Rational Functions

Opener

***Pre-Calculus 12*, pages 428–429**

Suggested Timing
30–40 min

Blackline Masters
BLM 9–1 Chapter 9 Prerequisite Skills
BLM U4–1 Unit 4 Project Checklist

Planning Notes

Explain to students that they will explore rational functions in this chapter. In section 9.1, they apply their knowledge of transformations to find the domain, range, and asymptotes of rational functions by comparing them to their base functions. In section 9.2, they explore points of discontinuity in the graphs of rational functions as opposed to vertical asymptotes. In section 9.3, students solve rational equations.

You may want to begin the chapter by discussing the images in the unit and chapter openers. The images in the unit opener show two geologists in northwestern British Columbia and biologist Sherri Fownes tracking tagged butterflies in Kananaskis country, in Alberta. The images in the chapter opener are of Pisew Falls in Manitoba, a ski plane in Yukon, and the lighthouse in Stanley Park in Vancouver.

Read the Key Terms with students. Ask them which terms they are already able to define, and have them give an example to illustrate their understanding. Then, have a discussion about what students know about functions and rational expressions. What do they think rational functions might be? Ask students how their understanding of these concepts might relate to the topics discussed in the opener.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter. Encourage students to use a summary method of their choice.

Unit Project

This project could be introduced at the beginning of the chapter. Students can preview the description of the project on page 427 and in the Project Corner on page 456. They can then keep this activity in mind throughout the chapter. Students can choose a topic they feel most comfortable with, or they can choose a topic with which they are less confident. They can use their work on the project as an opportunity to strengthen their understanding and learn how the concept applies to their lives.

Meeting Student Needs

- Provide students with a checklist containing the learning outcomes for this unit. Help them develop a sense of what they need to learn by the end of the chapter/unit.
- Students may wish to graph, separately, the functions given in the opener. For example, graph $f(x) = x^2$, $g(x) = x - 1$, and then $h(x) = \frac{x^2}{x - 1}$. Is there any relationship among the functions?
- Consider having students complete the questions on **BLM 9–1 Chapter 9 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out to students **BLM U4–1 Unit 4 Project Checklist**, which provides a list of all the requirements for the Unit 4 Project.

Enrichment

The chapter introduction describes how rational functions are used to model real-life phenomena, such as focusing light, travel times and conditions, and medicines entering the bloodstream. Challenge students to explain why these examples reflect rational expressions.

Gifted

The mathematics of division involves several unusual aspects that affect rational expressions. Ask students to speculate on what issues division creates in terms of functions, their existence in these circumstances, and why they exist in the first place. If they are having difficulty thinking of "unusual" aspects of division, lead them to consider why zero is the only number that results in the same answer when divided by any other number. Also, why is any number divided by zero undefined? It is best, however, to allow gifted students to arrive at these conclusions on their own.

Career Link

Ask students if they know anyone who is a chartered accountant. Do they know what a chartered accountant does? Many students believe that a chartered accountant is simply a bookkeeper. Many students will be interested to find out how pivotal the role of a chartered accountant can be to a company's business strategy. Their role can be integral to both short-term and long-term decision making. Suggest that students investigate this career further and, if they are interested, find out what kind of prerequisites they would need to pursue this career.

Exploring Rational Functions Using Transformations

9.1

***Pre-Calculus 12*, pages 430–445**

Suggested Timing

90–120 minutes

Materials

- graphing technology

Blackline Masters

BLM 9–2 Section 9.1 Extra Practice

Mathematical Processes for Specific Outcomes

RF14 Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials).

☑ Connections (CN)
☑ Reasoning (R)
☑ Technology (T)
☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–4, 6, 8, 10, 11, 13, 16 or 18
Typical	#1–3, 5–10, 12, 14 or 15, one of 16–18, C1–C3
Extension/Enrichment	#5, 7, 9, 17, 19–22, C1–C3

Planning Notes

There are a few concepts that you may want to revisit with students before beginning this section, including

- domain and range
- restrictions or non-permissible values
- the connection between the algebraic and graphical solutions
- how to find the intercept of functions, both algebraically and graphically
- the connection between intercepts/asymptotes and solutions

You may want to approach this lesson by relating rational functions to transformations.

Investigate Rational Functions

Students may have difficulty visualizing or predicting accurate speeds. You might demonstrate for students how fast each average speed is. Have a student ride a bike for a set distance, say 100 m, marking the start and end points. Students could record the time the rider takes to cover the distance, and then calculate the average speed in km/h. Based on this result, have students try to ride faster/slower and recalculate their speed. This activity will help students identify average speeds that are realistic in the Investigate.

