Logarithmic Functions

Opener

***Pre-Calculus 12*, pages 370–371**

Suggested Timing
45–60 min

Blackline Masters
BLM 8–1 Chapter 8 Prerequisite Skills
BLM U3–1 Unit 3 Project Checklist

Planning Notes

Acquiring an understanding that logarithmic functions and exponential functions are inverses of each other is a key outcome of section 8.1. This knowledge will help students to evaluate logarithms using a variety of methods. Encourage students to explore these various approaches to determine which method is best for them.

You may wish to introduce section 8.2 by refreshing students' understanding of transformations of functions. Ask students to explain the effect of each parameter. For example: What does a represent? If a is the vertical stretch factor, which coordinate does it affect? If there is a vertical stretch by a factor of 2, how many times larger will the y-coordinate be? If a is negative, what effect does this have on the graph of the function?

To introduce section 8.3, have a discussion about the laws of exponents. Focus especially on the product law and the quotient law. Ask students why they think a review of the law of exponents might help in learning the laws of logarithms. As students discover the laws, help them understand some of the common misconceptions or mistakes in applying them. For example, when adding two logarithms with the same base you do not add the arguments, you multiply them.

As students work through section 8.4, ensure that they understand the laws of logarithms and are comfortable with applying them to solve logarithmic equations. To handle the range of problems presented in this section, students must also be able to switch easily between the logarithmic form and exponential form.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter. Encourage students to use a summary method of their choice.

Unit Project

In the Project Corner that follows section 8.3, students work with data based on box office receipts for a popular movie. Their challenge is to determine the equation of an exponential function and a logarithmic function that fit the data. Then, they compare the models to decide whether one is better than the other and, if so, to explain why.

Meeting Student Needs

- Consider having students complete the questions on **BLM 8–1 Chapter 8 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out **BLM U3–1 Unit 3 Project Checklist**, which provides a list of all the requirements for the Unit 3 Project.

Enrichment

In the chapter introduction, references are made to naturally occurring models of logarithmic spirals. The collage includes images of the Whirlpool Galaxy, a snail, and a ram with its distinctive horns. The plant at the top is a spiral aloe. The leaves of a begonia escargot are shown at lower left. Encourage students to write a paragraph from the point of view of early mathematicians who viewed such models and perhaps were inspired to explore how to "solve" those natural phenomena using mathematics.

Gifted

Ask students to create descriptions of logarithmic functions that model nanotechnology, the technology of the super-small. What are some logarithmic spirals that might occur at the subatomic level?

Career Link

Discuss with students what they know about radiology and the specialized medical procedures that radiologists perform. Ask
- What do you know about radiology?
- What do you think radiologists study in addition to general medicine?
- How do you think radiologists might use mathematics? How might they use exponential and logarithmic functions?

Encourage students to go online to research careers in radiology. Careers in addition to radiologist include radiologist assistant, radiologic technician, and radiologic nurse.

8.1 Understanding Logarithms

***Pre-Calculus 12*, pages 372–382**

Suggested Timing

60–90 min

Materials

- graphing technology
- grid paper

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BLM 8–2 Section 8.1 Extra Practice

TM 8–1 How to Do Page 377 Example 4 Using TI-Nspire™ With Touchpad

Mathematical Processes for Specific Outcomes

RF7 Demonstrate an understanding of logarithms.

- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Reasoning (R)

RF9 Graph and analyze exponential and logarithmic functions.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Technology (T)
- ✓ Visualization (V)

RF10 Solve problems that involve exponential and logarithmic equations.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–10, 12, 13, 17
Typical	#1–9, 11–16, one of 18–20, C1–C2
Extension/Enrichment	#14–16, 18, 20–24, C1–C2

Planning Notes

Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.

Introduce the section by leading a discussion about inverses in mathematics. Ask students to explain what inverses are. How can you recognize an inverse graphically? How can you calculate an inverse algebraically? Why do you think the growth of a social networking site is exponential? Could the number of users accessing the site be modelled using a linear relation? Could the situation be modelled using a quadratic function? If an exponential function is the appropriate tool to predict the number of users as a function of time, what does the inverse of the exponential function describe?

Investigate Logarithms

Before students use a calculator in step 1, have them use their estimation skills. Will x be a whole number or a decimal value? What methods can you use to determine the approximate value of x? Can you do this algebraically? How would you determine the value of x graphically? Is there a value that 10^x will never be? What value does it approach but never reach? What is the largest value that 10^x will be?

How are the tables of values in steps 2 and 3 related? How are their graphs related? What do you notice about the domain and range for each question?

How do you know if a graph represents a function? What is the rule for functions? If it is not a function, what is it called? Can you tell from the equation whether something is a function?

Meeting Student Needs

- Haves students work in pairs to complete the investigation. While one student uses the calculator, have the other record the data and then draw the graph of the function in step 2. Switch roles to complete step 3. Have students discuss how to determine the graph of the inverse of a function.

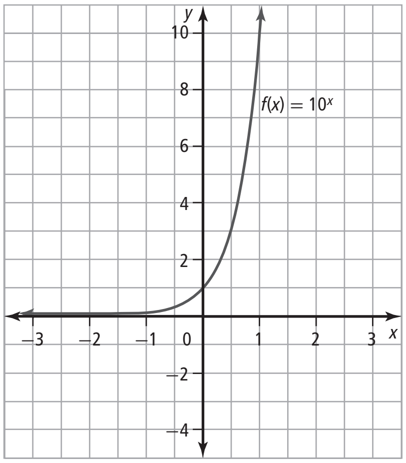
Answers

Investigate Logarithms

1. a) -0.3 **b)** 0.6 **c)** 0.9

2. a)

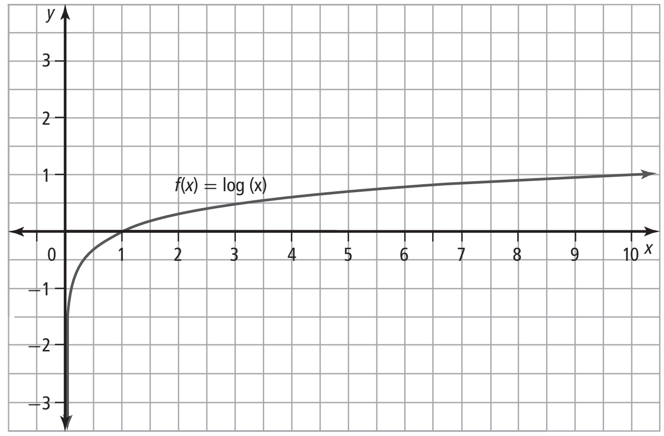
x	-1	-0.301	0	0.602	0.903	1
y	0.1	0.5	1	4	8	10



b) **i)** domain: $\{x \mid x \in \mathrm{R}\}$
ii) range: $\{y \mid y > 0, y \in \mathrm{R}\}$
iii) no x-intercept
iv) $y = 1$
v) $y = 0$

3. a)

x	0.1	0.5	1	4	8	10
y	-1	-0.301	0	0.602	0.903	1



b) **i)** domain: $\{x \mid x > 0, \in \mathrm{R}\}$ **ii)** range: $\{y \mid y \in \mathrm{R}\}$
iii) $x = 1$ **iv)** no y-intercept **v)** $x = 0$

4. a) -0.301 **b)** 0.602 **c)** 0.903; They are multiples of 0.301.

5. Both graphs have the same shape but are reflections in the line $y = x$.

6. Yes, the graph is a function. There is only one y-value for each x-value.

7. When 10 is raised to the power of $\log x$ the resulting answer is x. $\log x$ is the inverse of an exponential function.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • Remind students what an inverse is and show them the inverse form $x = c^y$. • Ensure that students are clear on how to test for a function.

Example 1

To evaluate logarithms you can use a couple of approaches. The first approach is by inspection and then by converting to exponential form. Ask students how to interchange between the two forms. For example, for $\log_7 49$ ask: The number 7 raised to what exponent is 49? Since $7^2 = 49$, then $\log_7 49 = 2$. This approach uses inverse properties, since logarithms and exponentials are inverse mathematical operations that undo each other.

The second approach is to write the logarithm as a ratio of two logarithms with common base 10. Logarithms can be evaluated by taking the logarithm of the argument and dividing it by the logarithm of the base:

$$\log_7 49 = \frac{\log 49}{\log 7}.$$

In $\log_c c^x = x$, the logarithm of an exponential, with the same base, equals the exponent, x. In $c^{\log_c x} = x$, an exponential raised to the logarithm of a number, with the same base, equals that number, x.

