Exponential Functions

Opener

***Pre-Calculus 12*, pages 332–333**

Suggested Timing
30–45 min

Blackline Masters
BLM 7–1 Chapter 7 Prerequisite Skills
BLM U3–1 Unit 3 Project Checklist

Planning Notes

Consider using small groups and partners whenever possible, to allow students to discuss and learn from each other. They can then share their understanding and strategies with the rest of the class. For the Examples, either have students work through one Example and then complete the Practise questions related to it before moving on to the next, or go through all of the Examples with the class and then complete the Check Your Understanding questions. Be aware that this chapter involves the use of graphing technology (graphing calculators or computer graphing programs), so you may need to reactivate some students' understanding of proper window settings, determining zeroes, and finding points where two functions intersect.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter and encourage them to use a summary method of their choice.

Unit Project

In the Project Corner at the end of section 7.2, students consider how exponential functions can be used to model real-world data. Have students work in pairs to discuss how different aspects of these types of functions are related to the graphs of the box office earnings of the three movies in the Project Corner.

Meeting Student Needs

- Consider having students complete the questions on **BLM 7–1 Chapter 7 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out **BLM U3–1 Unit 3 Project Checklist**, which provides a list of all the requirements for the Unit 3 Project.
- Provide students with a checklist containing the learning outcomes for this unit. Discuss specific terms with which student might be unfamiliar. This discussion will help students develop a sense of what they need to learn by the end of the unit.
- Students could browse through the chapter and create a list of key words, formulas, and variable designations that they are unfamiliar with and would like to learn more about.
- Use the visual aids, including video clips, to motivate students when starting the chapter.

Enrichment

After reading the chapter introduction, challenge students to create a list of examples of situations in which exponential functions might be more useful than other functions. Have them explain and record their thinking. These notes can be reviewed at the end of the chapter as a metacognitive exercise.

Gifted

Ask students to research and/or consider the similarities and differences between expansion of the universe and creation of a black hole—two opposing phenomena. Ask them to speculate about these cosmic events, and to record their thoughts about how the chapter subject matter, as seen in the introduction, might be related to them. These notes can be reviewed at the end of the chapter as a metacognitive exercise.

Career Link

Discuss with students what they know about chemistry and what chemists do. Ask
- What do you learn in your chemistry classes?
- In what kind of industries do you think chemists might work?
- In what areas do you think chemistry has played an important role in the past? Do you think that chemists' roles and the industries that they are involved in are changing? What fields do you think they might be involved in now and in the future that did not exist in the past?

7.1 Characteristics of Exponential Functions

***Pre-Calculus 12*, pages 334–345**

Suggested Timing

90–120 min

Materials

- graphing technology
- card stock

Blackline Masters

BLM 7–2 Section 7.1 Extra Practice

Mathematical Processes for Specific Outcomes

RF2 Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

RF3 Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

RF4 Apply translations and stretches to the graphs and equations of functions.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Visualization (V)

RF9 Graph and analyze exponential and logarithmic functions.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–6, 8, 9
Typical	#1–5, 7–11, C1, C2
Extension/Enrichment	#10–15, C1, C2

Planning Notes

By the end of this section, students should be able to describe the following aspects of any exponential function: its domain and range, its x-intercept and y-intercept, whether it is increasing or decreasing, and whether it has any horizontal asymptotes. Students should use an appropriate graphing technology to explore and develop an understanding of these characteristics. The technology they use must enable them to determine the values of any x-intercept(s) and to find points of intersection between two graphs. Students are also expected to use graphing technology to solve growth and decay word problems. Keep students involved and interested by first challenging them to attempt the Investigate or Example problems in pairs or groups of three. Allow for open discussion and coach students who are having difficulty by using leading questions that require the use of previous knowledge and skills.

Investigate Characteristics of Exponential Functions

This investigation uses graphing technology to help students gain an understanding of the basic shape of graphs of exponential functions of the form $y = c^x$. By examining the graphs of given exponential functions, students determine whether the function is increasing or decreasing, its domain and range, and the equation of any horizontal asymptotes. They then develop a generalization of how the value of c (the base of the exponent) affects the shape of the graph.

Before beginning the investigation, revisit some of the aspects of exponents with students. The goal here is not to get into too many specifics; the Investigate will help students explore exponents and exponential functions further. You simply want to get students thinking about the exponents and some of the “special cases” that they should be aware of. Ask

- What is another way to write 3^4?
- What is 1 raised to any exponent?
- What is the value of any base raised to 1?

If students are having difficulty with the Investigate, help them explore the function further. Consider using a table of values to show what happens to the number 2 when it is raised to an increasingly larger positive number. Ask

- How is this progression different from a linear progression?
- What happens as the exponent gets larger?
- How is this type of function different from a linear or quadratic function?
- How is $y = x^2$ different from $y = 2^x$?

Then ask students to relate this discussion to the shape of the curve. Ask students who are having difficulty to use their own words to summarize the shape of the graph of $y = 2^x$. Elicit a description of the domain and range in their responses. Also get students to recognize that the graph is increasing to the right or "opening" to the left.

Some students may mistakenly think that c^0 is the lowest point on the curve. Help students see the relationship between the y-intercept and $y = c^0$. Ask

- Can you have a negative exponent?
- Use your calculator to find the value of 2^{-1}, 2^{-2}, 2^{-3}, 2^{-4}, and 2^{-5}. What is happening to the value? Will it ever reach zero?

You may take this opportunity to discuss horizontal asymptotes and the use of "as the values of x get very small."

You can then discuss what happens when you vary the exponent of a base that is a positive number less than 1 (write this as $0 < c < 1$ so students become used to this notation). Again, for students having difficulty, use a table of values to apply a progression of exponents. For example, what is 0.5^1, 0.5^2, 0.5^3, 0.5^4, and 0.5^5? What is happening to the values? What are the implications for the shape of the graph? Have students graph this function and then relate their table to the graph.

For #3, ask

- If you use a value greater than 2 for c in $y = c^x$, does the graph curve upward faster or slower than $y = 2^x$?
- What general statement can you make about how the shape of the graph changes as the value of c increases?

For #5, you may want to discuss the concept of negative bases. If students are using a graphing calculator to graph the functions, they may have difficulty getting their technology to output a graph. If this is the case, suggest that they use a table of values to graph this function, noting any difficulties they encounter. Also, if students are attempting to use technology, ensure that they are entering the function correctly, using $(-c)^x$ and not $-(c)^x$. Once students have generated a graph, ask

- How does the shape of the graph change when $c < 0$?
- Does the graph look similar to exponential graphs in which $c > 0$?
- Is the graph a smooth continuous curve?
- Is this an exponential function?
- Explain why you think the graph looks as it does.

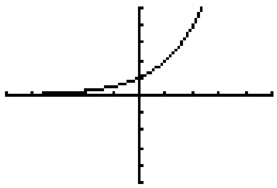
Meeting Student Needs

- Discuss the outcome(s) and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found in this section.
- Display a graph of an exponential function. Have students write down the characteristics of the graph, drawing on knowledge of characteristics of functions that have been developed in previous courses. Have them create an approximate table of values. Ask them what real-life situations may have these characteristics.
- If technology is not accessible, graph each equation on a separate overhead transparency for the Investigate. With the class, study $y = 2^x$ first. Place the graph of $y = 3^x$ on top of the first graph and compare and contrast the two graphs. Continue this process, illustrating five or six different functions. Ask students to describe what is happening to the shape of the graph and why.
- Some students will benefit from exploring the visual representation provided by graphing technology. Suggest that they try changing the parameters and values for $y = c^x$ to see the effects on the graph. Can they explain what they are seeing? Can they see any patterns? For example, they may change the sign of the power or base, or try graphing larger and smaller values of c. Suggest they graph these on the same set of axes. What happens?