This investigation is a good opportunity to discuss non-permissible values, particularly when students set up the equation to express time as a function of speed. Ask students about implied restrictions, such as speed and time must be greater than zero, and functional restrictions, such as division by zero.

You can discuss non-permissible values again when introducing domain and range. Discuss asymptotes with any student who prefers to find the domain and range algebraically. The vertical asymptotes can be found using the restrictions in the denominator; these values make up the domain.

For students who prefer to find the domain and range graphically, have them graph the functions and see if they can identify values the graph approaches but does not cross or touch. If students are unable to identify these points from the graph, encourage them to look in the table function of their calculator. Suggest that they scroll through the table, paying particular attention to the x-values and y-values. This will help them identify the domain and range.

Meeting Student Needs

- Discuss the outcome and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.
- Students could research the average speed of competitive and recreational cyclists. They might also compare the speed of each type of cyclist to the average speed of a marathon runner.

Common Errors

- Students sometimes forget to equate the denominator to zero to find the non-permissible values. For example, $f(x) = \frac{2}{x - 4}$ has a non-permissible value of $x = 4$, not $x = -4$.

R_x Encourage students to set the denominator to zero and then solve for x.

Answers

Investigate Rational Functions

1. a)

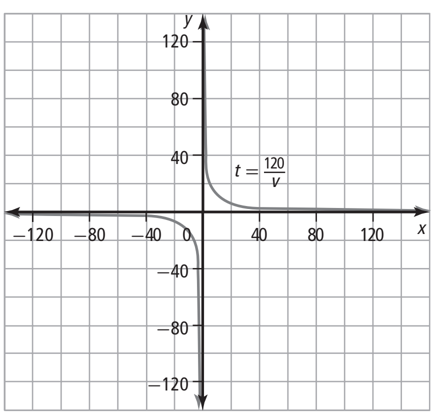
Average Speed (km/h)	Time (*h*)
1	120
2	60
3	40
4	30
5	24
6	20
8	15
10	12
12	10
15	8
20	6
24	5
30	4
40	3

b) average time gets longer; average time gets shorter

2. a) $t = \frac{120}{v}$

b) The zero value is not part of the domain or range. The zero value for the range is an asymptote.

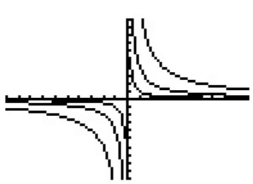
3. a)



b) The graph of the function does not intersect the x-axis or y-axis.
c) time approaches infinity

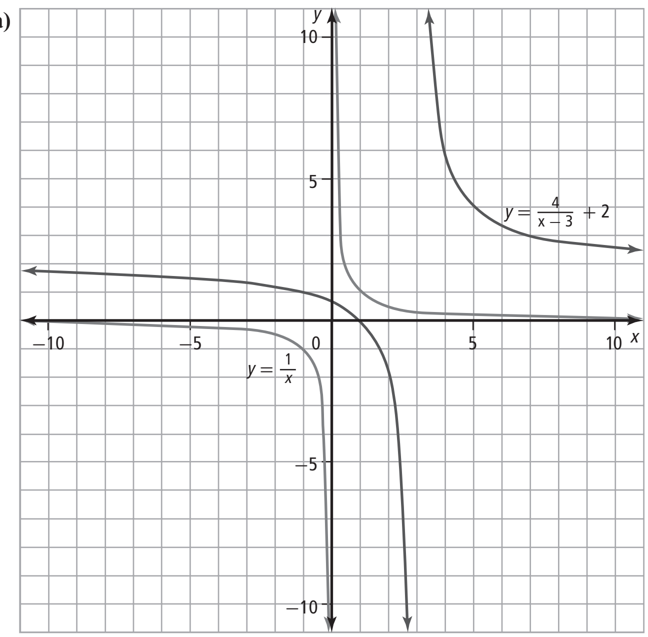
4. a) They are inversely related. The average speed dictates how quickly the graph decreases.
b) No. The domain is all real numbers except for $x = 0$.