To aid understanding, some students may find it helpful to compare combining powers and logarithms, with respect to the same base, to combining square and square root functions. For example, $\left(\sqrt{a^2}\right) = a$ and $(\sqrt{a})^2 = a$, $a > 0$.

Example 2

Ask students why the argument of a logarithm must always be positive. Use this as an opportunity to discuss restricted domain. Since the $\log_x 36 = 2$ has a restricted domain, the inverse $x^2 = 36$ also has a restricted domain. How is the domain restricted?

Example 3

What is the inverse of an exponential function? Can you graph it using a table of values? Can you graph the inverse by reflecting the exponential function? What is the line of reflection? Are there any restrictions for the domain and range of the function? What are the domain and range of the inverse? How can you determine if there are any asymptotes?

Example 4

Students may want to check their estimate using the LOG key on a calculator. Some calculators have a logarithm template with a field for the base. For calculators with only the common logarithm key, use the following approach to evaluate the logarithms of other bases.

$$\log_c x = \frac{\log_d x}{\log_d c}$$

Use the change of base formula for base $d = 10$.

$$\log_2 14 = \frac{\log_{10} 14}{\log_{10} 2}$$
$$\approx 3.8$$

Example 5

Ask students what they know about the Richter scale for measuring the magnitude of earthquakes. Ask students to explain the meaning of each magnitude in relation to other values on the scale. Encourage them to do some research to learn about some of the most powerful earthquakes measured to date.

For comparison, you might also discuss the newer system of measuring earthquakes developed in the 1970s, the moment magnitude scale. The symbol for the moment magnitude scale is M_w, with the subscript w meaning mechanical work accomplished. The moment magnitude, M_w, is a dimensionless number defined by $Mw = \frac{2}{3}\log_{10} M_0 - 10.7$.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Students could create a reference list of common squares and cubes. Practise writing these in both exponential and logarithmic form. Example: $2^3 = 8$, $\log_2 8 = 3$
- Students could work in small groups to create "dominoes," placing a question in exponential form on one end and a different question in logarithmic form on the other end. Each group could create 15 to 20 dominoes and then play against other students.
- You may wish to provide students with **TM 8–1 How to Do Page 377 Example 4 Using TI-Nspire™ With Touchpad**.

Common Errors

- Students may evaluate logarithms incorrectly. For example, students may evaluate $\log_2 8 = x$ as $2^8 = x$.

Rx Reinforce the idea that the base of a logarithm is the base of the exponent. The value of the logarithm is the exponent. For example, $\log_2 8 = x$ can be evaluated as $2^x = 8$.

Answers

Example 1: Your Turn

a) 5 **b)** $\frac{2}{5}$ **c)** 6 **d)** $2\frac{1}{2}$

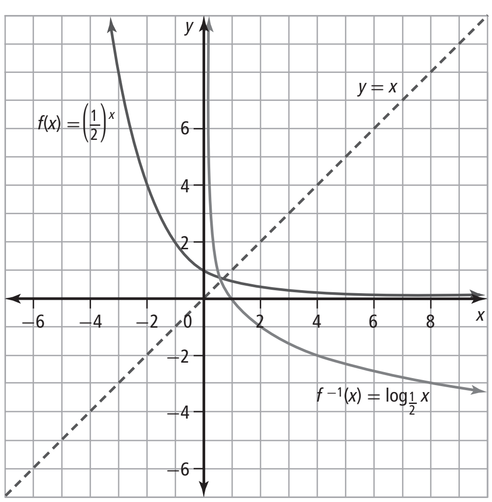
Example 2: Your Turn

a) $\frac{1}{16}$ **b)** $\frac{1}{2}$ **c)** 27

Example 3: Your Turn

a) $f(x) = \log_{\frac{1}{2}} x$

b)



domain: $\{x \mid x > 0, \in \mathrm{R}\}$;
range: $\{y \mid y \in \mathrm{R}\}$;
x-intercept: $x = 1$;
no y-intercept;
vertical asymptote at $x = 0$

Example 4: Your Turn

3.6

Example 5: Your Turn

25 times as great

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Encourage students to write the questions in exponential form. • Remind students that the base of the logarithm is a hint as to what the base should be when writing a question in exponential form.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Suggest that students write the questions in exponential form. • Ask what original form would be used for rational numbers in b) and c) (from radical forms).
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Students often have a difficult time understanding that $x = \frac{1}{2}^y$ and $y = \log_{\frac{1}{2}} x$ are just different ways to express the same thing. It is from $x = \frac{1}{2}^y$ that students write the exponential form. • Point out to students that the distances of all points on either side of the line $y = x$ must be the same when creating the inverse. The inverse points can be found by reversing the ordered pairs (x, y) to be new ordered pairs (y, x).
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs.
Example 5 Have students do the Your Turn related to Example 5.	• You may wish to have students work in pairs. • Assist students in setting up a comparison of the amplitudes. Ask them which exponents laws need to be used. This should give them sufficient information to complete the question.

Check Your Understanding

In #1, some students may forget how to determine the inverse of an exponential function from a graph. Ask students which key points are most useful in identifying the equation of an inverse. Suggest that students select a few key points and interchange the x- and y-coordinates. They could then use these points to determine the equation of the inverse. Some students may prefer an algebraic approach. Students could interchange x and y and determine the equation of the inverse using logarithms.

In #2 and 3, remind students that the base of an exponent is the same as the base of the logarithm. Ask students what other parts are analogous as well. What part of an exponent is the same as the argument of a logarithm? What part is the same as the answer of a logarithm? Understanding these patterns and similarities will help students easily convert between the two forms.

In #4, have students explain how to evaluate logarithms by switching to exponential form. One method is to set each logarithm equal to x and then switch to exponential form and solve for x. Alternatively, ask students how to evaluate logarithms by using a common logarithm with base 10. Why would 10 be the common base? Remind students that when they enter "log 3" into their calculator, the calculator actually evaluates $\log_{10} 3$. Based on the two methods, which one do students prefer?

If students are struggling with #10, ask them to graph each function and determine the key elements (intercepts, asymptotes, etc.) that are the same and those that are different. Ask them how to determine graphically if the functions are inverses of each other. Is there a way to check algebraically?

In #14, ask students what order they need to follow to evaluate each expression. Remind students to work from inside the brackets out, evaluating each expression one at a time.

In #15, ask students to identify the process of determining the x-intercept algebraically. Students can simply equate the y-value to 0 and solve for x by writing the logarithm in exponential form and then solving. For students who prefer a graphical approach, ask them how to enter this equation into their calculator. The easiest way to do so is to use a change of base approach (i.e., $y = \frac{\log (x + 2)}{\log (7)}$). How do you determine the intercept graphically?

In #20, students need to determine the value of x in the first equation before they can evaluate the second logarithm. Ask students how they can solve for x in the logarithmic equation. When they substitute x into the expression, what does the argument become? How can they evaluate the expression once they know x?

If students are struggling with #21, they could solve a simpler equation. How could you solve each logarithm using only one variable? How could you then create a system of equations to determine *m* and *n*?

Help students with #23 by asking how to simplify the question. Would writing the equation in exponential form help? Can you eliminate all the logarithm functions by switching to exponential form? Once you switch *x* and *y* to determine the inverse, how can you then solve for *y*?

In #24, ask students to recall how to solve systems of equations. Which method might be better suited to this question? How might the substitution method help? Which value would you choose to substitute into the other equation? Does it matter which value you select?

Meeting Student Needs

- Provide **BLM 8–2 Section 8.1 Extra Practice** to students who would benefit from more practice.
- Post five to seven questions on poster paper around the classroom. Students can rotate from question to question in small groups. Discuss the outcome or achievement indicator presented in each question and then solve the problem together

Enrichment

Ask student to test the statement that logarithmic and exponential functions are the same when the bases are inverses. Ask them to do this by graphing logarithmic and exponential functions whose bases are very close to being inverses and exploring differences between being "close to inverses" and being exactly inverses.