Answers

Investigate Characteristics of Exponential Functions

1. a)



b) The graph of the function $y = 2^x$ begins just above the x-axis and rises from left to right.

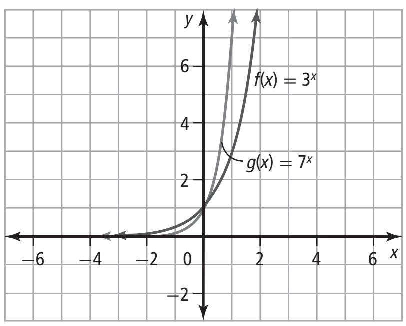
2. a) domain: $\{x \in \mathrm{R}\}$ because any value of x can be used as an exponent; range: $\{y > 0, y \in \mathrm{R}\}$ because the graph never reaches the x-axis

b) (0, 1) where the graph intersects the y-axis

c) none

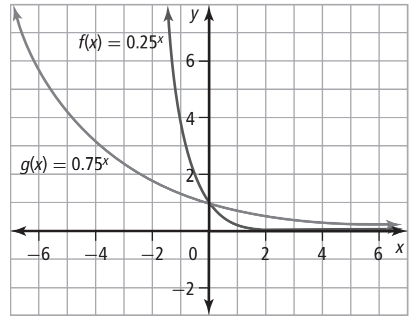
d) $y = 0$ because the value of y approaches zero as the value of x gets very small.

3.



Any values greater than 2 produces a graph similar to $y = 2^x$. The larger the value of c, the steeper the graph curves upward above the x-axis. All graphs share the same y-intercept, (0, 1).

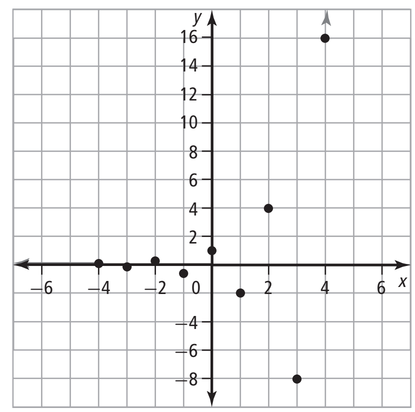
4.



The graphs decrease from quadrant II to quadrant I. They pass through the point (0, 1) and get closer to the x-axis as x increases.

5. If c is negative, a discrete graph is produced. The points on this graph alternate from negative to positive (or below and above the x-axis).

x	y
−4	$\frac{1}{16}$
−3	$-\frac{1}{8}$
−2	$\frac{1}{4}$
−1	$-\frac{1}{2}$
0	1
1	−2
2	4
3	−8
4	16



6. a) For $y = c^x$:
- When $c > 1$, the graph is smooth and continuous and curves upward from left to right, with a y-intercept of (0, 1). It has a horizontal asymptote of $y = 0$.
- When $0 < c < 1$, the graph is smooth and continuous and curves downward from left to right, with a y-intercept of (0, 1). It has a horizontal asymptote of $y = 0$.
- When $c < 0$, a discrete graph is produced, with points alternating above and below the x-axis.

b) A straight line is produced with an equation of $y = 1$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs. • Ask students to compare the graphs they completed in #3 and 4, discussing any similarities and differences. As a class, use these points to generalize a rule for #6.

Example 1

This example is used to help students compare exponential functions of the form $y = c^x$, where $c > 1$ and $0 < c < 1$. It illustrates the differences and similarities between the two basic types of exponential functions. Divide the class into groups of two or three and have the groups attempt part a). Ask students to discuss what they learned with the class. Then, have each group complete part b). Challenge all groups to graph each function by using a table of values and by using graphing technology. After the class has discussed both parts of the question, have the groups discuss how the graphs are similar and how they are different.

In the Your Turn question for Example 1, students graph the function without using technology and then describe the required characteristics of the graph. They then check their sketch by graphing the function with technology. Circulate amongst the groups while they work on the Your Turn question. If students are having difficulty, ask them to describe the their expectations for graph and how their results either support or differ from what they expect.

Example 2

Have students work in small groups. Before beginning, you may have a class discussion about strategies for approaching the question. If students are having difficulty, ask

- What equation do you get when you substitute the values of the point $(-1, 4)$ for x and y in the equation $y = c^x$? Given the equation $4 = c^{-1}$, what equation do you get if you raise both sides of the equation to the power of -1? What value of c makes the equation true?
- What equation do you get when you substitute the values of the point $(-2, 16)$ for x and y in the equation $y = c^x$? Given the equation $16 = c^{-2}$, what strategy can you use to solve for the value of c? What value of c makes this equation true?
- What do you notice about the two values that you calculated for c above? (They are the same, $c = \frac{1}{4}$.)
- What exponential function produces the given graph? ($y = \left(\frac{1}{4}\right)^x$)

Example 3

Have students read the student resource and define *exponential growth*, *exponential decay*, and *half-life* in their own words. For students who need coaching when working through this example, ask

- What mathematical fraction is implied by the word *half*?
- Given that a 1-kg substance has a half-life of two years, what mass of the substance remains after two years? four years? six years?
- For the sample of radium (Ra-225), the half-life is 15 days. How long does it take for the sample to decay to half of the original mass?
- How many half-lives are required for the sample to decay to $\frac{1}{4}$ of the original amount? How many days does this represent?
- As an exponential expression similar to $y = c^x$, what value of c best describes a half-life?
- What variable will you use to describe the amount of sample remaining?
- What exponential equation can you use to model the amount of radium remaining after n half-lives?
- How many half-lives are required to decay the original sample to $\frac{1}{32}$ of its original mass? If each half-life is 15 days, how many days are required for the sample to decay to $\frac{1}{32}$ of its original mass?

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Suggest that students begin to create a bookmark containing the Key Ideas for the chapter. Give each student half a sheet of card stock and instruct them to write down the definition and a quick note about increasing and decreasing functions. Students can continue to add to this sheet as new material is introduced throughout the chapter. They can refer to this card while working on the questions in each section.
- As an alternative to covering the Examples as a whole class, divide the class into groups of three. Each student in the group will be responsible for summarizing one example in this section. They will then prepare a presentation for the other two students in the group, including notes. Students can then work as a group to update their bookmarks.

Enrichment

Explain to students that there are animal populations that go through rapid expansion, followed by rapid collapse. For example, the population of snowshoe hares in the boreal forest. Tell students that a similar mathematical pattern occurs in the physical world. For example, radioactive material is brought to rapid decay by bringing the atoms close enough to sustain a nuclear reaction. As the fuel is used up, rapid decline in radioactive output occurs. Ask students to describe the mathematical similarities between these two circumstances. How are they related to what they studied in this section?

Gifted

Explain to students that a tsunami emanates outward from a single point, and has a rapidly expanding circumference. Ask students to consider the role of the inverse square law in terms of declining energy versus expanding circumference. Ask, "Can this case be described using exponential functions? Explain."

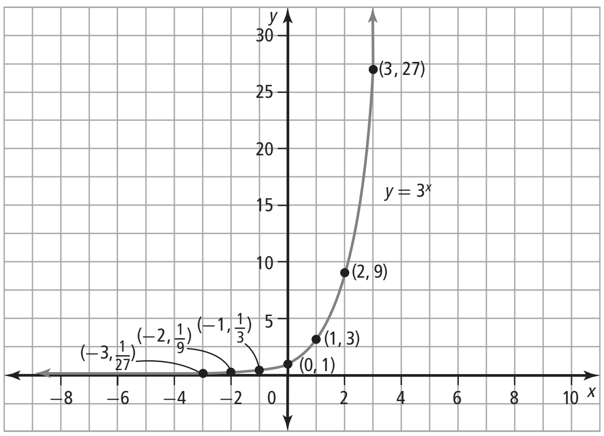
Common Errors

- Some students may not be able to determine the basic exponential function from a given graph.