5. a)



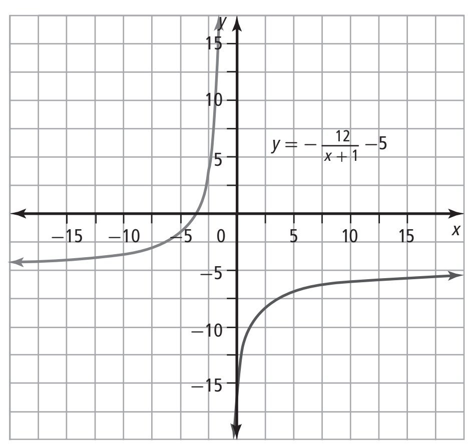
b) y-value becomes increasingly larger, approaching infinity
c) y-value becomes increasingly smaller approaching 0
d) If $a > 1$, then the graph is farther away from the origin. If $a < 1$, the graph is closer to the origin. The same is true for a negative a, but the graph is reflected on the y-axis. If $a = 0$, the graph is undefined.

6. a)



b) The 2 describes the vertical translation, the 3 gives the horizontal translation, and the 4 indicates a vertical stretch by a factor of 4.
c) $y = \frac{4}{x}$; The two graphs are congruent.

7. a) The negative sign indicates that the graph will be reflected on the x-axis. The graph will then be translated 5 units down, translated 1 unit left, and vertically stretched by a factor of 12.



b) $y = \frac{12}{x}$; The graphs have the same vertical stretch factor.

Answers

8. No. Only the translation of the function will change the asymptotes.

9. a) The vertical asymptote shifts the same amount as the function has been translated horizontally.

b) The values of the function approach infinity.

c) $y = \frac{8}{x}$; You could graph $y = \frac{8}{x}$, and then apply the vertical and horizontal translations.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs. • If students are having difficulty with the transformed graph, have them analyse the rational function graphed in step 6 and compare $y = \frac{4}{x-3}$ to $y = \frac{4}{x}$. They can then compare $y = \frac{12}{x+1}$ to $y = \frac{12}{x}$. After doing this, they may find it easier to make the comparison in the Reflect and Respond. • It may help students to verbalize a description of a rational graph.

Example 1

After graphing the function, some students may have difficulty identifying key characteristics of the graph. Guide their understanding by asking the following questions to help them identify key characteristics:

- What happens to the graph as x increases from 0?
- What happens to the graph as x decreases from 0?
- What happens to the graph when $x = 0$?
- What happens to the graph as y increases from 0?
- What happens to the graph as y decreases from 0?
- What happens to the graph when $y = 0$?

Ask the same questions for the Your Turn question if students are having difficulty. Ask what difference there is between having a numerator of 10 in the example, versus 6 in the Your Turn question.

Example 2

Suggest students use a table to organize the results of transformations and their effect on domain and range. Consider asking students the effects on domain and range, one transformation at a time.

	$f(x) = \frac{1}{x}$	$f(x) = \frac{6}{x}$	$f(x) = \frac{6}{x-2}$	$f(x) = \frac{6}{x-2} - 3$
Vertical Asymptotes				
Horizontal Asymptotes				

Discuss the asymptotes and plot them on the graph, which will split the graph into sections. Try to lead students to draw a good sketch of the function before entering the function into graphing technology. Discuss how making the denominator larger makes the range smaller.

For the Your Turn question, encourage students to sketch the shape of the graph and determine the asymptotes before using their graphing calculator. They should understand the graph and its important parts, and then use technology to verify their predictions.

Example 3

Discuss how to find the y-intercept algebraically, and why x is made to equal zero. Ask

- To find the x-intercept, why is the y-value zero?
- Why is only the numerator considered?
- Algebraically, the vertical asymptote can be found simply by equating the denominator to zero and solving for x. Why is only the denominator considered?
- Point out to students that they can determine the horizontal asymptote by isolating x in the equation and then identifying the restrictions on y.

  Rearranging the equation gives $x = \dfrac{5 - 2y}{4 - y}$.

  The denominator cannot be zero. So, $y = 4$ is the equation of the horizontal asymptote.

Encourage students to manipulate the expression on the right using long division by a binomial. You may want to work through the steps together:

$$\begin{array}{r} 4 \phantom{{}-5} \\ x - 2 \overline{)\,4x - 5} \\ \underline{4x - 8} \\ 3 \end{array}$$

The quotient is 4 and the remainder is 3, which are confirmed by the form in the student resource,

$$y = \frac{3}{x - 2} + 4.$$

Ask students how they could use a graphing calculator to find the vertical and horizontal asymptotes. Ask

- Can you identify them from the graph?
- How can you check if they are correct?
- How can you calculate the y-intercept?
- How can you calculate the x-intercepts?

