pi}{2} + n\pi$; $n \in I$

b) $\csc x$

Example 3: Your Turn

a) Left side $= 1 + \tan^2 \frac{3\pi}{4}$
$= 2$

Right side $= \sec^2 \frac{3\pi}{4}$
$= 2$

b) $\cos^2 x + \sin^2 x = 1$

$$\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$$1 + \tan^2 x = \sec^2 x$$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Ensure students understand how to identify the x-values for which $\sin x = 0$. • Some students may benefit from completing a unit circle to use for reference purposes.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Remind students of the importance of finding the non-permissible values prior to simplifying. • Encourage students to write the equivalent forms of the given trigonometric expression before trying to simplify.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Ensure students can explain why both sides are multiplied by $\frac{1}{\sin^2 x}$ before they move forward. Students should be able to readily identify the common denominator of equivalent terms and make appropriate changes.

Check Your Understanding

For #1, 3, and 4, encourage students who struggle with fractions within fractions to write the question out using the division symbol. For example,

$$\begin{aligned}\frac{\sin x}{\tan x} &= \sin x \div \tan x \\ &= \sin x \div \frac{\sin x}{\cos x} \\ &= \sin x \times \frac{\cos x}{\sin x} \\ &= \cos x\end{aligned}$$

For #5, if an equation is true for given values, ask students if it is true for all values. Have them explain how they know.

For #7b), have students evaluate their expression from part a) and the given expression for $\theta = \frac{\pi}{6}$, and then compare and explain the results.

Have students describe how they would enter the expression for #11 into their graphing calculator.

For #12, ask students
- If one value satisfies the identity, will all values work?
- If one value does not satisfy the identity, does that mean no values will?

For #14, encourage students to write out the multiplication to avoid making mistakes. For example, $(\sin x + \cos x)^2 = (\sin x + \cos x)(\sin x + \cos x)$

For #17, ask students to explain how they can simplify the left side of the equation. Is it possible to rewrite the left side as "something" plus $\sin x$?

Meeting Student Needs

- Provide **BLM 6–2 Section 6.1 Extra Practice** to students who would benefit from more practice.
- When simplifying trigonometric expressions, encourage students to express all ratios in terms of $\sin \theta$ or $\cos \theta$.
- Students may wish to use the color-coded cards containing the names of various trigonometric ratios. They could set up a question on the classroom floor and "substitute" until the expression is simplified.
- For #9, students should research the meaning of luminous intensity.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 3, 5, 7, 8, 10, and 11. Students who have no problems with these questions can go on to the remaining questions.	• Have students compare their non-permissible values as they work through the assigned questions. • #1–6 focus students' abilities to write equivalent reciprocal forms in order to reduce and simplify expressions. • Encourage students to list the reciprocal identities. • Some students may need support when isolating a variable as in #9. Provide them with an alternate expression for practice before they try #9. • Encourage students to link the graphical and algebraic solutions. • Encourage students to practise writing non-permissible values in both radians and degrees; if a domain is given, it determines the units for the non-permissible values.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C3. Students completing these may wish to complete the rest of the questions.	• Have students compare their response to C1 with that of a partner. • Have students work in pairs to complete C3. • It is important that students realize that expanding the domain may make the graphs appear different. Discuss with the class the need to see enough of a graph before drawing a conclusion, especially with absolute values.

6.2 Sum, Difference, and Double-Angle Identities

***Pre-Calculus 12*, pages 299–308**

Suggested Timing

90–120 min

Materials

- ruler
- protractor

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BLM 6–3 Section 6.2 Extra Practice

Mathematical Processes for Specific Outcomes

T6 Prove trigonometric identities, using:
- reciprocal identities
- quotient identities
- Pythagorean identities
- sum or difference identities (restricted to sine, cosine and tangent)
- double-angle identities (restricted to sine, cosine and tangent)

☑ Reasoning (R)

☑ Technology (T)

☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–4, 6–13
Typical	#1, 2, 5–9, 11, 14–16, 18 or 19, C1–C3
Extension/Enrichment	#14, 15, 17, 18 or 19, 20–24, C1–C3

Planning Notes

As a class, review how to measure angles with a protractor, the sum of interior angles of different polygons, and the parallel line theorems. The Investigate is an important tool for helping students develop the sum identities.

Meeting Student Needs

- Display posters showing the sum, difference, and double-angle identities.
- You may wish to collaborate with an arts teacher. Have students create Guilloché patterns and display them in the classroom.

Investigate Expressions for sin $(\alpha + \beta)$ and cos $(\alpha + \beta)$

Have students work in pairs to complete the Investigate.

Make sure students use a ruler and protractor to construct their diagram in step 1. Ask
- How do you know which angles are right angles?
- What is the sum of the interior angles in a triangle? How can it help you determine the missing angle in a triangle?

For step 2, ask students
- How can parallel lines help you answer this question?
- Which sides are parallel? How do you know?
- Which line segment serves as a transversal?

For step 3, have students write an expression for cos β, and then explain why AE can be labelled as cos β. Ask students to think about what would change if the length of the hypotenuse was not 1.

Have students consider which triangle(s) they would use to determine an expression for each line segment in step 4.

Have students verify their identities in step 5 by using the actual measurements of all the angles. Ensure students understand that there may be some variance within their identities due to measurement error. Then have students compare their measurements and verifications with those of a classmate.