R_x Ask students to consider a graph of the function $y = c^x$, where the point (1, 5) is on the curve. Ask: Can you substitute the values from the ordered pair for the variables x and y? What value of c satisfies the equation $5 = c^1$? Can you use a similar strategy to find basic exponential functions given the graph?

Answers

Example 1: Your Turn



domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y > 0, y \in \mathrm{R}\}$; no x-intercept and y-intercept is (0, 1); function is increasing and has a horizontal asymptote of $y = 0$

Example 2: Your Turn

$y = 5^x$

Example 3: Your Turn

a) domain: $\{t \mid t > 0, t \in \mathrm{R}\}$; range: $\{B \mid B > 1, B \in R\}$

b) $B(t) = 3^t$

c) $\approx$13.2 days

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Assist students with a table of values if they are having difficulty getting started. • Suggest that students compare their graph with that of a partner. • Provide students with a brief summary that indicates how to determine whether a function is increasing or decreasing. Allow students to include this summary in their graphic organizer.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • If students are having difficulty with the opening question, prompt them by asking them to consider $c > 1$ or $0 < c < 1$. Have them verbalize what each condition represents. • Encourage students to look for patterns. Have them focus on the y-values. • Suggest that students compare their equation with that of a partner.
Example 3 Have students do the Your Turn related to Example 3.	• As a class, discuss the meaning of tripling and what value would be used to represent it. • Before students begin, have them provide suggestions for what method they will use. This may assist students who are having difficulty knowing where to start. • Ensure that students do not arrive at the number of intervals and then fail to multiply it by the interval length of 7 days.

Check Your Understanding

If students are having difficulty with #1, ask them to use their own words to describe the format of an exponential function. What aspect of an exponential function does the unknown variable represent—the answer, base, or exponent? Which aspect of the function in each of the parts does the variable represent?

If students are having difficulty with #2c), suggest that they look at the graphs in #3, as well as the graphs throughout the section to see if there is a point that they all share. If they are still having difficulty, ask them to describe the significance of the y-intercept.

For #3, revisit with students which elements of an exponential function determine the shape of the graph. Which of the given graphs model an increasing function? Which models a decreasing function? Then, ask them what values of c in the exponential function $y = c^x$ indicate increasing and decreasing functions, and what aspect of an exponential function determines the rate at which a graph curves upward or downward.

For #4, ask students what the general form of an exponential function is. Which variables of the function $y = c^x$ are the same for both graphs? For part a), ask "Considering that this is an increasing function, would the base of the exponential function, c, have values less than or greater than 1?" You may also want to ask students to explain why the point (0, 1) cannot be used to solve for c. Then ask them to share one strategy to solve for c using either (1, 3) or (2, 9). Have a similar discussion for part b) if students are still having difficulty.

For #5, discuss what strategy students might use to graph each of the functions. How will they determine the domain and range from their graph? You might also want to reinforce the concept of c^0, and how this relates to the y-intercept for exponential functions of the form $y = c^x$. Discuss the equation of the horizontal asymptote for all exponential functions of the form $y = c^x$. Ask students how they can tell whether a given function is increasing or decreasing.

In #8, some students may have difficulty understanding 10% expressed as a decimal. Ask

- If a population of fish remains the same, what percent of the population remains from one year to the next?
- If the fish population grows by 10% in a year, what percent of the original population exists at the end of the year?
- How would you express this as a decimal?

Pursue a similar questioning strategy for part c), asking what percent of the original population remains at the end of the year if a population of fish decreases by 5% from one year to the next.

In #9, you may have to help some students recognize how many 10-metre descents a diver has made in a 25-m descent. Then, ask student what the value of d is for a descent of 25 m.

To assist students with #10, ask

- What value of the base of an exponential function is used to express a half-life?
- How many half-lives are required for the uranium to decay to 0.125 kg?
- If Uranium-235 has a half-life of 700 million years, how many years are required for the uranium to decay to 0.125 kg?
- If you continue to take one half of a substance, will it ever decay to 0?

In #11b), ask students how much a \$1 investment is worth after it triples in value. Then ask them how they could solve the equation $3 = (1.0175)^n$ using graphing technology. For part d), help students understand the amount of time in years that would be required to double an investment using the rule of 72. You might wish to use a different interest rate (such as interest rates in 1980) to examine the effect that the change in interest rate makes on the doubling of money. How could they use graphing technology to solve the equation $2 = (1.0175)^n$?

For #12, help students understand what percent of the population remains at the end of the year if the annual population growth rate is 1.27%. Some students may need help to write this percent as a decimal. Then ask

- Which variable in the expression $y = c^x$ is represented by a growth rate of 1.0127?
- How can you express 9 billion as a fraction of the 2011 population?
- What exponential expression can you use to represent a population growth of 1.27% per year that has grown by a factor of $\frac{9}{7}$?

Meeting Student Needs

- Consider having students work through the questions in pairs or groups of three.
- For #15, students could contact a financial institution and research the type of interest currently applied to the various accounts offered at the institution. Tell students that they can use this information to decide which institution and the type of savings vehicles they might consider for themselves.

- Provide **BLM 7–2 Section 7.1 Extra Practice** to students who would benefit from more practice.

Enrichment

Some student may wish to research the Krumbein phi scale to discover its use today. You could have them report back to the class to discuss what they have found.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–6, 8, and 9. Students who have no problems with these questions can go on to the remaining questions.	• You can use #1 to assess whether students can identify what an exponential function is. This is a basic understanding question. Students who have difficulty with this question should review additional questions before moving on. • Use #3 to 6 to assess whether students understand how the value of c affects the graph of an exponential function. Ensure that students are able to verbalize what they believe should be the results before beginning the question. Some students have difficulty remembering which values the domain and range represent. Help them develop a strategy for remembering this. • For #8, have students identify the key words in the information that help to explain the value of c (growth rate of 10% per year). The opposite reasoning is used in #9.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2.	• C1 is a basic question and important for students to use as an assessment of their understanding of different functions. If students are having difficulty with this question, complete the responses as a class and then provide students with a new set of equations to graph. You may wish to include a fourth graph in either set of examples, which has a c value of $0 < c < 1$. This would enhance students' opportunity to describe the effect of the c value on their function. • It may be valuable to place the summary of similarities and differences in a table format on the board for students to see. • Once students have completed C2, discuss how the graph they have created is different from those in C1. Have them suggest reasons for the difference.

Transformations of Exponential Functions

7.2

***Pre-Calculus 12*, pages 346–357**

Suggested Timing
90–120 min

Materials
- graphing technology
- grid paper

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 7–3 Section 7.2 Extra Practice

Mathematical Processes for Specific Outcomes
RF9 Graph and analyze exponential and logarithmic functions.
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–5, 7, 8, two of 9–12
Typical	#2–4, 6–8, three of 9–12, C1, C2
Extension/Enrichment	#12–14, C1, C2

Planning Notes

In this section, students apply their previous knowledge of transformations on exponential functions. After section 7.1, they should have a good understanding of the characteristics of the basic graphs of exponential functions. They will now experience how the graphs behave when reflected, stretched, or translated.

Before beginning the Investigate, discuss with students what they know about transformations. You can either have this discussion as a class or have students work in small groups to revisit these topics.
- What were the key ideas of transformations?
- What effects does changing the parameters of a, b, h, and k have on a graph?
- From your past experience with transformations, what do you think you might encounter in this section?

Investigate Transforming an Exponential Function

Have students work together in pairs or groups of three to discuss and share any Investigate discoveries. Students require graphing technology that allows them to graph a number of functions on the same axes. They will also need graph paper to sketch the graphs in their notebook (consider distributing **Master 3 Centimetre Grid Paper**). Suggest that they graph the base function by using technology and by sketching it on graph paper. They should then graph each new function in the set on the same set of axes, indicating which parameter, a, b, h, or k, has been changed and by what value. For each new graph, suggest that students write a note about how the new graph has been changed by the given transformation.