Example 4

Some students may struggle to enter the appropriate window settings when working with their graphing calculator. Ensure that they do not use "zoom fit," which may confuse them about which part of the graph is shown. One method you might suggest is for students to use estimation skills to determine where the x-intercepts and y-intercepts are located, and then adjust the window settings accordingly.

Once students have factored the denominators fully, ask them what transformations occurred to produce the new function. Function notation is confusing to some students, so it might help to discuss the changes first.

Example 5

Before working on this example, you may want to discuss with students different types of restrictions. Ask

- What are the given restrictions? (e.g., You cannot have a negative number of minutes.)
- What are the function restrictions? (e.g., x cannot be zero.)

You might also ask students to identify the important parts of the graph: intercepts, asymptotes, and point of intersection. Ask students to explain the meaning of these characteristics in the context of the original question.

Meeting Student Needs

- When creating a table of values, encourage students to use the form $xy = a$.
- Students could create individual Key Ideas notes, including points from the student resource. For example, the vertical asymptote is at $x = h$ and the horizontal asymptote is at $y = k$.
- In Example 2, work through the table as a class. Instruct students to substitute $x = 2$ or $y = -3$ into the equation to investigate why those values cannot be included in the domain or range.
- For Example 3, have students enter the function into their graphing calculator. If the graph produced is not the same as illustrated in the student resource, ask students to determine the mistake (which is usually entering the equation incorrectly). Ensure that students also work through the pencil and paper method because it can lead to a greater understanding of the function. In particular, work through the manipulation of the equation to arrive at the form $y = \frac{a}{(x - h)} + k$.
- For Example 5, students could research the costs of a cell phone with various providers, and then create an equation to represent the costs.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Enrichment

Ask students to create several rational functions that use transformations based on speed and time scenarios. Challenge students to make these functions interesting in terms of real-life situations involving motion. Then, challenge students to describe an experiment that reflects their function. For example, a scenario could involve the motion of toy cars along a track, with time measured using a stopwatch. Students could create a physical model of their speed–time function and run the experiment. In a sense, this is mathematical reverse engineering—creating a function and then building a physical representation of it.

Gifted

Einstein discovered a relationship between the speed of an object and time. Put simply, the closer an object travels to the speed of light, the slower time goes (time dilation). Challenge students to consider how time dilation affects speed and time calculations for real-life objects.

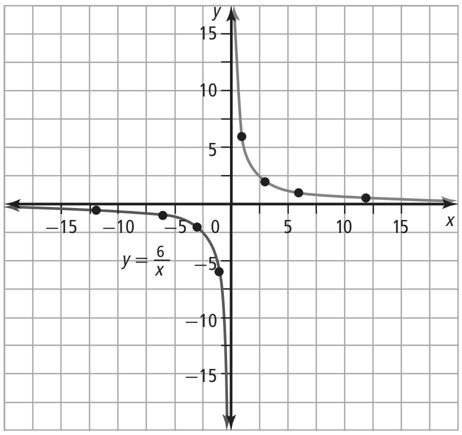
Common Errors

- Some students may struggle with the effects of vertical and horizontal translations on domain and range. For example, when considering $f(x) = \frac{1}{x} + 4$ and $f(x) = \frac{1}{x + 4}$ in relation to $f(x) = \frac{1}{x}$, one function affects the domain and the other function affects the range.

R_x Suggest that students find the domain and range of the original function and then graph the two other functions to see how the domain and range are affected.

Answers

Example 1: Your Turn

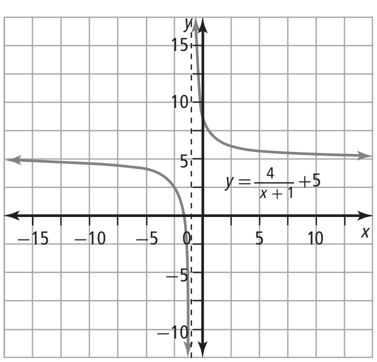


x	y
−60	−0.1
−12	−0.5
−6	−1
−3	−2
−1	−6
0	Undefined
1	6
3	2
6	1
12	0.5
60	0.1

Characteristic	$y = \frac{6}{x}$
Non-permissible value	$x = 0$
Behaviour near non-permissible value	As x approaches 0, $\lvert y \rvert$ becomes very large.
End behaviour	As $\lvert x \rvert$ gets larger, y approaches 0.
Domain	$\{x \mid x \neq 0, x \in \mathrm{R}\}$
Range	$\{y \mid y \neq 0, y \in \mathrm{R}\}$
Equation of vertical asymptote	$x = 0$
Equation of horizontal asymptote	$y = 0$