Gifted

It is said that some individuals are able to determine cube roots of numbers to many decimal places. Ask students to speculate how they might be able to do that, and then if and how they might be able to use similar skills to estimate the value of a logarithm.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do # 1–10, 12, 13, and 17. Students who have no problems with these questions can go on to the remaining questions.	• For #1, remind students that the distances of all points on either side of the line $y = x$ must be the same when creating the inverse, or ask students how the ordered pairs of the inverse points relate to the original (reversing the ordered pairs (x, y) to be (y, x) as new ordered pairs). • Students should be able to successfully complete #2 and 3 before moving on. If they are struggling, have them orally explain their process in order to correct any misunderstandings. Provide them with additional questions to ensure mastery. Some students may find it easier to write the exponential form for #4 and 12. • If students are having difficulty with #5, complete a sample question for them, such as $a < \log_2 32 < b$ $a < 5 < b$ $4 < 5 < 6$ Point out that the logarithm will be an approximation. Ask what their best approximation might be. • You may wish to prompt student thinking for #6b) by asking what would be the value of 4^{-2}. Have students provide a fractional value. Then, have them use this to solve the question.
Assessment *as* Learning	
Create Connections Have all students complete C2.	• For C2, encourage students to provide sufficient detail for their summary and review at the end of the term. Where possible, encourage them to develop their own examples and have a partner check their work. Have students review with a different partner for any information they might be missing.

Transformations of Logarithmic Functions

8.2

***Pre-Calculus 12*, pages 383–391**

Suggested Timing
60–90 min

Materials
- graphing technology

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BLM 8–3 Section 8.2 Extra Practice

Mathematical Processes for Specific Outcomes

RF7 Demonstrate an understanding of logarithms.
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Reasoning (R)

RF9 Graph and analyze exponential and logarithmic functions.
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Technology (T)
- ✓ Visualization (V)

RF10 Solve problems that involve exponential and logarithmic equations.
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2, 4, 5, 6a), b), 7, 8, 10, 12
Typical	#1, 3–5, 6c), d), 7, 9–11, 13 or 14, C1, C2, C4
Extension/Enrichment	#11, 14–17, C1, C2, C4

Planning Notes

Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.

Ask students why the base of a logarithmic function must be greater than 1. You may also wish to review the parameters *a, b, h,* and *k* before starting this lesson. Using a base function from previous units, ask students to explain the effects of each parameter on the function. Give students a transformed logarithmic function and ask them to predict what will happen to the function given what they have already learned about transformations. Why do you think the perceived differences in the magnitude of a stimulus get smaller when the magnitude increases? If students have difficulty thinking of situations involving logarithms, ask them to think of situations involving exponents and then describe the inverse situation.

Investigate Transformations of Logarithmic Functions

Ask students to select a couple of key points on the first graph and follow them as they work through all five graphs. One of the easiest points to work with might be the x-intercept. In graph 1, ask students to identify the x-intercept. Is the x-intercept in graph 2 the same? If it is, what transformation do you know has *not* occurred? What other point can you easily identify in graph 1? How has that point changed in graph 2?

You may wish to repeat this line of questioning from graph to graph to help students identify transformations. Has the point moved left or right? up or down? What points are affected when a function is reflected in the x-axis? What points are affected when a function is reflected in the y-axis? Which transformations affect x-coordinates? Which transformations affect y-coordinates? Which transformations affect domain or range?

Meeting Student Needs

- Have students reflect on a time that they worked hard to improve a skill, such as keyboarding, increasing the speed of throwing a baseball, or increasing the accuracy of hitting a target. At first, they might have recorded significant improvement. Over time, the rate of noticeable improvement would have been less and less, a situation that could be modelled by a logarithmic curve.

Answers

Investigate Transformations of Logarithmic Functions

1. Graph 1: $y = 3 \log x$; vertical stretch by a factor of 3
Graph 2: $y = 3 \log 2x$; horizontal stretch by a factor of $\frac{1}{2}$
Graph 3: $y = 3 \log 2x + 1$; vertical translation up by 1 unit
Graph 4: $y = 3 \log 2(x - 4) + 1$; horizontal translation right by 4 units

2. a) The a value would be negative. **b)** The b value would be negative.

3. a) The value h translates the domain left or right depending on the value. When the value of b is negative, the domain is the reflection of the previous domain in the y-axis.
b) The range is always all the real numbers for a logarithmic function.
c) Only h changes the position of the vertical asymptote.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • As a class, review what effects the parameters *a, b, h,* and *k* have on a logarithmic graph before assigning #3a)–c. This may assist students in describing the domain and range. • It may benefit students to copy the summary chart in the Link the Ideas into their graphic organizer for reference in their assignments.

Example 1

Students could begin by sketching the graph of $y = \log_3 x$ to get an idea what the base function looks like. Ask students to apply what they know about transformations to describe what each transformation will be. Then, have them select two or three key points on the graph of the base function and transform each to sketch the final graph.

Example 2

Why do you have to factor the 2? What effect would it have if you did not factor the 2?

Most students will be able to combine the transformations by using the single mapping, $(x, y) \rightarrow \left(\frac{1}{2}x - 3 - y\right)$, and graphing in one step. All the steps are shown to help those who need it.

Example 3

Ask students which method they prefer. If you had not been given the equation of the first function, how would you approach the question? Beginning with the point (4, 1), why do you look for a point with the same y-coordinate to compare and not a point with the same x-coordinate? Is there another equation that would represent this function? How can you check whether your equation describes the graph?

Example 4

What is the T-axis? Is there another way to solve this equation?

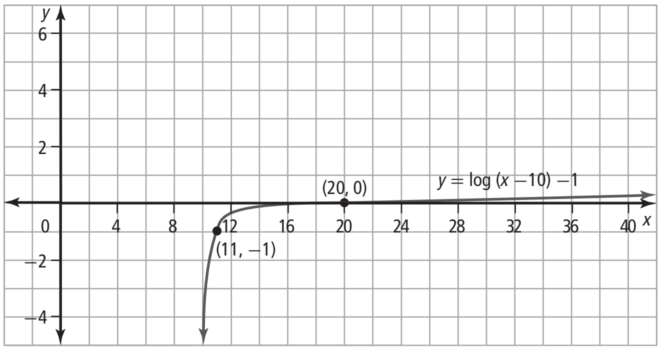
Meeting Student Needs

- Have students study the table at the beginning of the Link the Ideas section. Have them work in pairs to create four logarithmic equations, each containing one of the parameters a, b, h, and k. Then, have them create a table of values containing five points from the graph of $y = \log_c x$. Students can then transform the points to reflect the parameter in each of the equations.
- The diagram in the Key Ideas illustrates the effect of the parameters a, b, h, and k on the graph of the function $y = \log_c x$. Students should notice that the parameters outside the logarithm, a and k, are vertical transformations. The parameters inside the logarithm, b and h, are horizontal transformations.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Encourage students to differentiate a translation from a stretch using invariant points.

Answers

Example 1: Your Turn

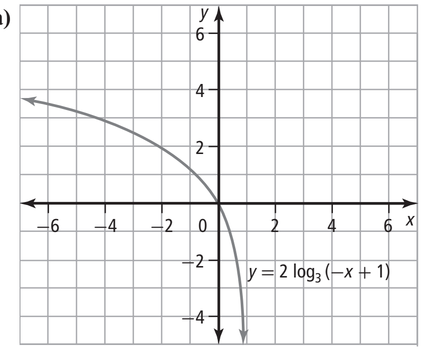
a)



b) **i)** $x = 10$
ii) domain: $\{x \mid x > 10, x \in R\}$; range: $\{y \mid y \in R\}$
iii) The y-intercept does not exist.
iv) $x = 20$

Example 2: Your Turn

2. a)



b) **i)** $x = 1$ **ii)** domain: $\{x \mid x < 1, \in R\}$; range: $\{y \mid y \in R\}$
iii) $y = 0$ **iv)** $x = 0$

Example 3: Your Turn

3. $y = -3\log_4 x$

Example 4: Your Turn

4. 1218 observations

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Have students verbally identify h and k and describe what effect they have on a graph. • Ask students to identify which intercept they will find first and what variable will need to be substituted with zero. • Remind students to change the equation into exponential form to help solve it.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Encourage students to set up a chart and to write points (ordered pairs) for each step of the transformation.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Before beginning, ask students if the x-intercept in the question has changed. What does this tell you about the type of stretch that has occurred? • Ask students to verbalize their thinking on how the lower graph is obtained. Ask how the points (4, 1) and (4, −3) are related.
Example 4 Have students do the Your Turn related to Example 4.	• Students may find it easier to rewrite the equation into the form $y = a \log x + c$. Have them identify the parameters and what they know these parameters do to a graph.

Check Your Understanding

For #2 and 3, ask students whether the order of the transformations matters. Have them apply the transformations in a different order to see if they end up with the same graph.