Some students may need coaching to recognize how the graphs have changed. For set A, ask students how the graph of $f(x) = 3^x$ changed when they graphed the function $f(x) = 3^x + 2$. How did it change when they graphed the function $f(x) = 3^x - 4$? Which parameter from the general equation $f(x) = a(c)^{b(x - h)} + k$ was changed in this set of graphs? How did changing the value of k affect the graph of $f(x) = 3^x$?

Have a similar discussion about the functions in set B:
- How did the graph of $f(x) = 2^x$ change when graphing the function $f(x) = 2^{x - 3}$? Did the graph move left or right when you changed x to $x - 3$?
- How did the graph of $f(x) = 2^x$ change when you changed x to $x + 1$?
- Which parameter, a, b, h, or k, did you change? How would you describe the effect of changing the parameter h in the exponential function $y = c^{x - h}$?

For set C, have students graph the original function and then identify the y-intercept and at least one other point on the graph. Then, ask "Which parameter, a, b, h, or k, did you change when writing the function $f(x) = \left(\frac{1}{2}\right)^x$ as $f(x) = 3\left(\frac{1}{2}\right)^x$?" Tell students to consider the point $(-2, 4)$ on the graph of $f(x) = \left(\frac{1}{2}\right)^x$ as a base point. Then, have them describe how the y-value is affected by varying the a parameter as follows, while keeping $x = -2$:
- $a = 3$: $f(x) = 3\left(\frac{1}{2}\right)^x$
- $a = \frac{3}{4}$: $f(x) = \frac{3}{4}\left(\frac{1}{2}\right)^x$
- $a = -4$: $f(x) = -4\left(\frac{1}{2}\right)^x$
- $a = -\frac{1}{3}$: $f(x) = -\frac{1}{3}\left(\frac{1}{2}\right)^x$

Once students have worked through this exercise, ask them to describe how the y-values of any point of an exponential function are affected by changing the parameter a in the function $f(x) = a(c)^x$. Then ask if the y-intercepts for all the functions graphed in set C are affected in the same way as any other point of the base function after a transformation involving parameter a. What two generalizations can you make about how changing the value of the parameter a affects the graph of the original exponential function?

For set D, ask students which parameter, a, b, h, or k, changes when writing the function $f(x) = 2^x$ as $f(x) = 2^{3x}$. Then, use the point (1, 2) as a base point and have students graph the line $y = 2$ and compare the x-values of all the intercepts of the graphs in set D and the line $y = 2$. Ask

- What are the x-values for the intersections of the line $y = 2$ and the functions $f(x) = 2^x$, $f(x) = 2^{3x}$, and $f(x) = 2^{\frac{1}{3}x}$?
- By what factor has the x-value been multiplied for the function $f(x) = 2^{3x}$? for $f(x) = 2^{\frac{1}{3}x}$?
- How have the x-values of the base function been affected by changing the value of the b parameter?
- When you graphed the function $f(x) = 2^{-2x}$, what changed besides a horizontal stretch?
- Does the graph of $f(x) = 2^{-\frac{2}{3}x}$ demonstrate the same effect on the graph of $f(x) = 2^x$?
- What two generalizations can you make about how changing the value of the b parameter affects the graph of the original exponential function?

Meeting Student Needs

- Discuss the outcome(s) and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found in this section.
- When working on the Investigate, students can work collaboratively in pairs or groups. Tasks can be assigned within each group, including an illustrator (using technology) and recorder.
- Encourage students to use graphing technology to see how changing parameters affects the graph of a function. What patterns do they see? Some students will benefit from changing one parameter at a time to see the effect on the graph.

Common Errors

- Some students have difficulty recognizing that an exponential expression of $(x - h)$ results in a horizontal translation of h units right when $h > 0$ and a translation of $|h|$ units left when $h < 0$.

R$_x$ Review the set B graphs from the investigation to make sure that they have been done correctly. Have students repeat this investigation using the basic exponential function $y = 3^x$. On the base function $y = 3^x$, have students identify the point (1, 3) by circling or highlighting. On the graph of the function $y = 3^{x + 1}$, what value of x produces a y-value of 3? Which way and how far has the original graph moved to translate the point (1, 3) to (0, 3)? On the graph of $y = 3^{x - 3}$, what value of x produces a y-value of 3? Which way and how far has the original graph been moved? Lead students to state an understanding of the effect of the sign of parameter h on the transformation of the original graph.

- Students often do not recognize that a negative value of the parameter a in the equation $y = a(c)^{b(x - h)} + k$ indicates a reflection of the original graph on the x-axis.

R$_x$ Have students graph the two functions $y = (3)^x$ and $y = -(3)^x$ on the same axes. Ask them to describe the change in the graph and to draw a generalization from their observations.

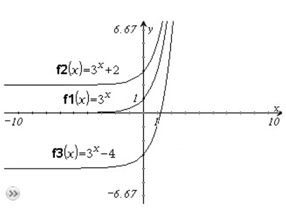
- Some students do not recognize that a negative value of the parameter b in the equation $y = a(c)^{b(x - h)} + k$ reflects the original graph on the y-axis.

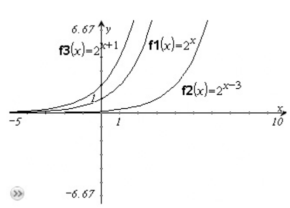
R$_x$ Have students graph the two functions $y = (3)^x$ and $y = (3)^{-x}$ on the same axes. Ask them to describe the change in the graph and to draw a generalization from their observations.

Answers

Investigate Transforming an Exponential Function

1. a)





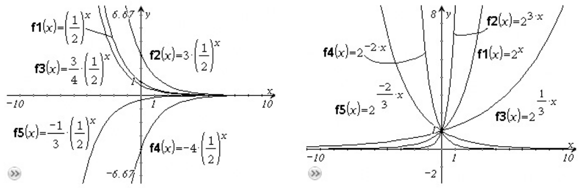
b) The parameter k represents a vertical translation, up for positive values and down for negative values.

c) The parameter h represents a horizontal translation of h units right when $h > 0$, and a translation of $|h|$ units left when $h < 0$.

2. The parameter h indicates a horizontal translation of h units right when $h > 0$, and a translation of $|h|$ units left when $h < 0$. The parameter k represents a vertical translation, up or down $|k|$ units.

Answers

3. a)



b) The parameter a represents two possible transformations:

i) a reflection on the x-axis if negative

ii) a vertical stretch of factor $|a|$, where every y-value on the original graph is multiplied by $|a|$ to stretch the graph vertically

c) The parameter b represents two possible transformations:

i) a reflection on the y-axis if negative

ii) a horizontal stretch of factor $\frac{1}{|b|}$, where every x-value on the graph is multiplied by the value $\frac{1}{|b|}$.

4. The parameter a represents a reflection on the x-axis if negative and a vertical stretch of a factor of $|a|$, where every y-value of the function is multiplied by a factor of $|a|$ units. The parameter b represents a reflection on the y-axis when negative and a horizontal stretch by a factor of $\frac{1}{|b|}$, where every x-value on the graph is multiplied by a factor of $\frac{1}{|b|}$ units.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • Discuss the results for #2 and 4 as a class, and summarize students' findings. Ensure that students are clear on the effects that each parameter has on the graph. • You may wish to use the chart in the Link the Ideas on page 348 of the student resource to review the findings from the investigation. Encourage students to summarize the chart in their graphic organizer.

Example 1

Group students into pairs to work on Example 1a). Have pairs share the strategies and suggestions that they used to answer the question. Discuss the strategies used for parts i) to iv).

Some groups may require some coaching. For part i), ask

- What is the mathematical relationship used in the student resource to describe transformations of exponential functions?
- Using the general formula $f(x) = a(c)^{b(x - h)} + k$, what values from the equation $y = 2(3)^{x - 4}$ correspond to the parameters a, b, h, and k?
- How does a value of 2 for the parameter a transform the graph of the function $y = (3)^x$?
- How does a value of -4 for h transform the graph of $y = (3)^x$?
- How does a value of 0 for k transform the graph of $y = (3)^x$?