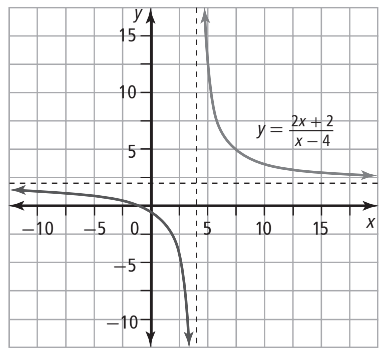
Example 2: Your Turn

To obtain a graph of $y = \frac{4}{x + 1} + 5$ from the graph of $y = \frac{1}{x}$, apply a vertical stretch by a factor of 4, and translate the graph 1 unit left and 5 units up.



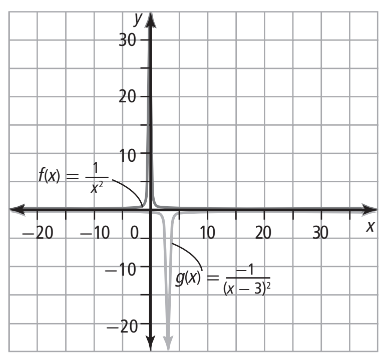
Characteristic	$y = \frac{4}{x + 1} + 5$
Non-permissible value	$x = -1$
Behaviour near non-permissible value	As x approaches −1, $\lvert y \rvert$ becomes very large.
End behaviour	As $\lvert x \rvert$ gets larger, y approaches 5.
Domain	$\{x \mid x \neq -1, x \in \mathrm{R}\}$
Range	$\{y \mid y \neq 5, y \in \mathrm{R}\}$
Equation of vertical asymptote	$x = -1$
Equation of horizontal asymptote	$y = 5$

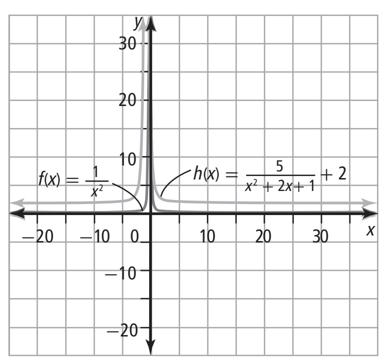
Example 3: Your Turn



The asymptotes are $x = 4$ and $y = 2$. The x-intercept is $x = -1$ and the y-intercept is $y = -0.5$.

Example 4: Your Turn





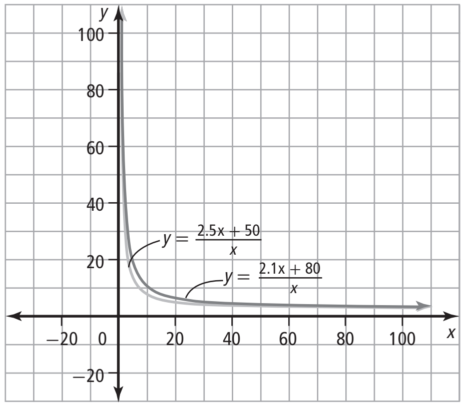
All of the graphs share the same general shape. The first and second functions are congruent, but the second is reflected and translated. The third is vertically stretched and translated.

Answers

Example 5: Your Turn

a) company A: $y = \frac{2.5x + 50}{x}$; company B: $y = \frac{2.1x + 80}{x}$

b)



c) The graph shows that the price per booklet is much higher if you buy small quantities, but gets progressively cheaper the more you buy.

d) For more than 75 booklets, Marlysse should buy from the second company. For fewer she should buy from the first.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Suggest that students use a table, as shown in the example, to help organize their thinking.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Suggest that students set up a chart similar to the one in the example. • Have students identify the values of h and k, and identify where these values are represented in the chart. • Ask students to describe verbally the effect of one of the parameters. Then, have them identify the effect of the second parameter on their own, before checking answers as a class.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • For students having difficulty with the concepts in this example, you may wish to review what an asymptote is and how a vertical or horizontal asymptote is written. • If students wish to solve the problem algebraically, help them get started by discussing how to change the function to the form $y = \frac{a}{x - h} + k$.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Suggest that students set up a chart to organize and present their thinking. • Assist students with factoring the denominator of $h(x)$. Then, help them use the chart of this function's characteristics to help with the characteristics of $g(x)$.
Example 5 Have students do the Your Turn related to Example 5.	• If students are having difficulty, assist them to set up the equation as a class. Then, students can use their skills to identify the characteristics of the graphs. • Clarify with students what they are looking for to determine which choice is better. Discuss this response as a class. • Refer students to Examples 2 and 3 for help determining the asymptotes from the function equation.