In #4, ask students to identify the base function in each part. Then, ask them to describe verbally the transformations that have occurred. They can then apply those transformations to the base function to sketch the graph. This approach encourages students to have an idea of what the graph will look like before they sketch it.

In #5, encourage students to identify as many characteristics as they can before sketching the graphs. Ask them how to recognize if a logarithmic function has a vertical asymptote or a horizontal asymptote. Have students determine some of the intercepts algebraically and some of them graphically. Ask them to identify the benefits of each method.

In #6, you might ask students to describe how the equation would change if the blue graph were a stretch of the red graph.

Before beginning #8, have students identify all the parameters and the transformations associated with them. Once all the parameters are summarized, students can then easily determine the equation of all the functions.

In #11, students must simplify the equation before they can compare it to the base function. Ask students how they can isolate the variable y. Have students recall that they must perform the inverse operation to both sides of the equation. Once they have two "$y =$" equations, they can identify the transformations that have occurred.

Ask students which parameter in the equation in #12 represents the magnitude. This will help them determine where in the equation the value of 7.0 would be substituted. How can you solve for E? What steps must be followed to isolate E? Why do you perform a reverse order of operations to isolate E?

In #13, why is the domain restricted to values greater than 56.1? What would happen if the value was less than or equal to 56.1?

In #14, what must be done to the equation before it can be compared to $h = \log m$? What value needs to be isolated? Ask students how accurate they think the predictions are. You might wish to discuss regression equations with students and ask if they think a logarithmic function would be the best model to choose. You could have students collect class data on mass and height, graph the data, and see if the logarithmic function is the best way to describe this relationship.

Once students have determined their equation in #16, ask them to sketch a graph of both functions using graphing technology. How can they use the table of values to check the reasonableness of their answer?

For #17, ask students if a and k affect the x-coordinate or the y-coordinate. If they are unsure, ask them to describe the effect of the parameter. For example, a is the vertical stretch factor. A stretch affects the value of the y-coordinate. Once students have determined that x is not affected, they can substitute x into the original equation to determine y. Once they know the original value of y and the "new" value of y, they can compare the two results to determine a and k.

Meeting Student Needs

- Students could sketch the graphs of the functions in #1 on transparency paper and lay each transparency individually on top of the graph of $y = \log_5 x$. Students could visually compare the graphs to make statements about the transformations. Have them take the same approach to solve #2 and 3.
- Provide **BLM 8–3 Section 8.2 Extra Practice** to students who would benefit from more practice.

Enrichment

The tracking of users of social networks can be used to create revenue for the otherwise free use of such networks. Ask students to suggest why the use of a particular social network might change, and then ask them to comment on how such changes might be reflected as transformations of logarithmic functions. How might revenue streams be affected mathematically by such transformations?

Gifted

Challenge students to describe how inductive reasoning in a mathematical context can be used to derive rules about the effect of h and k changes in comparing functions.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2, 4, 5, 6a), b), 7, 8, 10, and 12. Students who have no problems with these questions can go on to the remaining questions.	• Review the Key Ideas with students having difficulty before going on to #1 and 2. • For #4 or 5, students may find it easier to graph the exponential form first, then graph its inverse, and then sketch the transformation. This may take extra time, but it will provide struggling learners with an opportunity to review the effects of the parameters. This should then make it easier for them to do the reverse operations in #6. Similarly, students may wish to sketch a graph to assist them in writing the equations for #8. • Ensure that students can do #4 before moving on to any other question. • Encourage students to look back at #6 and how the graphs changed before starting #10.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C4.	• C1 is a basic understanding question and encompasses the section's concepts. It would be a good example for students to include in their graphic organizers. • You may wish to have students work with a partner for C3 and discuss their thinking. • Have students share with a classmate the work they have created in C4 and have them check each other's thinking.

Laws of Logarithms

8.3

***Pre-Calculus 12*, pages 392–403**

Suggested Timing
60–90 min

Materials
- graphing technology

Blackline Masters
BLM 8–4 Section 8.3 Extra Practice

Mathematical Processes for Specific Outcomes

RF7 Demonstrate an understanding of logarithms.
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Reasoning (R)

RF8 Demonstrate an understanding of the product, quotient and power laws of logarithms.
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Technology (T)

RF10 Solve problems that involve exponential and logarithmic equations.
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2, 4–8, 10, 11, 12a), b), 13, 14
Typical	#1, 3, 4–7, 9–11, 12c), d), 14, one of 15–17, C1–C4
Extension/Enrichment	one of 15–17, 18–20, C1–C4

Planning Notes

Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.

Ask student why they think converting multiplication involves addition while converting division involves subtraction. Since logarithmic functions are the inverse of exponential functions, ask students how they would multiply two powers. Why must the base be the same? Since the bases must be the same and you keep the base and add the exponents, how might that rule translate to logarithms? Give students a question to try, and discuss how they might approach the solution.

Then, ask students how they would divide two powers. Why must the base be the same? Since the bases must be the same and you keep the base and subtract the exponents, how might that rule translate to logarithms? Give students a question to try, and discuss how they might approach the answer.

Investigate the Laws of Logarithms

Step 1 is designed to steer students away from the temptation to multiply two logarithms with the same base by multiplying the arguments. Ask students to determine the value of log 1000 and then the value of log 100. Can they do this without using a calculator? They may need to be reminded of the common base. Then, ask students to multiply the two values:

$$\begin{aligned}\log 1000 \times \log 100 &= 3 \times 2\\ &= 6\end{aligned}$$

Now, have students evaluate log (1000 × 100):

$$\begin{aligned}\log (1000 \times 100) &= \log (100\,000)\\ &= 5\end{aligned}$$

Students can clearly see that the results are not the same. Since log (1000 × 100) is not equal to log 1000 × log 100, ask students what could have been done to get the correct answer.

Step 2 helps students understand that to divide two logarithms with the same base does not mean to divide the arguments. To illustrate, ask students to divide the values of log 1000 and log 100:

$$\begin{aligned}\log 1000 \div \log 100 &= 3 \div 2\\ &= 1.5\end{aligned}$$

Next, evaluate log (1000 ÷ 100):

$$\begin{aligned}\log (1000 \div 100) &= \log (10)\\ &= 1\end{aligned}$$

Since $\log (1000 \div 100)$ is not equal to $\log 1000 \div \log 100$, ask students if they see what could have been done to get the correct answer. This illustration is useful so that students do not make the mistake of dividing the logarithms.

Step 3 illustrates that the value of an argument raised to an exponent is not equal to the value of the entire logarithm raised to the same exponent. Ask students to square the argument and then evaluate the logarithm.

$$\begin{aligned} \log (1000)^2 &= \log 1\,000\,000 \\ &= 6 \end{aligned}$$

Now, evaluate $(\log 1000)^2$:

$$\begin{aligned} (\log 1000)^2 &= 3^2 \\ &= 9 \end{aligned}$$

Since $\log (1000)^2$ is not equal to $(\log 1000)^2$, ask students what could have been done to get the correct answer.

Meeting Student Needs

- Have students divide a page of their notebook into three sections, one for each of steps 1–3 of the investigation. At the bottom of each section, have them write a conjecture about the law related to that step.
- Have students create flashcards of the three laws of logarithms, including an example of each. Place the right side of the law on one side of the flashcard and the left side of the law on the other side of the flashcard.

Common Errors

- Students may apply the laws of logarithms incorrectly. Example: $\log 3 + \log 4 = \log 7$

R_x Have students use a calculator to evaluate their answers to see if they are equivalent. Evaluating the left side of the equation and then the right side will reveal that the values are not equivalent.

Answers

Investigate the Laws of Logarithms

1. a) $\log (1000 \times 100) \neq (\log 1000)(\log 100)$
$\log (100\,000) \neq 6$
$5 \neq 6$
b) i) 1.4771 **ii)** 1.3222 **iii)** 1.9956 **iv)** 1.9956 **v)** 1.3222 **vi)** 1.4771
c) $\log N + \log M = \log NM$
d) $\log 100\,000$

2. a) $\log \left(\frac{1000}{100}\right) \neq \frac{\log 1000}{\log 100}$
$\log (10) \neq 1.5$
$1 \neq 1.5$
b) i) 1.0792 **ii)** 0.8451 **iii)** 1.5563 **iv)** 1.5563 **v)** 1.0792 **vi)** 0.8451
c) $\log M - \log N = \log \frac{M}{N}$
d) $\log 10$

3. a) $\log 1000^2 \neq (\log 1000)^2$
$6 \neq 9$
b) i) 2.0969 **ii)** 1.6902 **iii)** 2.0969 **iv)** 1.2041 **v)** 1.2041 **vi)** 1.6902
c) $P \log M = \log M^P$
d) $\log 1\,000\,000$

4. a) $\log_6 36$ **b)** $\log_2 8$ **c)** $\log_9 81$

5. The relationships between product and addition, division and subtraction, and exponent and multiplication are the same.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • Discuss, as a class, the answers to #1c), 1d), 2c), and 2d) before assigning #4. • Have students provide oral responses to #5 and summarize them on the board. Have them include any information in their graphic organizer.