For part ii), ask

- Can you extend the table to include a column for stretches, and for horizontal and vertical translations?
- Which parameter, a, b, h, or k, determines the vertical stretch, horizontal stretch, horizontal translations, and vertical translations?
- How does each parameter affect a point on the graph of the base function? Can you use this knowledge to change the values of the given ordered pair, first for stretches and then for translations?

For part iv), ask

- If changing the parameter a causes the graph of the function to be stretched vertically and/or reflected on the x-axis, how will that change affect the domain and range of the base function?
- Does changing the value of a affect the x-intercept and y-intercept? Does it affect the horizontal asymptote?
- How do the values of b, h, and k affect the domain and range, x-intercept, y-intercept, and horizontal asymptotes of exponential functions?

After the groups have shared and gained new strategies, have them do part b) and then the Your Turn question. Be available to answer any questions or concerns as students complete this work.

Example 2

Students must use their knowledge of solving exponential decay problems combined with their understanding of transformations of exponential functions to determine the mathematical relationship expressed in the form $f(x) = a(c)^{b(x - h)} + k$. They first use the given information to determine the value of the base of the exponent, c. They then use this value to write an exponential function of the form $y = c^x$. Then, they convert the transformations into the parameters a, b, h, and k, and write the exponential function that describes the given graph.

If students are having difficulty, ask

- Is the temperature of the water increasing or decreasing?
- If the temperature decreases by 25% every 5 min, what percent of the original temperature remains after 5 min?
- If the base exponential function is represented by $y = c^x$, what base exponential function represents the situation expressed in this question?
- If each change occurs over a 5 min interval, how many temperature changes occur in 10 min? 20 min? How can you represent the number of temperature changes in terms of the total time taken?
- What expression best represents the parameter b of the function $f(x) = a(c)^{b(x - h)} + k$?
- What is the horizontal asymptote of the base function? What is the horizontal asymptote of the given graph? What is the vertical translation of the original asymptote to the asymptote of the given graph? What parameter represents vertical translation? What is the value of k?
- If the base graph, $y = \left(\frac{3}{4}\right)^x$, and the given graph share the same horizontal asymptote, are they the same graph? Why? What kind of stretch has transformed the base graph to the given graph?
- If you move the given graph down 20 units to make up for the vertical translation, what is the value of the new y-intercept? What is the y-intercept of the original function? How many times larger is the y-intercept of the given graph than the y-intercept of the graph of $y = \left(\frac{3}{4}\right)^x$?
- What is the vertical stretch factor from the base graph and the given graph? Which parameter, a, b, h, or k, represents a vertical stretch? What, then, is the value of a?

Meeting Student Needs

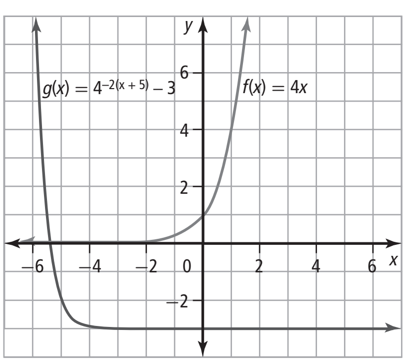
- As students work through the section, they can add information to the bookmark they started in section 7.1. Suggest that they include the effects of the parameters a, b, h, and k on a graph.
- Consider using overhead transparencies or presentation software to provide a visual representation of the effects of each variable on an original graph. Ensure that each graph is created with a different colour for easy comparison.
- Considering having students work in pairs for Example 1. Each student can study either part a) or part b), and then walk their partner through their strategy for solving it. The partners can then work cooperatively through Example 2, discussing the application of a transformation to a situation.
- Ask students to verbally summarize the information presented in the entire section. Then, working individually, they might write the summary either in their notebook or on their bookmark from memory. They can check their work against the Key Ideas in the student resource.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Answers

Example 1: Your Turn

The function $y = 4^{-2(x+5)} - 3$ represents a transformation of the function $y = 4^x$, with these transformational parameters:

- $a = 1$, so there is no vertical stretch or reflection on the x-axis
- $b = -2$, so the graph is reflected on the y-axis and every x-value of the original function is multiplied by a factor of $\frac{1}{2}$
- $h = -5$, so there is a horizontal translation of 5 units left
- $k = -3$, so there is a vertical translation of 3 units down



The domain remains the same: $\{x \in R\}$. The range changes from $\{y \mid y > 0, y \in R\}$ to $\{y \mid y > -3, y \in R\}$, and the horizontal asymptote changes from $y = 0$ to $y = -3$. The change is a result of applying a vertical translation of 3 units down. If there is no vertical stretch, then the range is affected in the same way as the horizontal asymptote and moved up or down k units. The original graph has no x-intercepts because it never crosses the x-axis, but due to a vertical translation down 3 units, the new graph has an x-intercept of $(-5.4, 0)$. The y-intercept has changed from $(0, 1)$ to $(0, -2.99999)$ due to a combination of a reflection on the y-axis, a horizontal stretch by a factor of $\frac{1}{2}$, and a vertical translation of 3 units down.

Example 2: Your Turn

a) $A = 200\left(\frac{1}{2}\right)^{\frac{t}{432}}$

b) The a-value of 200 indicates a vertical stretch by a factor of 200, and the b-value of $\frac{1}{432}$ indicates a horizontal stretch by a factor of 432.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Encourage students to review the statement just before the beginning of this Example about the order in which the parameters are completed. While most student will be able to link to their previous work, struggling learners may find this summary helpful. • Assist students with revisiting these concepts, prompting them if they are unable to answer the side question associated with part b) iii). • Students may find that using a chart or table makes it easier to complete the question and organize their responses. • Some students may find it helpful to use technology to step through the transformation of $y = 4^x$ to $y = 4^{-2(x + 5)} - 3$ one parameter at a time. This will allow them to see the effect that each parameter has on the graph.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Assist any students who are having difficulty writing the basic half-life exponential function. • If students are having difficulty, prompt them to verbalize the changes that are evident in the graph, and to state which parameter they believe is responsible for each noted change. Go through the example orally before having students complete the question on their own.

Check Your Understanding

For #1 and 2, revisit the parameters a, b, h, and k in the functions $f(x) = a(c)^{b(x - h)} + k$ and what effect they have on transforming an exponential function $f(x) = c^x$. Students may want to summarize these transformations on a card or poster, but encourage them to work as much as possible without these tools.

For #3, some students may want to create a table with their response for each question part. How can they use the function $f(x) = a(c)^{b(x - h)} + k$ to determine the parameters a, b, h, and k from the given exponential functions?

For #4, encourage students to describe in words what parameters will result in a graph of the given shape. Suggest that they consider each of the parameters, a, b, h, and k, and describe the effect that each has on the graph.

Some students may have difficulty with #5b). Ask

- What transformation of the function $y = 4^x$ does the equation $y = 4^{-x}$ indicate?
- How would a reflection on the y-axis transform the point $\left(-2, \frac{1}{16}\right)$?
- What new transformation is indicated by $y = \left(\frac{1}{2}\right)(4)^{-x}$?
- How will a vertical stretch of $\frac{1}{2}$ affect the point $\left(-2, \frac{1}{16}\right)$?
- What horizontal and vertical translations are indicated by the equation $y = \frac{1}{2}(4)^{-(x - 3)} + 2$?
- How will a horizontal translation of 3 units right and a vertical translation of 2 units up affect the point $\left(-2, \frac{1}{16}\right)$?

Ask students to use these types of questions to formulate a strategy to determine the transformations on the remaining ordered pairs listed in the table.