Check Your Understanding

For #1 to 3, encourage students to use their knowledge of transformations to describe what the graph looks like compared to the base function. Ask them to draw a sketch of the graph based on this understanding. They can then use their graphing calculator to check their sketch.

For #7, ask students if they can identify the base function on which each graph is based. Once they have identified the base function, ask them to describe the transformations in their own words, and to identify which parameter each one affects.

Ask students what they should do with each point in #8. They should see that by substituting the coordinate into the original equation for x and y, they will end up with two equations in two unknowns. Ask students if they recall the method for solving a system of equations in two unknowns. Students could use an algebraic approach or a graphical approach to find the values of a and k.

In #9, ask students "What can you determine about a function if its graph has a vertical asymptote at $x = 2$?" Once students have identified that this affects the root of the denominator, they can focus on the other asymptote. What do they know about a function with a horizontal asymptote of $y = -3$? You may need to help students recognize the type of function with this particular asymptote.

For students who struggle to find the error in #10, encourage them to solve the equation themselves and then compare their solution to Mira's.

In #11, ask students what they think the purpose is for writing the function in this form. What information do they know about the graph of the function in this new form that they did not know in the original form?

Ask students how to determine the intercepts of the function algebraically in #12. Why must $y = 0$ to find the x-intercept(s)? Most students prefer to find the intercepts graphically. Students should understand the connection between the x-intercept of the function and the root of the numerator. Consider having "calculation races" in which one side of the class finds the intercepts on the calculator and the other side finds the intercepts algebraically. Students should see the speed benefit of calculating the equations algebraically. Ask

- Why do you only need to focus on the numerator of the function?
- How do you determine the vertical asymptotes of the function? Why do you only need to focus on the denominator of the function?
- How do you determine the horizontal asymptotes from a graphing calculator?
- How can a table of values help you approximate the asymptote?

Horizontal asymptotes are the most difficult ones to identify, and thus students will need the most direction with them.

In #13, help students understand that there are two types of restrictions in this question: implied and functional. Ask students to identify the variables in the equation. What are the implied restrictions on these variables in the context of the question?

- What value cannot represent the number of buyers, N?
- The average price, p, cannot be what value?

Then, ask students to look at the equation. Without thinking about what each variable represents, what restrictions does the function have? What does this mean in relation to the question?

For #14, ask students what the smallest and largest possible values are for the length and width of the rectangle. This should help students identify the restrictions on each variable. Discuss the types of relationships that are possible between two variables. If two variables are inversely proportionate, then as the value of one variable increases the other decreases. If two variables are directly proportionate, as the value of one variable increases or decreases, the other does the same. What type of relationship do the length and width of the rectangle have?

For #16, as a form of enrichment or class discussion, ask students if there are other factors to consider when buying a freezer. Is it better to spend more money up front to reduce the monthly cost? You could also discuss the concept of a break-even point. Finding this point will allow students to determine when the cost of the more expensive freezer (with the lower monthly costs) will equal the money invested in the less expensive freezer (with the higher energy costs). Discuss the timeframe before and after the break-even point, and how this might help students in their purchasing decisions.

Students find the break-even point in #18 as well. Based on this point, students can determine which store is less expensive up to the break-even point, and which store is less expensive after the break-even point. Ask students how this information might factor into a decision on which store to rent the bike from.

For #19, ask students to identify the variables in this situation and what implied restrictions there are. What are the restrictions based solely on the function? How do these restrictions relate to the domain and range of the function?

In #21, students often get confused finding the inverses because there are two x-values. Ask students to interchange x and y, and they will then have two y-values. Once they have eliminated the fraction by multiplying each term by the lowest common multiple (LCM), they can then collect all the terms with y in them on one side of the equation. Since both terms have y in them, students can factor out y, and then divide both sides of the equation by the remaining factor. Once they have found the inverse, ask them how they can check whether their inverse is correct. Students could graph the original function and the inverse function and check whether each graph is the mirror image of the other about the line $y = x$.

For #22, ask students how they can determine the vertical and horizontal asymptotes algebraically. Would it help to simplify the expression algebraically before graphing it? Ask students to simplify the equation by adding the fractions together and then identifying the asymptotes. Can they also identify the intercepts?