Example 1

Students should go through the question one step at a time. For parts b) and d), students should recall how to switch from radical form to exponential form. For part c), encourage students to try this question using the quotient law. Write the expression as $\log_6 1 - \log_6 x^2$. Ask students how to evaluate $\log_6 1$ (i.e., The number 6 to what exponent has a value of 1?). Do they arrive at the same answer as the method shown? Which method do they prefer?

For part d), the placement of the brackets in the second step is very important. Students need to remember they are subtracting the entire expression for $\log y\sqrt{z}$.

Example 2

Some students might leave the answer in part a) as $\log_6 36$. Encourage students to look at the base of the log, 6, and see if the argument, 36, can be written as a power of the base. If it cannot, then they can leave the question at this step. If it can, however, they have to simplify further.

For part b), an alternative solution using the power law for exponents is possible.

$$\begin{aligned}\log_7 7\sqrt{7} &= \log_7\left(7 \times 7^{\frac{1}{2}}\right)\\ &= \log_7 7^{\frac{3}{2}}\\ &= \frac{3}{2}\end{aligned}$$

Example 3

Encourage students to state the restrictions before they start to simplify each expression. They will need to recall the laws of exponents to simplify part a). For part b), why it is important to factor the numerator and/or denominator fully? In discussing the restrictions on the logarithmic functions (the argument must be greater than 0), you might also point out the additional restriction for the rational function (x cannot be 1 or -3). While this restriction is accounted for in the logarithmic restrictions, it is a good idea to discuss it as well, so students do not forget that rational functions also have restrictions.

Example 4

You may wish to show students an alternate approach for each part by calculating individual intensities.

a) To determine the intensity of the chainsaw, substitute $\beta = 85$ and $I_0 = 10^{-12}$ into

$$\beta = 10\log\left(\frac{I}{I_0}\right).$$

$$\begin{aligned}85 &= 10\log\left(\frac{I}{10^{-12}}\right)\\ 8.5 &= \log\left(\frac{I}{10^{-12}}\right)\\ 10^{8.5} &= \frac{I}{10^{-12}} \quad \text{Write in exponential form.}\\ 10^{8.5}\times 10^{-12} &= I\\ 10^{-3.5} &= I\\ 0.000316 &= I\end{aligned}$$

The intensity of the chainsaw is $10^{-3.5}$ or 0.000316 W/m^2.

To determine the intensity of the portable media player, substitute $\beta = 110$ and $I_0 = 10^{-12}$ into

$$\beta = 10\log\left(\frac{I}{I_0}\right).$$

$$\begin{aligned}110 &= 10\log\left(\frac{I}{10^{-12}}\right)\\ 11 &= \log\left(\frac{I}{10^{-12}}\right)\\ 10^{11} &= \frac{I}{10^{-12}} \quad \text{Write in exponential form.}\\ 10^{11}\times 10^{-12} &= I\\ 10^{-1} &= I\\ 0.1 &= I\end{aligned}$$

The intensity of the portable media player is 10^{-1} or 0.1 W/m^2.

If $I_2 = 10^{-1}$ and $I_1 = 10^{-3.5}$, then the ratio of the two intensities is

$$\begin{aligned}\frac{I_2}{I_1} &= \frac{10^{-1}}{10^{-3.5}}\\ &= 10^{2.5}\\ &\approx 316\end{aligned}$$

This means that a maximum volume level of the portable media player is approximately 316 times as intense as the sound of the chainsaw.

b) Let β_2 be the decibel level and I_2 be the intensity of sounds considered safe.

$$\begin{aligned}\beta_2 &= 10\log\left(\frac{I_2}{I_0}\right)\\ \frac{\beta_2}{10} &= \log\left(\frac{I_2}{I_0}\right)\end{aligned}$$

So, $I_2 = I_0\,(10)^{\frac{\beta_2}{10}}$ (equation 1)

Substitute $\beta_1 = 20$ for the decibel level of a whisper with an intensity of I_1.

$$\begin{aligned}\beta_1 &= 10\log\left(\frac{I_1}{I_0}\right)\\ 20 &= 10\log\left(\frac{I_1}{I_0}\right)\\ 2 &= \log\left(\frac{I_1}{I_0}\right)\end{aligned}$$

So, $I_1 = I_0(10^2)$ (equation 2)

Divide equation 1 by equation 2.

$$\begin{aligned}\frac{I_2}{I_1} &= 10^{\frac{\beta_2}{10}-2}\\ 100\,000 &= 10^{\frac{\beta_2}{10}-2}\end{aligned}$$

Then, solve for β_2.

$$\begin{aligned}10^5 &= 10^{\frac{\beta_2}{10}-2}\\ 5 &= \frac{\beta_2}{10}-2\\ 7 &= \frac{\beta_2}{10}\\ 70 &= \beta_2\end{aligned}$$

Meeting Student Needs

- Students can work through the examples in pairs or small groups. Have each group create a set of ten questions involving the three laws of logarithms. Exchange questions with another group. Answer the questions and return them to the original group to have answers checked. Discuss any problems encountered in determining the correct solution.

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Students may forget to apply the power law or apply it incorrectly.
  Example: $\log 3 + 2 \log 4 = 2 \log 12$ or
  $3 + 2 \log 4 = \log 12^2$

R_x Remind students that they can apply the product and quotient laws only on logarithms that have no value in front of them. They must apply the power law *first* before applying the product or quotient law.

Answers

Example 1: Your Turn

a) $\log_6 x - \log_6 y$ **b)** $\frac{\log_5 x + \log_5 y}{2}$ **c)** $\log_3 9 - \frac{2\log_3 x}{3}$

d) $5 \log_7 x + \log_7 y - \frac{\log_7 z}{2}$

Example 2: Your Turn

a) 2.5 **b)** 3 **c)** 2

Example 3: Your Turn

a) $\log_3 x, x > 0$

b) $\log_2\left(\frac{x+3}{x+2}\right), x < -3$ or $x > 3$

Example 4: Your Turn

a) 12 589 times more acidic **b)** 3.10

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• This Your Turn offers students an opportunity to use all of the laws of logarithms. Although they are familiar with similar rules from exponents, they often find radicals and exponents in logarithms difficult to break down. Before assigning the questions, review orally the steps that were used to solve part d) in Example 1. Have students verbalize these steps. Have them complete the questions and compare with a partner. Having students place their work on the board and reviewing it as a class may help clarify students' thinking.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Point out to students that the base in the logarithm is often an indicator of what base needs to be used to break down the expression. • Remind students that they are working backward for parts a) and c). Review the meaning of a subtraction and addition sign in logarithms.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • You may wish to prompt students by asking what the value of $\frac{1}{2}$ would have represented. Remind them that they are again working backward to find a simplified original question. • Some students may need a quick review of factoring the difference of squares.
Example 4 Have students do the Your Turn related to Example 4.	• Have students complete part a) with a partner, or assist students in setting up the equation. Have them complete part b) independently and verify their solution with a partner.

Check Your Understanding

If students are struggling with #1, they will need to review the rules. Remind students that when arguments are multiplied, logarithms are added. When arguments are divided, logarithms are subtracted. This question is asking students to expand the expressions.

In #2, students must write the expression as a single logarithm. Since all the bases are the same, students can simplify by combining logarithms. Students may need to be reminded that they should apply the power law first before attempting to apply any other law.

For #3b), students might not know what to do with the denominator of 2. Remind students that dividing by 2 is the same as multiplying by $\frac{1}{2}$. For part c), remind students to simplify what is inside the brackets and write it as a single logarithm. Then, deal with $\frac{1}{5}$.

In #5, students may be thrown off seeing a logarithm as an exponent. Ask them to simplify the logarithm first and then place its value as the exponent in order to evaluate the original expression. For example, $\log_2 40 - \log_2 5 = \log_2 8$ or 3. So, the original expression becomes 3^3.