Answers

Investigate Expressions for sin $(\alpha + \beta)$ and cos $(\alpha + \beta)$

2. a) From $\triangle ABE$, $\angle BEA = 90° - \alpha$; from $\triangle AEF$, $\angle FEA = 90°$.
$$\begin{aligned}\angle CEF &= \angle BEC - \angle BEA - \angle FEA \\ &= 180° - (90° - \alpha) - 90° \\ &= \alpha\end{aligned}$$

b) $\angle DAF = 90° - \alpha - \beta$; $\angle AFD = \alpha + \beta$; $\angle AFE = 90° - \beta$; $\angle CFE = 90° - \alpha$

3. Since cosine is the ratio adjacent:hypotenuse, and the hypotenuse has length 1, the adjacent side to β is $\cos \beta$.

4. $AB = \cos \alpha \cos \beta$, $BE = \sin \alpha \cos \beta$, $EF = \sin \beta$, $CE = \cos \alpha \sin \beta$, $CF = \sin \alpha \sin \beta$, $AD = \sin \alpha \cos \beta + \cos \alpha \sin \beta$, $DF = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

5. $\angle AFD = \alpha + \beta$; $\sin (\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha$; $\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$

6. b) It works for obtuse angles as well. There are no restrictions on the domain.

7. $\sin 2\alpha = 2 \sin \alpha \cos \alpha$; $\cos 2\alpha = \cos^2 \alpha - \sin 2\alpha$

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigation. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to provide students with a diagram of rectangle ABCD with inscribed triangle AEF. Review the lengths and angle measures for step 4 before having students proceed to step 5. • Students should compare their results in step 6 and draw a correct conclusion for obtuse angles in step 6b). Coach students to determine whether obtuse angles are covered by the identities. • You may wish to have a student write their solution for $\sin 2\alpha$ on the board so others can use this as a model for $\cos 2\alpha$.

Example 1

Ask students how they know which identity to focus on. Once students identify the correct identity, have them write the general identity and then the identity with the specific angles under it.

Example 2

Use this example to demonstrate how to manipulate identities to write them in different forms. Ask students how to isolate $\sin^2 A$ in the equation $\cos^2 A + \sin^2 A = 1$. Ask students to explain why $\sin^2 A$ was isolated rather than $\cos^2 A$.

Example 3

Use this example to reinforce the process of identifying non-permissible values. Students also have the opportunity to use trigonometric identities to simplify expressions.

Have students consider which trigonometric ratio they think the expression will simplify to and explain their choice.

Example 4

In this example, students apply the sum or difference identities and the special angles to find exact values. Have students list the special angles in degrees and in radians for reference. In each case, students rewrite the angle as a sum or difference of two special angles, and then apply the appropriate identity. Students may use a calculator to check their answers.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Students may wish to make a list of all angles that can be evaluated in exact form using either the sum or difference of two angles. For example, $15° = 60° - 45°$ and $\frac{\pi}{12} = \frac{\pi}{3} - \frac{\pi}{4}$.

Enrichment

Encourage students to create questions (and their solutions) that require them to write expressions as single trigonometric functions.

Gifted

Ask students to explain, using an example, whether it is possible to use sum and difference identities to find angles for which exact trigonometric values cannot be determined.

Common Errors

- Students may mix up the identities. For example, they may use $\sin (A + B)$ where they should use $\sin (A - B)$.

R_x Help students to develop a way to help them remember the sum and difference identities. Alternatively, encourage students to write each of the identities on a formula sheet and refer to the formula sheet as they work. Encourage students to write out each step of their work. For example, writing $\sin 165°$ as $\sin (120° + 45°)$ before substituting the identity.

Answers

Example 1: Your Turn

a) $\cos 53°$

b) $\sin \frac{\pi}{6}$

Example 2: Your Turn

$\cos 2A = 1 - 2\sin^2 A$

Example 3: Your Turn

a) $x \in R, x \neq 90° + 180n; n \in I$

b) $\tan x$

Example 4: Your Turn

a) $\frac{-1-\sqrt{3}}{2\sqrt{2}}$ **b)** $-\frac{\sqrt{3}-1}{1+\sqrt{3}}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Coach students to work backward by assigning the given angle measures an α and β and determine whether the resulting pattern belongs to a sum or difference formula.
Example 2 Have students do the Your Turn related to Example 2.	• Have students identify equivalent forms of the Pythagorean identities and write them in a graphic organizer. This may assist students in determining equivalent forms of identities when trying to rewrite them.
Example 3 Have students do the Your Turn related to Example 3.	• Coach students to solve $\cos x = -1$ and determine where on the unit circle this is true. Ask students to identify the domain, and then have them use it to find the solution to $\cos 2x = -1$. • Have students use the equivalent forms they derived in the previous example. • Have students compare their results to those they obtain when using technology, and model the solution for the class.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Ensure that students understand that many of these questions can be approached and solved correctly in more than one way. • Encourage students to try to solve this problem in more than one way.

Check Your Understanding

For #1, have students consider the clues to look for to know whether they are working with an angle sum or difference or a double-angle identity. Have students explain how they know if the single function will be sine, cosine, or tangent.

For #4b), have students explain how the number 6 inside the brackets affects the answer.

For #6, ensure students understand that a single substitution that results in Left Side $\neq$ Right Side is sufficient to show the inequality is true. However, any number of substitutions for the variable that result in Left Side = Right Side is *not* sufficient to prove an identity is true.