Students may have difficulty with the form of the functions in #7. Remind them of the general form and relate it to the transformed function. Given the general form of transformations, $y = af[b(x - h)] + k$, which transformational parameters are represented by the function $y = f(x - 2) + 1$? What transformations are indicated when $h = 2$ and $k = 1$? If students are still having difficulty, do a similar walkthrough of part b) before letting students work on parts c) and d) independently.

Before students attempt to sketch the graphs in #8, you may want to have a discussion about the order in which students should apply the transformations of reflections, stretches, and translations.

Before students work on #9, ask them if the scenario described, drug absorption and persistence, would involve exponential growth or decay. Ask how this is reflected in the mathematical function. Ask them how long it takes for the mass to decay to 79% of the original amount. If it takes 3 h, how much is left after 6 h? 9 h? 12 h? What strategy did they use to determine these amounts?

If students are having difficulty with #10a), ask

- If the initial temperature was 95° and the final temperature was 20°, what is the value of $(T_i - T_f)$? Which parameter, a, b, h, or k, is represented by $(T_i - T_f)$?
- What is the value of T_f? What parameter, a, b, h, or k, is represented by T_f?
- What is the value of the parameter b?

Once students know the values of a, b, and k, help them see how they can describe how to transform the original function $T(t) = (0.9)^t$ to $75(0.9)^{\frac{t}{5}} + 20$. Suggest that they describe, either to you or a partner, the strategy they will use to determine the temperature of the coffee after 100 min. Also discuss how they might solve this problem by graphing.

For #12, you might want to discuss with students whether this is a problem that involves exponential decay or growth, and what this means for the graph. Even before looking at the specifics, ask students what type/shape of graph they would expect. If students are having difficulty setting up the function, ask: How does the fact that Carbon-14 has a half-life of 5730 help to determine the expression for the exponent in the exponential decay equation? You are asked to determine the percent of the original sample remaining—how does this affect the exponential decay equation?

If students are having difficulty with #13, ask

- If the colony of bacteria doubles every 10 h, what is the value of the base of the exponent for this exponential growth? What is the value of the exponent if the colony doubles every 10 h?
- What was the original area of the bacterial growth? What was the final area of the bacterial growth? What function describes an exponential growth where the colony doubles every 10 h and changes from covering 100 cm^2 to 200 cm^2?

Some students may have difficulty with the unit conversion in #13b). Help them by asking what units are being used to describe the area in the stem of the question, and how this differs from part b). Ask

- How can you express the radius of 6378 km in centimetres? What is the surface area of Earth in square centimetres?

Meeting Student Needs

- For #1 to 3, encourage students to check their answers using technology.
- For #9, students may wish to research the time it takes for various forms of common over-the-counter pain medications to be absorbed. Are there products that absorb quicker or last longer? How do these time-based considerations figure into the cost of the product?
- Provide **BLM 7–3 Section 7.2 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–5, 7, and 8. Students who have no problems with these questions can go on to the remaining questions.	• For #1 to 5, students deal with the identification or results of different parameters. In #1 to 3, students encounter a basic level of question that they should be able to complete successfully before moving forward. If students are having difficulty with #4, have them verbalize the changes they see in the functions and how each change might affect the graph. They could check their own response using technology. • Ensure that students are successful with #1 to 5 before assigning #7 and 8. These two questions could also serve as a self-check of students' ability to describe the effects of a parameter on a graph. It may help some students to identify the variable associated with each parameter before describing the effects. Ensure that they can do #4 before moving on to any other question. If they are unable to complete this question, spend time identifying and remediating their difficulty.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2.	• For C1, some students may have difficulty linking the question's application with the math. Before working on this question, describe a situation in which there are three mice in a box with one dish of food. Ask the class to discuss what would happen if each mouse had six babies, and then each baby has six babies, and so on. Would the mice continue to multiply indefinitely if the food and box size remained the same? • You may wish to suggest that students graph an exponential function of their choosing to help them visualize and answer C2.

Solving Exponential Equations

7.3

***Pre-Calculus 12*, pages 358–365**

Suggested Timing

90–120 min

Materials

- graphing technology

Blackline Masters

BLM 7–4 Section 7.3 Extra Practice

Mathematical Processes for Specific Outcomes

RF10 Solve problems that involve exponential equations.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 3, 4, 6, 7a), b), e), 8, 10
Typical	#2, 4–7, 9, 10, one of 11–14, C1, C2
Extension/Enrichment	#12, 15–18, C1, C2

Planning Notes

In the investigation, students explore how some exponential expressions can be expressed using the same base value. Then, they explore strategies to solve exponential equations algebraically. To help students reactivate their knowledge of expressing values of numbers written as exponents taken to the same base, have them evaluate the terms of the sequence of exponents $2^{-5}, 2^{-4}, \ldots, 2^4, 2^5$ before doing the investigation.

Investigate the Different Ways to Express Exponential Expressions

Consider making copies of the table on page 358 in the student resource and handing them out to students. Some students may prefer to create and fill this table out on the computer.

Divide students into small groups of no more than three students to work through the table. After the groups have completed their tables, suggest that they circle or highlight all the expressions that result in a final expression of 2^3.

If students are having difficulty, ask

- Does the expression 2^3 show up anywhere in the first column? What was its original expression using $\frac{1}{2}$ as the base? Do you see the expression 2^3 anywhere else in the table? Can you write these expressions using the original base as listed at the top of each column?
- What do you notice about the original values of each exponent's base? Can you write all expressions in the form of an equality statement?
- What relationship exists between the bases of $\frac{1}{2}$, 2, and 4? Can these values be written as powers of 2? Can you name three more powers of 2 greater than 4 and three more powers of 2 less than $\frac{1}{2}$?
- What do you notice about the sign of exponents for powers of $\frac{1}{2}$ compared to the signs of exponents with powers of 2 and 4? Why does this occur?

To help students with #2, ask if they can find in their tables other exponential expressions for 2^{-4} that have different bases. What are they? Can you follow the same procedure for all other expressions in the same column as 2^{-4}?

When students are working on the Reflect and Respond questions, discuss what the phrase "bases are the same" means. Some students may need assistance reducing 8^{x-1} to 2^{3x-3}. Once students have successfully performed this operation, help them recognize that if the bases on each side of equal sign are the same, the value of the exponents must be the same as well. They can then solve for x by solving the equation $x = 3x - 3$. Make sure that students see the relationship between this algebraic solution and the intersection point in part c). You can then extend this discussion into #4. Ask students how they might use graphing to solve an equation that does not involve common bases. Have them use technology to graph the two terms and find the point of intersection. They can then substitute this value in the equation to see that this is, in fact, the solution.

Meeting Student Needs

- Discuss the outcome(s) and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found in this section.

- Have students explore the types of financial calculators discussed in the opener. They might access online mortgage calculators or investment calculators at competitive financial institutions. Students could research the cost of a home they would like to own, enter the cost into the calculator, and research and enter the current mortgage rate and length of amortization to understand the real costs. They can then calculate the cost of credit and discuss their thoughts on this cost.
- For #3, some students may benefit from exploring common bases visually using technology. Suggest that they use technology to graph $y = 8^{x-1}$, $y = 2^{3(x-1)}$, and $y = 2^{3x-3}$, and describe what they see.

Common Errors

- Some students may not recognize values that share the same base.

R*x* Have students write out the values of powers of 2, 3, and 4, using exponents -5 to $+5$. Have them list them in order of value and share the patterns that they observe.

- Some students may forget how to solve equations by graphing.

R*x* Lead students through the process of solving an equation. Ask: Can you write the left side of the equation as a graphing function? Can you write the right side of the equation as a graphing function? Can you graph both of these functions on the same axes? Do the graphs intersect with each other? Which of the values of the shared point represents the solution to the equation?