Meeting Student Needs

- Students could prepare a review lesson outlining the Key Ideas with examples. They could then make a presentation to the class or submit it for evaluation. The examples should represent the concepts and learning indicators presented in the section.
- Suggest that students create a generic template to indicate characteristics of the graphs. This may be particularly helpful for #3 to 6.
- Discuss #17 with the physics teacher, or consider having him/her present this question to the class. The physics teacher may want to explore more of these types of questions in the physics class while students are also working with them in math class.
- Provide **BLM 9–2 Section 9.1 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–4, 6, 8, 10, 11, 13, and 16 or 18. Students who have no problems with these questions can go on to the remaining questions.	• Have students work in pairs to discuss their reasons for their choices in #1. They can verify their answers with technology. • Students may benefit from using a chart similar to those used in the examples to complete #2, 3, 4, and 6. Doing so may assist in focusing their attention on each parameter in a transformation. If you plan to assign #7, check for student understanding of the previous questions or have students discuss their results with a partner. • Again, students may find it easier to set up a chart and verbalize what they see in each graph for #7. They can then complete the chart to find the values of a, h, and k. • Some students may benefit from a review of systems of equations and methods that can be used to solve them. Have them identify their method of choice. • Ask students to identify what the method in #10 is trying to achieve before having them look for the error. Taking this approach will assist students in solving #11. • Encourage students to identify methods they could use to solve #13 (graphing, table of values) before beginning.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C2.	• C1 gives students an opportunity to reflect on their own thinking about transformations and how they have applied them to rational expressions. If some students are still having difficulties with transformations, it would be helpful to review the areas that are causing difficulties before moving on to the next section. • C2 provides an opportunity to apply all the concepts learned in the section to a real-world scenario. It also allows students to identify any areas of weakness that they might have.

Analysing Rational Functions

9.2

***Pre-Calculus 12*, pages 446–456**

Suggested Timing
60–90 min

Materials
- graphing technology

Blackline Masters
BLM 9–3 Section 9.2 Extra Practice

Mathematical Processes for Specific Outcomes
RF14 Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials).
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2a), b), 3, 4a), b), 5, 6a), b), 7, one of 13–15
Typical	#1, 2c), d), 3, 4c), d), 6–15, one of 16–19, C1–C3
Extension/Enrichment	#8, 10, 11, 16–23, C1–C3

Planning Notes

Begin the section by having a discussion about the opening paragraph and accompanying image. Discuss how the speed of an airplane is affected by the wind. Ask, "What is the difference between a head wind and tail wind? Which slows the speed of the plane down? Which helps the plane travel faster?"

Then, before moving on, talk to the class about restrictions and the importance of finding restrictions before simplifying rational functions. You should also have a discussion about points of discontinuity in a graph, and the difference between a point of discontinuity and an asymptote in a rational function. Ask students if it is possible to have both a point of discontinuity in the graph and an asymptote. Help students make the connection that when a graph has a point of discontinuity in it, the numerator and denominator of the original function have a common factor.

Investigate Analysing Rational Functions

Ask students to predict the graph of the function before entering it into the calculator. When they do graph the function, ask
- How is the graph of this function different than the graphs of the functions in the last section?
- What type of function does the graph represent?
- How is it possible that a rational function produces the graph of a linear function?
- Look at the table of values for this function. Is there a value of x that produces an error? Is this value shown as being an error when you look at the graph?
- Why do you think the graph looks as though it is continuous? Why does the graphing calculator not show the point of discontinuity in the graph?
- How can you recognize whether a function will have a point of discontinuity in the graph?

If students are having difficulty with the restrictions, ask
- Once you have found the restriction for the x-value, how do you find the restrictions for the y-value?
- Why does this function not have asymptotes?
- Why is it important to fully factor the numerator and denominator?
- Why should restrictions be stated before the function is simplified?

Meeting Student Needs

- Discuss the outcome(s) and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.
- Students may wish to research more about airspeed and groundspeed. You might consider having a pilot of a small plane make a short presentation about these terms, as well as other terms relative to the occupation, including how to get a pilot's licence.
- Students can be encouraged to simplify the rational expression to determine the equation.

Common Errors

- Some students may miss the point of discontinuity in the graph because it does not show up on their graphing calculator.