For #7, have students simplify the expression using a difference identity and then have them simplify the expression using a sum identity and compare their results.

Ask students how they can verify their answers for #8. Ask

- Why are exact values more accurate?
- What situations might necessitate exact answers?
- When might it be more useful to have a decimal approximation?

As an extension to #9, ask students to research the latitudes of other locations. Challenge students to find a location that has the latitude found in part c).

For #11, ensure students understand how the quadrant in which an angle is located affects the values of the trigonometric ratios. Have students determine the exact values if θ is in quadrant III.

For #13, discuss as a class which double-angle identity could be used. Have students explain what to do with the variables v and g.

For #14, have students expand and simplify $(m + n)^2$ before they expand and simplify $(\sin x + \cos x)^2$. Have students examine the expanded expression and determine if they can regroup the terms to resemble one of the identities.

For #17, students should first determine which quadrant angle x lies in. Then they can determine the sign of $\cos x$ and of $\cos (\pi + x)$.

For #18, have students explain how to verify the value of k numerically.

Meeting Student Needs

- Have students work in groups of two or three to complete the Project Corner on page 308.
- Provide **BLM 6–3 Section 6.2 Extra Practice** to students who would benefit from more practice.
- Students should refer to the laminated list of trigonometric identities provided at the beginning of the chapter when working on the questions.
- Students should be encouraged to draw sketches where possible. For example, students would benefit by drawing a sketch for question #11.
- Students may wish to make more cards containing trigonometric substitutions and then practise some of the questions by laying the cards out on the floor.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–4 and 6–13. Students who have no problems with these questions can go on to the remaining questions.	• Encourage students to use patterning for #1, 2, 4, and 9. You may need to coach students through #1e), since they may have difficulty determining how to use the coefficient 8 properly. • Students may need help to determine the identity to use for #7. Assist them to assign A and B in place of 90 and x. This should provide support in choosing the correct identity. • Remind students that exact values are the values for the unit circle coordinates and not approximated calculator values. • Encourage students to use exact values where possible, including for #12
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2.	• You may wish to have students work in pairs. • For C1, encourage students to draw the given $\cos x = \frac{-5}{13}$ on a Cartesian plane in the correct quadrant as indicated by the domain and determine the missing side length first. This will allow them to find the sine value and proceed with the question. • Encourage students to sketch C2. However, a graphing calculator could be used to help make a comparison easier. • Ensure students remember the effect that parameter b has on a function and how it can be used to answer C2b).

6.3 Proving Identities

***Pre-Calculus 12*, pages 309–315**

Suggested Timing
60–90 min

Materials
- graphing technology

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BLM 6–4 Section 6.3 Extra Practice

Mathematical Processes for Specific Outcomes
T6 Prove trigonometric identities, using:
- reciprocal identities
- quotient identities
- Pythagorean identities
- sum or difference identities (restricted to sine, cosine and tangent)
- double-angle identities (restricted to sine, cosine and tangent)

☑ Reasoning (R)
☑ Technology (T)
☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–5, 7–10, 12, 13
Typical	#1–9, 12–14, 15, C1, C2
Extension/Enrichment	#11, 15–18, C1, C3

Planning Notes

You may wish to coordinate a cross-curricular activity with a physics teacher. Alternatively, you could talk to a physics teacher to coordinate your units so you can support each other in dealing with trigonometric functions. Discuss angles of elevation and depression with students prior to beginning the Investigate.

Investigate the Equivalence of Two Trigonometric Expressions

Ask students how they might simplify each expression in step 1. Students should also consider which value for θ they will use to check the numerical equivalency of the two expressions. Have students explain whether showing the expressions are equal for a given θ-value proves they are equivalent.

For step 3, have students discuss which identities would be useful in simplifying each expression.

For step 5, have students consider whether there are cases in which numerical/graphical verification would suffice. Ask students if numerical/graphical verification can *disprove* an identity, and ask them to explain their answer.

Meeting Student Needs

- You may wish to have students work in groups of four. One pair of students could work through one equation; the other pair the second equation. The pairs compare their results after each equation has been simplified.
- Alternatively, have one group solve the equations graphically and the other group solve them algebraically. Then have students share their results.
- Proving trigonometric identities can be compared to balancing chemical equations at their various stages of chemical reactions, before the final equations are arrived at.

Enrichment

Ask students to consider the formula $d = \frac{v^2 \sin 2\theta}{g}$.
Have students see if the formula models the horizontal distance travelled by the model rocket. Have students test key points such as the initial velocity to test the formula's validity. As each variable changes, ask students if the results make sense.

Gifted

Challenge students to create an experiment to collect data for a video game. The game's players simulate throwing a football or baseball while sensors track their movements. Have students consider what variables might be measured and how they might use data to create a formula.

If possible, have students collect the data from real-world experimental results.

Common Errors

- Students often misuse the equal sign in proving identities. For example, they may add/subtract or multiply/divide terms across the equal sign.

R_x Emphasize that each side of a potential identity must be simplified separately to show that they are equivalent. You might suggest students use a T-chart and manipulate each side separately until they match.