Answers

Investigate the Different Ways to Express Exponential Expressions

1. a)

n	$\left(\frac{1}{2}\right)^n$	2^n	4^n
-2	$\left(\frac{1}{2}\right)^{-2} = (2^{-1})^{-2} = 2^2$	2^{-2}	$4^{-2} = 2^{-4}$
-1	$\left(\frac{1}{2}\right)^{-1} = 2$	2^{-1}	$4^{-1} = 2^{-2}$
0	$\left(\frac{1}{2}\right)^0 = 2^0 = 1$	$2^0 = 1$	$4^0 = 2^0 = 1$
1	$\left(\frac{1}{2}\right) = 2^{-1}$	2	$4 = 2^2$
2	$\left(\frac{1}{2}\right)^2 = 2^{-2}$	2^2	$4^2 = 2^4$

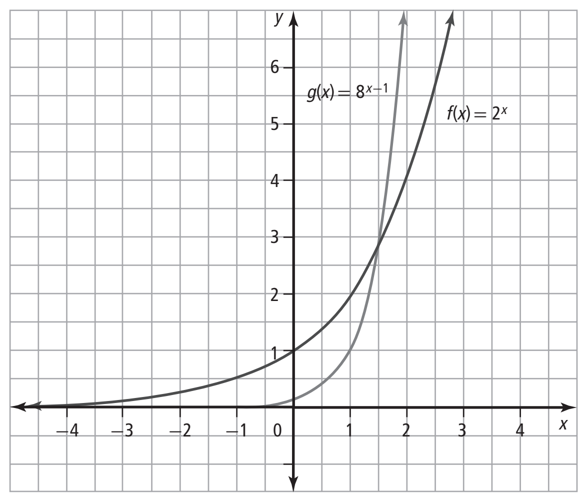
b) Equivalent expressions share bases that can be expressed as a power of 2.

2. $2^{-2} = \left(\frac{1}{2}\right)^2 = 4^{-1}$, $2^{-1} = \left(\frac{1}{2}\right)$, $2^0 = \left(\frac{1}{2}\right)^0 = 4^0$, $2 = \left(\frac{1}{2}\right)^{-1}$, $2^2 = \left(\frac{1}{2}\right)^{-2}$

3. a) Reduce each side to its lowest base. The term 8 can be written as 2^3, so $8^{x-1} = (2^3)^{x-1} = 2^{3x-3}$.

b) Express both sides of the equation using exponential functions with the same bases. Since the bases are the same and the expressions are equivalent, the exponents must be equal. Write and solve an equation in which the exponents are equal, and then solve for x.

c)



Since the point of intersection shares the same ordered pair (x, y) and we are solving for x, then the x-value of the shared point is the solution to the equation.

4. a) No, because the two given exponential expressions cannot be written using common bases.

b) They cannot be solved using common bases, but you can solve them by graphing.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • If students are having difficulty with #3a) and b), solve the question as a class. Ensure that you show the steps of changing 8 to base 2. Have students complete parts b) and c) on their own, and then compare their answers. You may wish to follow-up with a similar question to check for understanding. For example, ask student to solve $3^x = 27^{x-1}$. • Ask students to complete #4 independently, and then discuss their responses with a partner or in small groups. Write a generalized statement on the board that will help remind students of how to determine the limitations.

Example 1

Ask students to write the equivalent values of powers of 3 from 3^{-5} to 3^5. Discuss how they can recognize a power of 3. If students are having difficulty, ask

- Can 27 be written as an exponent? What is the base of the exponent?
- Can 9 be written as a power with a base of 3?
- Can the values 27 and 81 be written as powers with the same base? What is the common base for 27 and 81?
- Can you use your exponent laws to simplify the new expression? Which laws did you use?

Example 2

Have students work in pairs for this Example. One partner can solve part a) while the other solves part b). Each partner should use a different strategy. Then, they can check each other's work. Some pairs may need coaching. Ask

- Can each side of the equation be expressed as exponential expressions with the same base?
- Is there an algebraic strategy to solve exponential equations? What strategy will you use to solve this equation using algebra?
- Can you solve this equation by graphing? What strategy would you use to solve by graphing? What aspect of your graph provides the solution to the equation?

Example 3

You might want to have a discussion with students about where they may have seen the formula for compound interest before. Help them recall what the formula is. Perhaps provide the formula $A = P(1 + i)^n$ and ask what the parameters A, P, i, and n represent. You can then ask what the values of A and P are from the word problem. Ask

- If the interest rate is 6.12% annually, what is the interest rate for each of the 4 compounding periods? Note that some students may need help to recognize the percent written as a decimal (0.0612).
- What is the value of $(1 + i)$? Can you substitute the given values into the formula $A = P(1 + i)^n$?
- What parameter are you solving for?
- Can you solve the equation $1.18 = 1.0153^n$ using common bases?
- Which number system do you expect the value of n to be a member of: whole, integer, rational, irrational, or real? Why?

You might also discuss different strategies for solving this problem. Ask students if they could use trial and error to test for possible values of n. Or, ask if they can graph $y = 1.18$ and $y = 1.0153^n$ on the same axes, and whether the graphs share any points of intersection. Ask: What is the whole number value closest to the value you found by graphing? If n has a value of 12 and there are 4 compounding periods per year, how many years will it take for Christina's \$5000 investment to grow to \$5900?

Meeting Student Needs

- Students could create a chart of bases 2, 3, 4, and 5 with exponents between −4 and +4. These are numbers to look for in the questions in this section.
- Revisit with students the concept of converting from radical form to exponential form.
- Ensure that students complete the Your Turn after Example 2, using graphing technology to check their answer. Doing so will build an understanding that the intersection of the two values is the solution to the question.
- For Example 3, students may be interested to compare saving to purchase a vehicle versus borrowing to make the purchase. Ask them to state whether they would be willing to wait in order to save the cost of the interest.
- For Example 3, have students verbalize how the equation for the graphing calculator was determined. Then, discuss why the solution is found at the x-intercept.
- Students can update their bookmark with the Key Ideas.

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Students may forget how to express a radical number as an exponent.

R_x Review radicals and fractional exponents with students, making sure that you discuss how square roots and cube roots are expressed differently, using fractional exponents.

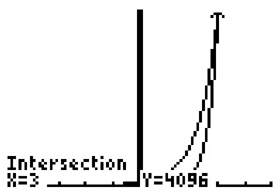
Answers

Example 1: Your Turn

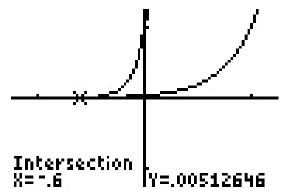
a) 2^6 **b)** 2^{-3} **c)** 2^8

Example 2: Your Turn

a) $x = 3$



b) $x = -\frac{3}{5}$



Example 3: Your Turn

5 years

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• To assist students with these types of questions, you may wish to create a table on the board that has the values 2, 3, 4, 5, and 6 down the left side, and the values of 1, 2, 3, 4, and 5 across the top. Have students fill in each square, using the column value as the base and the row value as the exponent. Then, ask students to write the power and its value. These are values that are frequently used, and students will find it easier to identify numbers with common bases once they become more familiar with these values. • Prompt students by asking what base all three questions have in common. How could they use their chart to determine value?
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • If students are having difficulty with solving, have them first isolate the bases on both sides and write their equivalent forms with equal bases. They can then substitute the values back into the equation. • You may wish to review the exponent rules with students before having them work on this example.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • If students are having difficulty with the Your Turn question, prompt them to verbalize what the values of 8.3%, \$1000, and \$1490 represent in the formula $A = P(1 + i)^n$.

Check Your Understanding

For #1, some students may need to revisit the exponent rules. For example, you might want to ask students to simplify the expression $(2^2)^6$. Then, ask them if they can use a similar approach to write the remaining expressions in this question as a power of base 2.

Help students determine the lowest common base for #2a). They can then attempt the rest of the questions on their own.