If students are struggling with #7, suggest they substitute values for x and y and evaluate each side to determine if the sides are equivalent. They could use a common base of 10 ($c = 10$) for all the questions. The importance of c is that the base is the same for both sides of the equation. This question is a good one to reinforce some of the rules while also providing an opportunity to discuss common misconceptions.

In #10 and 11, discuss with students the possible restrictions they might encounter. For example, the argument of a logarithm must be greater than zero, and division by 0 is not allowed. In #11, discuss the importance of factoring the numerator and denominator fully to check for common factors. Remind students to list restrictions before removing any common factors.

In #12, students should work out each side of the equation to see if the left side equals the right side. The base of c merely means that both sides have a common base. Ask students why it is so important that logarithms have the same base. They should know that the laws of logarithms can be applied only if the base is the same.

Assign #13 if you assign 14. Ask students to determine the sound intensity of 20 dB and then of 10 dB. Is the 20 dB sound twice as loud as the 10 dB sound? How much more intense is it?

In #17, ask students what it means to have a mass ratio of 1.06. What is one possible set of values for m_o and m_f to give a ratio of 1.06? Substitute those values into the rocket equation. Try another set of values for m_o and m_f to give a ratio of 1.06. Substitute those values into the equation. Do you get the same answer?

To answer #20, students need to use the laws of exponents, the laws of logarithms, and the change of base formula. Suggest that students treat the left side and the right side of each identity as separate questions. They should work on one side of the equation and try to make the other side of the equation have the same value.

Meeting Student Needs

- Ask students to recite or write the three laws of logarithms without referring to either the student resource or their notebook.
- Have students determine and compare the equations of an exponential function and a logarithmic function that model the data in the Project Corner at the end of section 8.3.
- Provide **BLM 8–4 Section 8.3 Extra Practice** to students who would benefit from more practice.

Enrichment

Ask students to show by example how laws of logarithms were used by early calculators to perform calculations, and to explain why they are useful in this context.

Gifted

Challenge students to create a flowchart that would allow a student to solve a logarithmic problem by directing that student to select and use the laws of logarithms correctly.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2, 4–8, 10, 11, 12a), b), 13, and 14. Students who have no problems with these questions can go on to the remaining questions.	• Students should be successful in #1 and 2 before moving on. If necessary, complete these questions as a group and then have students complete the remainder independently. • For #6, students may wish to refer to the summary chart they prepared in the previous section of the effects of each parameter on an equation. • Students may find it easier to answer #7 by inserting values for x and y. This allows them to use the technology to verify their thinking. Encourage students to write the values given in #8 in terms of factors of 3 and 5 in order to apply the laws of logarithms. • Questions #10–12 require students to work backward and simplify. Remind students that restrictions are determined before any cancelling occurs. • Suggest that students write the number in #13a) in scientific notation before beginning.
Assessment *as* Learning	
Create Connections Have all students complete C1, C2, and C4.	• Remind students to think of the laws of logarithms when answering C1 in terms of $y = \log x$. • Discuss with students what the addition sign in C2 implies. They may find it helpful to have a copy of the unit circle values available. Note that the logarithm is base 2, which should help students to realize that the product will be a value that can be written with base 2. • Encourage students to use their own examples for C4. They may wish to clone some questions they completed in their assignments. Have them share their questions with a partner.

8.4 Logarithmic and Exponential Equations

***Pre-Calculus 12*, pages 404–415**

Suggested Timing

90–120 min

Materials

- graphing technology

Blackline Masters

BLM 8–5 Section 8.4 Extra Practice

Mathematical Processes for Specific Outcomes

RF7 Demonstrate an understanding of logarithms.

- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Reasoning (R)

RF8 Demonstrate an understanding of the product, quotient and power laws of logarithms.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Technology (T)

RF9 Graph and analyze exponential and logarithmic functions.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Technology (T)
- ✓ Visualization (V)

RF10 Solve problems that involve exponential and logarithmic equations.

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–8, 10–12, 17
Typical	#1–9, two of 11–14, 15, 18, 19, one of 20–22, C1–C5
Extension/Enrichment	#9, 13, 14, 16, 20–22, C2–C5

Planning Notes

Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found in this section.

Ask students to describe the shape of an exponential graph. Sketch an exponential graph and ask students what sort of situations it might represent. Talk about the difference between exponential growth and exponential decay. You might also wish to discuss storage space on devices such as a computer or a portable media player, so students can understand the units of measurement. Ask students to research how big a song on their media player is or how big a picture on their camera is so they have an idea of how large each unit of measurement really is.

Investigate Logarithmic and Exponential Equations

In step 1, ask students what laws they could use to simplify $2 \log x$. Once they have applied the power law, ask students what they know about two logarithms with the same base that are equal. If the logarithms are equal and they have the same base, the arguments must be equal as well. Once they make the arguments equal, ask students why there are two solutions. If students need a visual representation, they could graph the quadratic and see that there are in fact two x-intercepts.

In step 2, if you were to solve the original equation graphically, there are two approaches they could take. How can you solve it using an intersection method? What value would you look for? How could you solve using intercepts? What would you need to do to the equations to use this method?

You might wish to show students an alternative way to complete Sarah's method, beginning in the second line of step 4:

$$
\begin{aligned}
x \log 25 &= \log 125 - \log 25 \\
x \log 25 &= \log (125 \div 25) \\
x \log 25 &= \log 5 \\
x &= \frac{\log 5}{\log 25} \\
&= \frac{\log 5}{\log 5^2} \\
&= \frac{\log 5}{2 \log 5} \\
&= \frac{1}{2}
\end{aligned}
$$

Students could also evaluate the log by plugging $\frac{\log 5}{\log 25}$ into their calculator. This is useful information for logarithms that cannot be simplified with a common argument.

You might discuss with students how Adam's method is only useful if the powers can be written with a common base. Sarah's method, however, can be used for any question in which the exponent is the unknown value. Discuss with students that the whole purpose of logarithms is to solve equations in which the unknown value is the exponent.

Meeting Student Needs

- Ask students to model exponential growth by considering a sum of money that doubles in value each day. Suppose you have 1 penny on the first day of a month. If the sum doubles each day for 30 days, how much money would there be at the end of the period?
- Discuss exponential growth in the context of living organisms. What factors affect population growth?

Answers

Investigate Logarithmic and Exponential Equations

1. a) $\log x^2 = \log 36$
b) It is possible to eliminate the logarithm on both sides of the equation, which leaves you with a quadratic equation.
c) $x = -6, 6$

2. a) Sketch a graph of $y = 2 \log x - \log 36$ and solve for the x-intercepts.
b) $x = 6$

3. $x = 6$; the logarithm of negative numbers is not real

4. Step 1: They both divide both sides by 2.
Step 2: Adam finds a common base of 5 for both sides, while Sarah uses logarithms to lower the exponent.
Step 3: Adam uses laws of exponents to determine a single power for the left side, while Sarah uses laws of logarithms to bring the power in front of the logarithm.
Step 4: Adam solves the resulting linear equation, while Sarah uses laws of logarithms to rearrange the left side of the equation and then solves for x.

5. Adam could have taken the logarithm of both sides and then used laws of logarithms to solve for x. Sarah could have divided by log 25 and then subtracted 1 from both sides.

6. Answers will vary.

7. Adam's method only works for equations in which a common base can be found easily. Sarah's method generally works for all equations.

8. Yes, she could have used any base.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Remind students that in step 1a) applying the laws of logarithms requires them to work backward to determine the original equation. Discuss the answers they obtain for step 3. Discuss whether the two answers are acceptable answers in the logarithmic form. Have students suggest reasons why or why not, but allow them to complete the rest of the investigation before verifying whether two solutions exist. • Some students may find it difficult to find alternative approaches to step 4 in either of the samples, as the processes shown may be the method of choice of the student. Offer students who have found alternative approaches an opportunity to show their work on the board or share it with a partner. • Encourage students to look for a common base when answering step 8.

Example 1

In part a), remind students that if two logarithms are equal and they have the same base, then the arguments must be equal as well.

Students cannot equate the logarithms in part b) because there is another term. The rule only works if there are only logarithms on either side with no other terms. Students have to work to simplify the right side of the equation. Ask students to identify the base of each logarithm. How can they write 1 as a logarithm with base 10? If they are struggling, set up an equation to help them understand: $1 = \log_{10} x$. They can solve the equation by writing it in exponential form or they might recall the rule $\log_c c = 1$. Once they have two logarithms on the right side of the equation, students should be able to simplify them into one logarithm using the product law.