Left Side	Right Side

Answers

Investigate the Equivalence of Two Trigonometric Expressions

1. Yes, graphing the function gives identical graphs. Using identical angles and velocities gives the same answer.
$20 \sin 2\theta = 20 \times 2(\tan\theta - \tan\theta \sin^2\theta)$

2. Both formulas contain the factor $\frac{(v_0)^2}{g}$, which simplifies to 20 for the given values.

3. $\sin 2\theta = 2(\tan\theta - \tan\theta\sin^2\theta)$

Left Side $= \sin 2\theta$	Right Side $= 2\tan\theta(\cos^2\theta)$
	$= 2\sin\theta\cos\theta$
	$= \sin 2\theta$

5. The algebraic method allows us to show both sides of the identity in equivalent forms, which proves that the two sides are equal. The numerical method may show that the two sides are equal for a given value of the variable, but that does not tell us if they are equal for all possible values. Sometimes the graph of a function does not show values that are undefined. For example, the graphs of $y = \sin x$ and $y = \cos x \tan x$ appear the same, but their values are not equal when $x = 90° + 180°n$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigation. Encourage them to generalize and reach a conclusion about their findings.	• Students may have difficulty writing equivalent forms algebraically. You may wish to complete this as a class and discuss how the graph or a numerical substitution may fail to prove an identity true. You may wish to link it to an algebraic example before completing the trigonometric one. You could use the example of $x^2 = 4$ and $x = 2$ as an example. Ask students whether this is the complete solution.

Example 1

Ensure students understand the importance of always identifying the non-permissible values first. Have students consider which x-values would be good to use to verify the equation and which would not.

Ask students to suggest reasons why it is a good idea to rewrite trigonometric ratios in terms of sine or cosine.

Example 2

Continue to discuss with students the difference between verifying and proving an identity. Ensure students understand and can articulate the necessary conditions for proving an identity.

Example 3

As students work to prove more complicated identities, ensure they understand that there is usually more than one way to prove an identity. Ask students to explain why both the numerator and the denominator of the right side of the equation were multiplied by $1 - \cos x$.

Example 4

Students may struggle factoring $2\cos^2 x - \cos x - 1$. You may wish to have students factor $2x^2 - x - 1$ and use the result to help them factor $2\cos^2 x - \cos x - 1$.

Ask students if there are any non-permissible values.

Meeting Student Needs

- Students may need to make a list of some strategies from previous chapters and previous years. For example, when dividing rational expressions, multiply by the reciprocal.
- Have students review the different types of factoring. For each question, provide an example of factoring a polynomial then have an equivalent trigonometric question. For example, factor $x^2 - 5x - 6$, then factor $\sin^2 x - 5\sin x - 6$.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Students sometimes apply the identities incorrectly. They may struggle with a starting point when it comes to proving identities.

R_x Encourage students to begin by expressing all the trigonometric ratios in terms of sine and cosine. Then students should have a list of all the trigonometric ratios and identities so they can easily compare what they know to what they need to prove. Students will become more proficient with identifying patterns with additional practice.

- Some students incorrectly divide out factors and terms in trigonometric identities that involve rational expressions. For example, students may think $\frac{\cos 2x}{\cos x} = 2$ or $\frac{\cos x}{(1 + \cos x)} = \frac{1}{1 + 1}$.

Rx Have students work with related numerical examples to help them visualize the process.

Answers

Example 1: Your Turn

a) $x \in \mathrm{R}, x \neq 90° + 180°n; n \in \mathrm{I}$

b) For $x = 60°$:

Left Side $= \frac{\tan 60° \cos 60°}{\csc 60°}$
$= 0.75$

Right Side $= 1 - \cos^2 60°$
$= 0.75$

c) Left Side $= \frac{\tan x \cos x}{\csc x}$
$= \tan x \sin x \cos x$
$= \sin^2 x$

Right Side $= 1 - \cos^2 x$
$= \sin^2 x$

Example 2: Your Turn

Left Side $= \frac{\sin 2x}{\cos 2x + 1}$
$= \frac{2 \sin x \cos x}{2 \cos^2 x}$
$= \frac{\sin x}{\cos x}$
$= \tan x$

Right Side $= \tan x$

Example 3: Your Turn

Left Side $= \frac{1}{1 + \sin x}$

Right Side $= \frac{\sec x - \sin x \sec x}{\cos x}$
$= \frac{\sec x(1 - \sin x)}{\cos x}$
$= \frac{1 - \sin x}{\cos^2 x}$
$= \frac{1 - \sin x}{1 - \sin^2 x}$
$= \frac{1 - \sin x}{(1 - \sin x)(1 + \sin x)}$
$= \frac{1}{1 + \sin x}$

Example 4: Your Turn

Left Side $= \frac{\sin 2x - \cos x}{4 \sin^2 x - 1}$
$= \frac{2 \sin x \cos x - \cos x}{4 \sin^2 x - 1}$
$= \frac{\cos x(2 \sin x - 1)}{4 \sin^2 x - 1}$
$= \frac{\cos x(2 \sin x - 1)}{(2 \sin x - 1)(2 \sin x + 1)}$
$= \frac{\cos x}{2 \sin x + 1}$

Right Side $= \frac{\sin^2 x \cos x + \cos^3 x}{2 \sin x + 1}$
$= \frac{\cos x}{2 \sin x + 1}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Remind students of the importance of determining the non-permissible values before simplifying. • It might be easier for some students to write the fraction as a division expression, especially if the denominator is a fraction. For example, $(\tan x \cos x) \div \csc x$ becomes $\left(\frac{\sin x}{\cos x} \times \cos x\right) \div \frac{1}{\sin x}$. • Remind students that one method alone will not be sufficient to prove an identity. Allow students to use all three approaches in the order of their choice. • Remind students that showing with numbers only that both sides of an identity equation are equal represents that it works with certain numbers, but not all permissible numbers.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • As a class, discuss which equivalent identities or expressions could be used in the proof. • Students often try to convert the expression to too many different equivalent forms, and thus never achieve the goal of making both sides the same.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • This example provides students with the realization that some questions require changes to both sides in order to prove the identity.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Encourage students to substitute equivalent forms into either or both sides of the question to try to limit the values to sine, cosine, or tangent. • Encourage students to try substituting into the double-angle parts of the expression.