For #3, revisit how to express a radical as an exponent. Ask: How would you express $\sqrt{16}$ as an exponent? Can you rewrite the exponent using a power of base 4? Have a similar discussion about parts b) and c). How do you express $\sqrt{2}$ as an exponent? Can you write 2^4 as a power of 4?

For #4 and 5, revisit the concept of solving using a common base. Ask students what strategies they could use to check their answers (substitution or graphing).

For #6a), ask students to verbalize their strategy for determining their first guess for the value of x. For part c), ask what exponent makes the value of 1.2^n equal to 0.5. Given the expression $1.2^{x-1} = 1.2^n$, what value of x makes the equation true?

Discuss with students how they could use graphing to solve #6. You may want to have a discussion about how to set the graphing window so that the intersection points can be viewed. Which value of the ordered pair (x, y) represents the solution to the equation?

For #8a), some students may have difficulty determining the possible values for the domain and range needed to set the window settings on their graphing technology. Help them by asking

- What is the spoilage rate when the temperature is 0°?
- What is the spoilage rate when the temperature is 25°?
- What are the possible values for the range of R?
- What are the given values for the domain?
- Can you use this information to determine the window settings required to see the graph in the given temperature range?

For #9, discuss how students can determine the base value of the exponential function from the fact that the bacterial culture is doubling. Then ask how knowing that the bacteria double every 0.75 h helps determine the exponent for this exponential function. If the original culture has 2000 bacteria and the final count is 32 000, how many times has the bacteria doubled during this growth period? What strategies can you use to solve the equation $16 = 2^{\frac{t}{0.75}}$?

For #11a), review the formula for solving compound interest problems. Suggest that students first identify the parameters that are provided in the problem. Some students will benefit from a discussion of the effects on i when the interest compounds quarterly, rather than annually as it did in #10. For part b), ask: If the $1000 is invested for 4 years and is compounded quarterly, how many compounding periods are there in the 4 years? Then, for part c), discuss what strategies students might use to solve for n in the equation $2 = (1.02)^n$.

For #12, discuss the nature of questions involving half-life and exponential decay. Ask

- Given that Cobalt-60 decays using half-life, what is the value of the base of the exponent for the exponential function that describes the decay?
- How does knowing that Cobalt-60 has a half-life of 5.3 years help determine the exponent for the function that describes the decay?
- What strategy can you use to determine the fraction remaining of Cobalt-60 after 26.5 years?
- What strategies can you use to solve the equation $\frac{1}{512} = \left(\frac{1}{2}\right)^{\frac{t}{5.3}}$?

Before students work on #15, discuss the strategy of solving through graphing. Ensure that students are clear about the significance of the intersection point. Ask

- Is the graph of $y = 2^{3x}$ above the graph of $y = 4^{x+1}$ before or after the point of intersection? Through what values of x is the graph of $y = 2^{3x}$ above the graph of $y = 4^{x+1}$?
- What is the solution of the inequality $2^{3x} > 4^{x+1}$? Can you use a similar strategy to solve the inequality $81^x < 27^{2x+1}$?

For #16, students use exponent rules to write 4^{2x} as $(4^x)^2$. When factoring, they may need prompting to replace the 4^x with a variable in order to recognize that they can factor the expression as $(4^x + 3)(4^x - 1)$.

If students are having difficulty with #18, ask

- From the given information, what values can you use to replace the parameters PMT, PV, i, and $(1 + i)$ in the formula $PMT = PV\left[\frac{i}{1 - (1 + i)^{-n}}\right]$?
- How can you rearrange the equation $813.90 = 150\,000\left[\frac{0.0025}{1 - (1.0025)^{-n}}\right]$ so that $(1.0025)^{-n}$ is isolated on one side of the equation?
- What strategies can you use to solve the equation $(1.0025)^n = 1 - \frac{150\,000(0.0025)}{813.90}$?

Meeting Student Needs

- Students can use the chart of bases they prepared to assist with the questions.
- Have students compare methods for solving #3. This will illustrate that there is more than one method that will lead to the correct solution.
- For #8, students could research the spoilage rates of various products. What is the process used to ensure a longer shelf life? What is the alternative? Students could make educated decisions about what they will look for when shopping for foods.
- For #13, research the current interest rate for a Canada Savings Bond. Determine the value of a $500 bond after 5 years at the current interest rate.
- For #14, students could research RESP investments. What are the terms of use? How much does the government contribute yearly? How much would need to be invested monthly in order for the grandson to have $20 000 in 18 years?
- Provide **BLM 7–4 Section 7.3 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 3, 4, 6, 7a), b), e), 8, and 10. Students who have no problems with these questions can go on to the remaining questions.	• In #1 to 3, students are required to recognize what numbers have common bases. Students could use the chart they developed for an earlier example to assist them in identifying the common bases. They should be able to solve these questions comfortably and correctly before moving onto #4, which requires them to find a common base and solve. Remind students that once they have a common base and have simplified using exponent rules, the bases can be dropped and the equations solved for the variable. • In #6 to 8, students' ability to solve uncommon bases graphically is assessed. Remind students to check their window settings on their graphing calculator before beginning so that they are able to view the intersection points. Ensure they understand that the intersection point is the solution. • For #10, prompt students to identify what parameters are covered by the values given in the problem for the equation $A = P(1 + i)^n$. Ask students whether this should be solved algebraically or graphically.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2.	• Students may use the chart of bases they developed for an earlier example to help with C1a). You may also suggest using a factor tree to develop the bases. These approaches could then be applied as a strategy for identifying the steps used to solve C2. Have students compare their explanations in C2b) with a partner for consistency.

Chapter 7 Review and Practice Test

7

***Pre-Calculus 12*, pages 366–369**

Suggested Timing
90–120 min each

Materials
- graphing technology
- grid paper

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 7–2 Section 7.1 Extra Practice
BLM 7–3 Section 7.2 Extra Practice
BLM 7–4 Section 7.3 Extra Practice
BLM 7–5 Chapter 7 Study Guide
BLM 7–6 Chapter 7 Test

Planning Notes

Have students who are not confident with the concepts discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test. You may wish to provide students with **BLM 7–5 Chapter 7 Study Guide**, which links the achievement indicators to the questions on the Chapter 7 Practice Test in the form of self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum that will meet the related curriculum outcomes: #1–10.

Meeting Student Needs

- Encourage students to list five to eight key ideas from the chapter. Allow them to refer to their bookmark to correct any mistakes in their list and to add any key ideas that they missed.
- Students can work through the practice test individually, using the bookmark for assistance. Suggest that they place an asterisk beside each question for which they had to use the bookmark. Students will then have a solid indication of the material that needs to be learned. They can place a check mark beside every question they completed without assistance.
- Suggest that students create a visual organizer of their own choosing to illustrate their understanding of the ideas of this chapter. Components could include examples of problems they have solved and their worked solutions, definitions, real-world examples of applications of exponents, and graphs showing key points. This tool could be used to reactivate their knowledge of the chapter for the final exam.
- Students who require more practice on a particular topic may refer to **BLM 7–2 Section 7.1 Extra Practice**, **BLM 7–3 Section 7.2 Extra Practice**, and **BLM 7–4 Section 7.3 Extra Practice**.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 7 Review The Chapter 7 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #5–12	• You may wish to have students work in small groups of similar ability. • Encourage students to use **BLM 7–5 Chapter 7 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 7 Study Guide This master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. The minimum questions suggested are questions that students should be able to confidently answer. Encourage students to try additional questions beyond the minimum. • Consider allowing students to use any summative charts, concept maps, or graphic organizers in completing the practice test.
Assessment *of* Learning	
Chapter 7 Test After students complete the practice test, you may wish to use **BLM 7–6 Chapter 7 Test** as a summative assessment.	• Before the test, coach students in areas in which they are having difficulty. • You may wish to have students refer to **BLM 7–5 Chapter 7 Study Guide** and identify areas they need reinforcement in before beginning the chapter test.