Example 2

In part a), some students may think of Method 2 in a different way. You may wish to share the following method using the inverse property of logarithms:

Both sides of the equation can be expressed with a common base using the inverse property of logarithms, $c^{\log_c x} = x, x > 0$.

$$\begin{aligned} 4^x &= 605 \\ x &= \log_4 605 \\ x &\approx 4.62 \end{aligned}$$

In part b), remind students before they take the logarithm of both sides that there must be a power on one side of the equal sign and a number of the other side. Students must eliminate the 8 on the left side before taking the logarithm of both sides.

In part c), ensure students apply the distributive property and multiply each term of the exponent by the logarithm. Students may be confused to have x on both sides of the equation. Tell them to collect all the terms with x in them on one side of the equation. Since x is a common factor, students can factor the x and then isolate. Students can then use their calculators or use logarithm rules to simplify further. Example:

$$\begin{aligned} x &= \frac{2 \log 3 + \log 4}{2 \log 4 - \log 3} \\ &= \frac{\log 3^2 + \log 4}{\log 4^2 - \log 3} \\ &= \frac{\log 36}{\log \frac{16}{3}} \end{aligned}$$

The benefit of simplifying further is that there is less chance of error entering data into the calculator.

Example 3

You might wish to show students an alternative approach to solving this question:

$$\begin{aligned} 3.6022 \log_{10} 0.78 &= \log_{10} m - 3.444 \\ \log_{10} 0.78^{3.6022} &= \log_{10} m - \log_{10} 2782.2747 \\ \log_{10} 0.4086 &= \log_{10} \frac{m}{2782.2747} \\ 0.4086 &= \frac{m}{2782.2747} \\ 1136.8374 &= m \end{aligned}$$

Example 4

Ask students why the variable in the exponent must be divided by 5730. Why must the percent be changed to a decimal?

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- When the exponent is a binomial and the power law is applied, students may forget to apply the distributive property. Example:

$$\begin{aligned} 4^{2x-1} &= 3 \\ \log (4^{2x-1}) &= \log (3) \\ 2x - \log 4 &= \log 3 \end{aligned}$$

R_x Have students place brackets around the exponent when applying the power law. Example:

$$\begin{aligned} 4^{2x-1} &= 3 \\ \log (4^{2x-1}) &= \log (3) \\ (2x - 1)\log 4 &= \log 3 \\ 2x \log 4 - \log 4 &= \log 3 \end{aligned}$$

Answers

Example 1: Your Turn

a) $x = 3$ **b)** $x = 8$ **c)** $x = -1, 9$

Example 2: Your Turn

a) $x = 11.29$ **b)** $x = 7.62$ **c)** $x = 1.35$

Example 3: Your Turn

1.21 m

Example 4: Your Turn

21 min

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Point out to students that in all the solved examples a common base was sought or the problem was changed to find an equivalent one with a common base. • Ensure students are clear on why they verify their solutions. • Suggest students go through each question and identify any parts of the question that should be written differently before solving.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Point out to students that the base in the logarithm is often an indicator of the base that needs to be used to break down the expression. • Review the meaning of a subtraction sign and an addition sign in logarithms.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • You may wish to prompt students by asking them the known given in the question. They may need assistance in clarifying the meaning for each parameter. • Some students may need a quick review of factoring the difference of squares.
Example 4 Have students do the Your Turn related to Example 4.	• Half-life and doubling are concepts students should be familiar with and have sufficient practice in for further assessments.

Check Your Understanding

For #1, 4, 7, and 8, allow students to choose either an algebraic or graphical approach. Encourage students to develop expertise in both approaches. If they use a graphical approach, encourage them to try the intersect method and the intercept method.

Students may not readily see the mistake in the solution in #3. While the logarithmic expression appears to simplify to the correct numeric value, remind students to check the restrictions of the original logarithms. A value of $x = 5$ gives an original equation of $\log(-3) - \log(-1)$. Ask students if logarithms can have a negative argument. Even though it simplifies to a positive argument eventually, the equation was not defined to begin with.

Ask students to identify any restrictions of the logarithms as an initial place to start #4. Do the possible roots satisfy the restrictions? If so, check them in the original equation to see if they satisfy the equation.

In #5, ask students what laws they will use to satisfy each question. They should have a game plan in mind before they attempt to solve the question.

If students have difficulty identifying the errors in #6, ask them to solve each question and compare their solutions to the students' solutions to determine the error.

Students may struggle in #7d) with what to do with the number 4 in the equation. One possible strategy would be to write 4 as a power of 7 and apply the product law for exponents on the left side of the equation.

In #8, remind students to state the restrictions. They will have logarithmic restrictions as well as restrictions for any rational arguments.

In #11, you might discuss the equation with students and what each part means. What does the 10 000 represent? What does the base 1.035 represent? The first two parts of this question can be answered simply by understanding what the equation means. Students can determine either graphically or algebraically the time it takes the northern pike population to double. What assumptions are being made when determining the time to double? Why is this an approximate value? What other factors need to be considered to come up with a more realistic approximation?

In #13, students may need help determining the correct values of i and n. The value of i is not the annual interest rate but rather the interest rate per compounding period. So, 6% interest compounded semi-annually means that in each compounding period the interest is 3%. Students could check their answers using a financial application on their calculator.

Students may struggle with #15 and need help setting it up. Remind them that the percent should be written as a decimal. Time, in years, will be the unknown variable in the exponent that they will be solving for. Ask them how they take into account the half-life period of 5730 years.

If one logarithm equals another logarithm in value, then the arguments can be equated. Why is this not true for inequalities such as those in #19? Ask students if they can find examples where $\log a < \log b$ and $a < b$. Why is this not always the case? Since the common base is 10, ask students to evaluate each logarithm to see why.

In #20, students have to simplify the left side of the equation and write it as a single logarithm. Once a single logarithm is on the left side and a numeric value is on the right side, students can switch to exponential form to solve for x. In part c), however, there will be a logarithm on either side, so students can merely equate the arguments.

In part a), you might want to remind students to consider the restrictions on the denominator in the exponent, i.e., $\log x \neq 0$.

Meeting Student Needs

- If students are making the same mistakes as the students in #6, you might provide additional instruction to deal with any misconceptions students are still having.
- Have students conduct research and make a brief presentation to the class about one of the topics in #9 to 17. The variety of subjects should allow each student to find something of interest that involves logarithmic equations.
- Provide **BLM 8–5 Section 8.4 Extra Practice** to students who would benefit from more practice.

Enrichment

In a nuclear fission reaction, the difference between a nuclear reactor and a nuclear bomb is a matter of growth and decay rates. Encourage students to speculate on how logarithmic and/or exponential equations might be used to model and track such reactions.

Gifted

Nuclear fusion reactions, or events such as the Big Bang, are characterized by events of decay and then growth in the context of fractions of seconds. Challenge students to research the role of the mathematics of logarithmic and exponential equations in the modelling of such events.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–8, 10–12, and 17. Students who have no problems with these questions can go on to the remaining questions.	• Students should be successful at completing #1 before they go on to any additional questions. These questions can be solved without the use of technology. Encourage students to write them in exponential form to solve. • The reverse is true for #2, where students should write the questions in logarithmic form to solve. Remind them to simplify first. You may wish to complete #2c) with students as a sample, and then have them complete the rest independently. • For #3 and 6, students must identify an error. Encourage students to solve the questions independently before attempting to find potential errors. Have them compare their work to a partner's. • Remind students that once an equation is written in logarithmic form, they are following the basic rules of solving an equation. Students often lose sight of this because the logarithmic notation confuses them into believing that other rules must be used. • Questions 10–12 provide students with the equations. Remind students to identify what each of the variables represents in the problem. This should help them identify the unknown. Listing what is known is a good visual way for students to identify what they are solving for. You may want to assign Essential students #15 so they have exposure to both half-life and doubling times.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3 and C5.	• Students will approach C1 differently and not necessarily the way Fatima started. Encourage them to solve it their own way first and then compare it with their partner's work. Ensure that they are exposed to the process that Fatima used at some point. • C2 and C3 allow students to use formulas they have seen before and solve for missing terms using logarithms. Some will find the use of logarithms a much easier approach. • C5 gives students an opportunity for self-assessment. Encourage them to develop their own examples even if it means they clone an existing question. Encourage them to share with a partner and check each other's work.