Check Your Understanding

It is always good practice to remind students that when proving identities, they should start with the side of the equation that looks more complex.

To simplify the expressions in #1, students need to factor the numerator and denominator and look for common factors. Remind students to determine all non-permissible values *before* simplifying.

Students who struggle with #3 could do a parallel question to help them understand and apply the steps.

Students could solve a simpler question, such as $\frac{2}{3} + \frac{3}{4}$, and then follow the same steps to add/subtract the trigonometric fractions.

For #4, ask students to explain why rewriting the expression using sine and cosine works in simplifying many trigonometric functions.

For #5, have students explain why a graphical representation is not considered proof of an identity.

Have students solve #6 two ways—first by expanding the expression and then writing it in terms of sine and cosine, and second by writing it in terms of sine and cosine and then expanding—and ask which method they prefer.

For #7, prompt students to use what they know about sin $2x$ to write a trigonometric identity for csc $2x$.

Have students identify the non-permissible values for each of the expressions in #10. Ask students to describe what the non-permissible values look like graphically.

For #12, have students determine the value of sin 90° and explain how this can help prove the identity.

For #14, ask students how many examples it would take to prove an identity is true. How many examples would it take to disprove an identity?

Have students use what they know about double-angle identities to simplify sin $4x$ and cos $4x$ in #16.

For #18, students should determine the non-permissible values first. Have them rewrite the numerator in terms of one trigonometric ratio. Some students may benefit from factoring a simpler expression to help them factor the denominator.

For C3, have students graph both sides of the equation to help determine the non-permissible values and to determine whether it is an identity.

Meeting Student Needs

- Provide **BLM 6–4 Section 6.3 Extra Practice** to students who would benefit from more practice.
- Students can work in pairs. One partner gives directions, the other partner solves the problem. Students alternate roles.
- For questions #12 to 14, have students work in pairs. Each student simplifies one side of the equation, then they compare their results.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–5, 7–10, 12, and 13. Students who have no problems with these questions can go on to the remaining questions.	• #1–4 give students an opportunity to simplify expressions by substituting parts of the expression with basic trigonometric and reciprocal identities. Students should be able to handle these questions accurately before moving on to #5 and 7. • Coach students through #8 to assist them in determining which equivalent identities should be used to simplify the expression. Working through #8 should assist them in their selection in #9. • Encourage the use of patterning for students having difficulty with #11 and 12. Have them discuss with a partner which of the identities should be used before they begin to simplify. Have them compare their work and discuss any differences.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• C1 revisits concepts from the Investigate. Students should have completed sufficient questions and examples in order to respond. Discuss this response as a class and have students complete the response in their graphic organizer for future reference. • In C2, students should realize that only one side of the equation will be manipulated. They have already been told that they are using a difference identity. Have students expand the difference and compare what they have written with what a partner has before simplifying it.

6.4 Solving Trigonometric Equations Using Identities

***Pre-Calculus 12*, pages 316–321**

Suggested Timing
60–90 min

Materials
- graphing technology

Blackline Masters
BLM 6–5 Section 6.4 Extra Practice

Mathematical Processes for Specific Outcomes

T5 Solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians.

- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

T6 Prove trigonometric identities, using:
- reciprocal identities
- quotient identities
- Pythagorean identities
- sum or difference identities (restricted to sine, cosine and tangent)
- double-angle identities (restricted to sine, cosine and tangent)

- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–8, 10
Typical	#1–6, 9, 10, 12, two of 13–17, C1, C2
Extension/Enrichment	#11, 12, 15, 16, 19, 20, C2, C3

Planning Notes

Invite students to bring school-appropriate music to class. Listen to some of the selections and discuss how fading is used. As a class, discuss how students think a sine wave could model sounds for musical instruments. Ask
- How might the graph of a sine function show the sound frequency of a guitar string being plucked?
- Would the function remain the same throughout the entire graph?
- What part of the graph would gradually change?
- What are other examples of situations that could be modelled using the sine function? You may wish to go into more detail describing the different methods of fading a song.

Investigate Solving Trigonometric Equations

Ask students
- What parts of the graph should be included in your sketch?
- Can you use words such as *amplitude* and *period* to describe your graph?

For step 2, have students identify the first few zeros of the function to the right of the origin and look for patterns. Encourage students to describe the pattern in words before trying to determine an algebraic expression for it. Have students check whether the pattern is true for zeros to the left of the origin.

Gifted

Ask students to explore mathematically the statement that "the instrument with the purest sound wave is the flute."

Common Errors

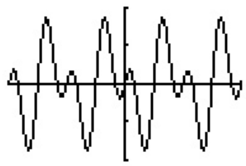
- When solving $\sin 2x - \sin x = 0$, some students write the first step as $\sin x(\sin x - 1) = 0$.

R_x Remind students that $\sin 2x$ is not the same as the product $(\sin x)(\sin x)$, which is $\sin^2 x$. Students should use the double angle identity $\sin 2A = 2\sin A \cos A$.

Answers

Investigate Solving Trigonometic Equations

1.
```
WINDOW
 Xmin=-720
 Xmax=720
 Xscl=180
 Ymin=-2
 Ymax=2
 Yscl=1
 Xres=1
```



The graph is periodic; each period has two peaks, the first greater than the second.

2. $x \in \mathrm{R}, x = 0° + 180°n, 60° + 360°n, 300° + 360°n; n \in \mathrm{I}$

3. $x \in \mathrm{R}, x = 0° + 180°n, 60° + 360°n, 300° + 360°n; n \in \mathrm{I}$

4. The algebraic method is more analytic and gives an exact solution. With a good understanding of the function it is easy to find all the zeros.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigation. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs. • Encourage students to explain why they chose their method for step 4.

Example 1

Use this example to demonstrate how to solve trigonometric equations by substituting trigonometric identities and factoring. Ask

- Why must one side of the equation be set to zero before you can solve it by factoring?
- When you solve an equation algebraically, how do the solutions correspond to a graphical solution?
- If the domain were not restricted, what is an expression that could be used to state the solutions for all real numbers?
- What two things must be done to the equation in part b) before you are able to factor it?

Example 2

When students check the solution graphically, it appears from the graph that there is a solution at $x = 0$. Since $\cot x = \frac{\cos x}{\sin x}$, we know that there is no solution at $x = 0$ because it is a non-permissible value. Ask students why they think that the graphing calculator does not show this restriction. The graphing calculator appears to be graphing only the trigonometric function $\cos x$. Have students use the table for this graph to determine the value of the function when $x = 0$. Remind students that when a trigonometric function is simplified, the graph may not show all solutions.

Example 3

Students might not see where the solutions are coming from. Ask them which of the special triangles could be used to solve $\sin x = \frac{\sqrt{2}}{2}$. Since $\sin \frac{\pi}{4} = \frac{\sqrt{2}}{2}$, the reference angle is $\frac{\pi}{4}$. From the unit circle, the sine ratio is also positive in quadrant II. If the reference angle is $\frac{\pi}{4}$, what angle in quadrant II will have a sine ratio of $\frac{\sqrt{2}}{2}$? For $\cos x = 0$, ask students where $\cos x = 0$ on the graph. The first point occurs where $x = \frac{\pi}{2}$. The second point is where $x = \frac{3\pi}{2}$. So, cosine has a ratio of 0 starting at $x = \frac{\pi}{2}$ and repeating every π.

Example 4

Ask students the following:

- Why is there no solution for $\sin x = 3$? Explain.
- How can you tell looking at a trigonometric equation whether it has a solution?

Meeting Student Needs

- Give each student a sheet of paper. Instruct them to write as many key ideas as they can remember learning in the chapter. Then pair students and have them group their ideas into either 3 or 4 main categories. Have students jigsaw with another group, synthesizing the ideas generated each time. Refine the heading titles if necessary until they all come to a general consensus.

- As students work through each example, highlight the key trigonometric identities targeted in the example.
- For Example 3, discuss the concept of a *general solution* prior to working through the example.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Enrichment

Ask students to defend this statement: "When solving trigonometric equations using identities, non-permissible values must be identified carefully."

Answers

Example 1: Your Turn

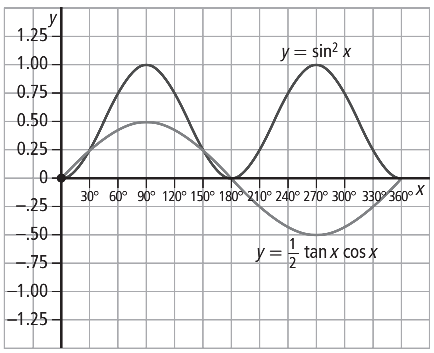
a) $x = \frac{\pi}{6}, x = \frac{\pi}{2}, x = \frac{5\pi}{6}, x = \frac{3\pi}{2}$

b) $x = 0$

Example 2: Your Turn

a) $x = 30°, 150°$

b)



Example 3: Your Turn

$x \in \mathrm{R}, x = 2\pi n, \frac{2\pi}{3} + 2\pi n, x = \frac{4\pi}{3} + 2\pi n; n \in \mathrm{I}$

Example 4: Your Turn

$x \in \mathrm{R}, x = 2\pi n; n \in \mathrm{I}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Remind students that it is much easier to factor when all the terms share a common trigonometric ratio. If more than one ratio is required (for example, sine and cosine), check for common factors that can be taken out of each term. For example, algebraically the process of factoring $xy + x = x(y + 1)$ is the same process used to factor $\cos^2 x + \cos x = \cos x(\cos x + 1)$.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Some students may find it easier to move all values to one side and graph one equation. Others will prefer to write the left and right sides as separate equations and graph them separately. Ask students how they would read the graphs. • Remind students to identify the non-permissible values.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Ask students whether both sides need to be simplified or just one. • Ask students whether it would be better to write the left side in terms of cosine or sine, and have them explain their choice. • If students are having difficulties, have them factor $2x^2 - x - 1 = 0$ first.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Ask students to identify whether both sides need to be simplified or just one. If just one, which one? • Ask whether it would be better to write the left side in terms of cosine or sine, and have students explain their choice. • Remind students to determine the non-permissible values before simplifying.