Chapter 8 Review and Practice Test

8

***Pre-Calculus 12*, pages 416–420**

Suggested Timing
60–90 min

Materials
- graphing technology

Blackline Masters
BLM 8–2 Section 8.1 Extra Practice
BLM 8–3 Section 8.2 Extra Practice
BLM 8–4 Section 8.3 Extra Practice
BLM 8–5 Section 8.4 Extra Practice
BLM 8–6 Chapter 8 Study Guide
BLM 8–7 Chapter 8 Test

Planning Notes

Have students who are not confident discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test. You may wish to provide students with **BLM 8–6 Chapter 8 Study Guide**, which links the achievement indicators to the questions on the Chapter 8 Practice Test in the form of self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum that will meet the related curriculum outcomes: #1–12.

Meeting Student Needs

- Students should select questions in the review depending on the outcomes that require further study.
- You may wish to have students complete the practice test prior to working through the review questions. Once students mark the practice test, they can focus on completing just the necessary questions in the review
- Students who require more practice on a particular topic may refer to **BLM 8–2 Section 8.1 Extra Practice**, **BLM 8–3 Section 8.2 Extra Practice**, **BLM 8–4 Section 8.3 Extra Practice**, and **BLM 8–5 Section 8.4 Extra Practice**.

Enrichment

- Ask students to create a Venn diagram or mind map that represents visually the understandings they have come to as a result of the work done this chapter.

Gifted

- Ask students to select an example from the natural world and explain how it is that a seemingly almost perfect logarithmic spiral can be created. For example, how did the nautilus that created the shell in the chapter opener create a shell of such mathematical beauty with no understanding of the laws of logarithms?

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 8 Review The Chapter 8 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #1–14, one of 15–17, 18–20	• You may wish to have students work in small groups of similar ability. • Encourage students to use **BLM 8–6 Chapter 8 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 8 Study Guide This master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. The minimum questions suggested are questions that students should be able to confidently answer. Encourage students to try additional questions. • Consider allowing students to use any summative charts, concept maps, or graphic organizers when completing the practice test.
Assessment *of* Learning	
Chapter 8 Test After students complete the practice test, you may wish to use **BLM 8–7 Chapter 8 Test** as a summative assessment. **Minimum:** #1–12	• Before the test, coach students in areas in which they are having difficulty. • You may wish to have students refer to **BLM 8–6 Chapter 8 Study Guide** and identify areas they need reinforcement in before beginning the chapter test.

Unit 3 Project Wrap-Up

***Pre-Calculus 12*, page 421**

Suggested Timing
60–90 min

Blackline Masters
Master 1 Holistic Project Rubric
Master 2 Ana-Holistic Project Rubric
BLM U3–1 Unit 3 Project Checklist

Mathematical Processes
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

General Outcome
Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes
RF2 Demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.
RF3 Demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations.
RF4 Apply translations and stretches to the graphs and equations of functions.
RF7 Demonstrate an understanding of logarithms.
RF8 Demonstrate an understanding of the product, quotient and power laws of logarithms.
RF9 Graph and analyze exponential and logarithmic functions.
RF10 Solve problems that involve exponential and logarithmic equations.

Planning Notes

Ensure students are aware of the Unit 3 Project information provided on page 331, the Project Corners on pages 357 and 403, and the Unit 3 Project Wrap-Up on page 421.

Have students use **BLM U3–1 Unit 3 Project Checklist** to make sure that all parts of their project have been completed. As a class, brainstorm different ways students can do their presentations. You may wish to limit the time each student is allowed to present.

You may wish to work with the class to create a specific rubric for the project using either **Master 1 Holistic Project Rubric** or **Master 2 Ana-Holistic Project Rubric** as a template. Review the general holistic points within the 1–5 scoring levels. Discuss with students how they might achieve each level in the Unit 3 Project. A completed rubric in each style for this project is available at www.mcgrawhill.ca/school/learningcentres by following the links. Note that these are just samples; your class rubric may have more detail.

Ask questions such as the following:
- What are the big ideas in the unit? (For example, one big idea is that a logarithmic function is the inverse of an exponential function and logarithms allow you to solve any exponential equation.)
- Which of the big ideas are involved in the project?
- What part of the project could you complete or get partially correct to indicate that you have a basic understanding of what was learned in the two chapters? (Should you get a pass mark if you can develop a function (or functions) to model the movie's cumulative box office revenues?)
- What would be on a level 1 project? What might you start on correctly? What could be considered a significant start?
- What would be expected for a level 5 project? What should it include? Try to help students realize that a level 5 project may have a minor error or omission that does not affect the final result.
- Knowing the expectations of levels 1, 3, and 5 projects, what would be expected for a level 4? Help students to understand that this is still an honours level and therefore the work should be reflective of this. However, even an honours project may have a minor error or omission. Discuss the difference between a major conceptual error and a minor miscalculation or omission. Understanding this point will help clarify for students the expectations and differences between a pass and an above–average result, and may encourage some students to work toward the highest level. Repeat the process for level 2.

Use the rubric to ensure that students understand the criteria for an acceptable level, as well as what would warrant either an unacceptable or an honours grading.

Assessment	Supporting Learning
Assessment *of* Learning	
Unit 3 Project This unit project gives students an opportunity to apply and demonstrate their knowledge of the following: • demonstrating an understanding of logarithms and exponents • demonstrating an understanding of the product, quotient, and power laws of logarithms • graphing and analysing exponential and logarithmic functions • solving a problem by modelling a situation with an exponential or logarithmic equation Work with students to develop assessment criteria for this project. **Master 1 Holistic Project Rubric** and **Master 2 Ana-Holistic Project Rubric** provide descriptors that will assist you in assessing students' work on the Unit 3 Project.	• You may wish to have students use **BLM U3–1 Unit 3 Project Checklist**, which provides a list of the required components for the Unit 3 Project. • Reviewing the Project Corner boxes at the end of some sections of Chapters 7 and 8 will assist students in developing appropriate data presentations. • Make sure students recognize what is expected for the minimum requirements for an acceptable project as well as the difference between level 5 and level 4. • Clarify the expectations and the scoring with students using **Master 1 Holistic Project Rubric**, **Master 2 Ana-Holistic Project Rubric**, or the rubric you develop as a class. It is recommended that you review the scoring rubric at the beginning of the project, as well as intermittently throughout the project to refresh student understanding of the project assessment.

Cumulative Review and Test

UNIT 3

***Pre-Calculus 12*, pages 422–425**

Suggested Timing
60–90 min

Materials
• graphing technology

Blackline Masters
BLM U3–2 Unit 3 Test

Planning Notes

Have students work independently to complete the review, and then compare their solutions with those of a classmate. Alternatively, you may wish to assign the cumulative review to reinforce the concepts, skills, and processes learned so far. If students encounter difficulties, provide an opportunity for them to share strategies with other students. Encourage them to refer to their notes, and then to the specific section in the student resource. Once they have determined a suitable strategy, have students add it to their notes. Consider having students make a list of questions they found difficult. They can then use the list to help them prepare for the unit test.

Meeting Student Needs

- Have students review the checklist containing the learning outcomes for Unit 3. Students who require more practice on a particular topic may refer to the extra practice blackline master for the relevant chapter and section.
- Encourage students to review their own summary of the key ideas, including examples, presented in each of Chapters 7 and 8.

Assessment	Supporting Learning
Assessment *for* Learning	
Cumulative Review, Chapters 7 and 8 The cumulative review provides an opportunity for students to assess themselves by completing selected questions pertaining to each chapter and checking their answers against the answers in the back of the student resource.	• Have students review their notes from each chapter to identify topics they had problems with, and do the questions related to those topics. Have students do at least one question that tests skills from each chapter. • Have students revisit any chapter section they are having difficulty with. • You may wish to have students review the study guide blackline masters from Chapters 7 and 8 as well as practice tests and chapter tests they completed to help identify any skill areas that still require reinforcement.
Assessment *of* Learning	
Unit 3 Test After students complete the cumulative review, you may wish to use the unit test on pages 424 and 425 as a summative assessment.	• Consider allowing students to use their graphic organizers. • You may wish to have students complete **BLM U3–2 Unit 3 Test**, which provides a sample unit test. You may wish to use it as written or adapt it to meet the needs of your students. The answers to this unit test can be found on **BLM 8–8 Chapter 8 BLM Answers**.