Check Your Understanding

For #8, remind students that if the domain of the function is not specified, they should write a general solution.

Ask students to describe the steps they would follow, starting with the given equation, to find the solution for #12. Then ask students to reverse that process to describe how to find the equation given its solution.

Students could create an equation for #13 and then trade with a partner to solve each other's equations. Students should check their partner's solution to ensure it is correct.

For #19, students may need to review the laws of exponents. Additionally, some students may need this hint: Express all terms using a common base (2) and then equate exponents. Give students a similar question so they can practise writing powers using the same base.

Meeting Student Needs

- Make two large circles with the desks. The desks in the outer circle face in, matched with a desk from the inner circle facing out. Students each take one seat. Working with the facing pair, work through one question. The students in the outside desks then rotate one desk clockwise to meet a new partner to answer the next question. If both students are stuck on a question they may ask assistance from a group on their left or right.
- Provide **BLM 6–5 Section 6.4 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–8 and 10. Students who have no problems with these questions can go on to the remaining questions.	• For most questions, students can write the questions in equivalent terms of x and y if it makes it easier to solve algebraically, and then switch back to the equivalent trigonometric identity. • Coach students through #10. You may wish to begin by having them solve an algebraic question such as $(x + 2)(x + 3)(x - 1) = 0$. Have students verbalize the process and what the results mean before beginning #10.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2.	• Encourage students to discuss C1a) with a partner before solving. Once they have agreed on which function to use, have students complete the question independently. Have them include all their work in their graphic organizer. • Use a similar approach for C2. You may wish to write the equation as $3x^2 + x - 1 = 0$ and ask students to identify any problems they have in trying to factor it. Review the quadratic formula with students as needed.

6 Chapter 6 Review and Practice Test

***Pre-Calculus 12*, pages 322–324**

Suggested Timing
60–90 min

Materials
- graphing technology

Blackline Masters
BLM 6–2 Section 6.1 Extra Practice
BLM 6–3 Section 6.2 Extra Practice
BLM 6–4 Section 6.3 Extra Practice
BLM 6–5 Section 6.4 Extra Practice
BLM 6–6 Chapter 6 Study Guide
BLM 6–7 Chapter 6 Test

Planning Notes

You may wish to provide students with **BLM 6–6 Chapter 6 Study Guide**, which links the achievement indicators to the question on the Chapter 6 Practice Test in the form of self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1–13.

Meeting Student Needs

- Allow students to work with a partner on Chapter Review #6. You may need to explain the shape of a sawtooth wave. If students have difficulty determining the next terms in the series, encourage them to look for a pattern and to see what changes between the terms.
- Students should work through the complete list of outcomes provided at the beginning of Chapter 6. Which outcomes and indicators do they know? Which outcomes and indicators require extra study and practice?
- Students should complete the practice test individually, without books or assistance. Ensure that each student has the list of trigonometric identities provided at the beginning of the chapter.
- Students who require more practice on a particular topic may refer to **BLM 6–2 Section 6.1 Extra Practice**, **BLM 6–3 Section 6.2 Extra Practice, BLM 6–4 Section 6.3 Extra Practice**, and **BLM 6–5 Section 6.4 Extra Practice**.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 6 Review The Chapter 6 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #1–4, 7–9, 11–16, 18, 19, 21	• Ask students to revisit any section from the review that they are having difficulty with, prior to working on the practice test. • Encourage students to use **BLM 6–6 Chapter 6 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 6 Study Guide This master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. The minimum questions suggested are questions that students should be able to confidently answer. Encourage students to try additional questions beyond the minimum. • Consider allowing students to use any summative charts, concept maps, or graphic organizers when completing the practice test.
Assessment *of* Learning	
Chapter 6 Test After students complete the practice test, you may wish to use **BLM 6–7 Chapter 6 Test** as a summative assessment. **Minimum:** #1–13	• Before the test, coach students in areas in which they are having difficulty. • You may wish to have students refer to **BLM 6–6 Chapter 6 Study** Guide and identify areas they need reinforcement in before beginning the chapter test.

Unit 2 Project Wrap-Up

UNIT 2

***Pre-Calculus 12*, page 325**

Suggested Timing

90–120 min

Blackline Masters

Master 1 Holistic Project Rubric
Master 2 Ana-Holistic Project Rubric
BLM U2–1 Unit 2 Project Checklist

Mathematical Processes

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

General Outcome

Develop trigonometric reasoning.

Specific Outcomes

T1 Demonstrate an understanding of angles in standard position, expressed in degrees and radians.
T4 Graph and analyze the trigonometric functions sine, cosine and tangent to solve problems.
T6 Prove trigonometric identities, using:
- reciprocal identities
- quotient identities
- Pythagorean identities
- sum or difference identities (restricted to sine, cosine and tangent)
- double-angle identities (restricted to sine, cosine and tangent)

Planning Notes

You may want to book some time in the computer lab for students to research the topic of their choice. Students could present a few key ideas, based on their research, to the class. This would allow everyone to hear about all the options even if they did not choose that option themselves. You might also wish to get experts in the field to join the class for the presentations. This way they can add to the information and students could have an opportunity to ask them questions.

Ensure students are aware of the project information provided on page 163, the Project Corners on pages 205, 281, and 308, and the Unit 2 Project Wrap-Up on page 325. If students did not complete the Project Corners in Chapters 4, 5, and 6, have them do that now and then use the information to prepare their presentation.

Have students use **BLM U2–1 Unit 2 Project Checklist** to make sure that all parts of their project have been completed. As a class, brainstorm different ways students can do their presentations. You may wish to limit the time each student is allowed to present.

You may wish to work with the class to create a specific rubric for the project using either **Master 1 Holistic Project Rubric** or **Master 2 Ana-Holistic Project Rubric** as a template. Review the general holistic points within the 1–5 scoring levels. Discuss with students how they might achieve each of these levels in the Unit 2 Project. A completed rubric in each style for this project is available on the *McGraw-Hill Ryerson Pre-Calculus 12* web site. Note that these are just samples; your class rubric may have more detail.

Ask questions such as the following:

- What are the big ideas in the unit?
- Which of the big ideas are involved in the project?
- What part of the project could you complete or get partially correct to indicate that you have a basic understanding of what you learned in Chapters 4 to 6?
- What would be on a level 1 project? What might you start on correctly? What could be considered a significant start?
- What would be expected for a level 5 project? What should it include? Try to help students realize that a level 5 project may have a minor error or omission that does not affect the final result.
- Knowing the expectations of levels 1, 3, and 5 projects, what would be expected for a level 4? Help students to understand that this is still an honours level and therefore the work should be reflective of this. However, even an honours project may have a minor error or omission. Discuss the difference between a major conceptual error and a minor miscalculation or omission. Understanding this point will help clarify for students the expectations and differences between a pass and an above-average result, and may encourage some students to work toward the highest level. Repeat the process for level 2.

Use the rubric to ensure that students understand the criteria for an acceptable level, as well as what would warrant either an unacceptable or an honours grading.

Meeting Student Needs

Enrichment

Have students list some television programs that highlight the use of math or science skills. Challenge students to create an interesting problem (and correct solution) that requires aspects of what they have learned this unit. Perhaps the problem might relate to a medical drama, a crime mystery, or a science fiction plot.

Gifted

Ask students to review the Unit 2 Project Wrap-Up. Encourage them to find the relationships between the different options. What are the mathematical connections between them?

Assessment	Supporting Learning
Assessment *of* Learning	
Unit 2 Project This unit project gives students an opportunity to apply and demonstrate their knowledge of the following: • angle measurement • trigonometric identities • trigonometric equations • trigonometric functions Work with students to develop assessment criteria for this project. **Master 1 Holistic Project Rubric** and **Master 2 Ana-Holistic Project Rubric** provide descriptors that will assist you in assessing students' work on the Unit 2 Project.	• You may wish to have students use **BLM U2–1 Unit 2 Project Checklist**, which provides a list of the required components for the Unit 2 Project. • Reviewing the Project Corner boxes at the end of some sections of Chapters 4 to 6 will assist students in developing appropriate data presentations. • Make sure students recognize what is expected for the minimum requirements for an acceptable project as well as the difference between level 5 and level 4. • Clarify the expectations and the scoring with students using **Master 1 Project Rubric, Master 2 Ana-Holistic Project Rubric**, or the rubric you develop as a class. It is recommended that you review the scoring rubric at the beginning of the project, as well as intermittently throughout the project to refresh students' understanding of the project assessment.

UNIT 2

Cumulative Review and Test

***Pre-Calculus 11*, pages 326–329**

Suggested Timing
60–90 min

Materials
- graphing technology
- grid paper
- ruler

Blackline Masters
Master 3 Centimetre Grid Paper
BLM U2–2 Unit 2 Test

Planning Notes

Have students work independently to complete the review, and then compare their solutions with those of a classmate. Alternatively, you may wish to assign the cumulative review to reinforce the concepts, skills, and processes learned so far. If students encounter difficulties, provide an opportunity for them to discuss strategies with other students. Encourage them to refer to their notes, and then to the specific section in the student resource. Once they have determined a suitable strategy, have students add it to their notes. Consider having students make a list of questions that they found difficult. They can then use the list to help them prepare for the unit test.

Meeting Student Needs

- Students should work through the complete list of outcomes provided at the beginning of Unit 2. Which outcomes and indicators do they know? Which outcomes and indicators require extra study and practice?
- Students who require more practice on a particular topic may refer to the relevant Extra Practice master from Chapter 4, 5, or 6.

Assessment	Supporting Learning
Assessment *for* Learning	
Cumulative Review, Chapters 4–6 The cumulative review provides an opportunity for students to assess themselves by completing selected questions pertaining to each chapter and checking their answers against the answers in the back of the student resource.	• Have students review their notes from each chapter to identify topics that they had problems with, and do the questions related to those topics. Have students do at least one question that tests skills from each chapter. • Have students revisit any chapter section they are having difficulty with. • You may wish to have students review the study guide blackline masters from Chapters 4 to 6 as well as practice tests and chapter tests they completed to help identify any skill areas that still require reinforcement.
Assessment *of* Learning	
Unit 2 Test After students complete the cumulative review, you may wish to use the unit test on pages 328 and 329 as a summative assessment.	• Consider allowing students to use their graphic organizers. • You may wish to have students complete **BLM U2–2 Unit 2 Test**, which provides a sample unit test. You may wish to use it as written or adapt it to meet the needs of your students. The answers to this unit test can be found at the end of **BLM 6–8 Chapter 6 BLM Answers**.