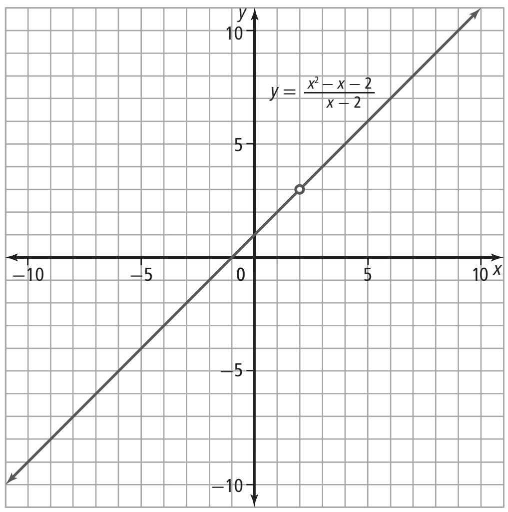
R_x Discuss with students that rational functions may have a vertical asymptote and/or a point of discontinuity in the graph. Ask them to attempt to find or create a rational function that does not have an asymptote or point of discontinuity. For example, $y = \frac{1}{x^2 + 1}$.

Answers

Investigate: Analysing Rational Functions

1. a) When $x = 2$ there is division by 0, so there is no value for $x = 2$.

b)



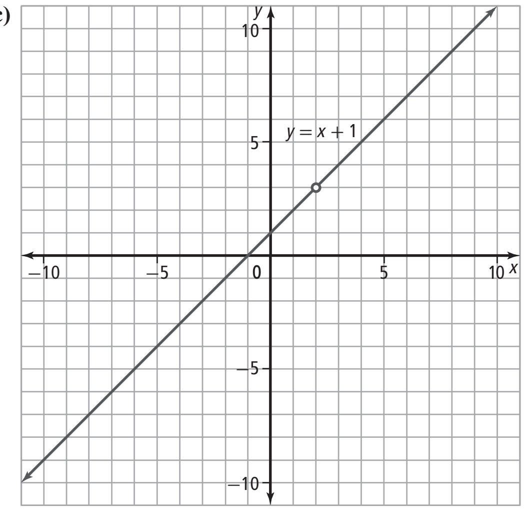
x	0	1	2	3	4	5	6
y	1	2	does not exist	4	5	6	7

c) Example: I would expect that the function would look like a rational function, but since the x-values and the y-values, increase by the same value for all values other than $x = 2$, the shape of the function is linear.

2. a) domain $\{x \mid x \neq 2, x \in \mathrm{R}\}$

b) divide the numerator and denominator by $x - 2$; $y = x + 1$

c)



x	0	1	2	3	4	5	6
y	1	2	does not exist	4	5	6	7

The graph and table of values match the ones for the original function.

d) Focusing on the discontinuity at (2, 3) would be a good way to show the differences.

3. a) With the graph of the earlier rational function, the graph got nearer and nearer to the non-permissible value without reaching it. In this example, there is no indication that the function is approaching a discontinuity.

b) The rational part of the function cancels out in all cases except when $x = 2$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Some students may need assistance in factoring the quadratic or identifying that factoring may need to be a step in the process. Use discussion and questioning to help lead students to this realization. • Discuss #3b) as a class to assist students in identifying whether their thinking will lead to a correct approach for future questions.

Example 1

Ensure that students understand that they must state the restrictions before they cancel like terms in the numerator and denominator. Ask them why this is important. Then, ask students to describe how they know if there is a vertical asymptote or a point of discontinuity in the graph. Ask, "Once you know what x cannot equal, how do you find what y cannot equal?"

Example 2

Some students may still be confused about how to distinguish between a point of discontinuity and an asymptote in the graph of a function. Ask students how they can tell by just looking at a rational function if there will be a point of discontinuity in its graph. Help them recognize that if the numerator and denominator of a rational function have common factors, there will always be a point of discontinuity in the graph. If the numerator and denominator of a rational function do not have common factors, there will be a vertical asymptote. Ensure that they recognize that a function may have either a point of discontinuity and/or an asymptote.

Example 3

Make sure students understand that they need to fully factor both the numerator and the denominator, and why this is done. You may need to help some students with factoring. Help them recognize that there will be a point of discontinuity in the graph whenever there is a common factor in the numerator and denominator, but that this does not eliminate the possibility of a vertical asymptote. Ask, "What are the key parts of a graph that can help you identify the matching function?"

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Suggest that students write a short explanation of the steps to follow when evaluating a rational function. Their description should include an explanation of when there will be a vertical asymptote and when there will be a point of discontinuity.
- Have students work in pairs. One partner can present Example 1 to the second partner, and then they can reverse roles for Example 2. Students can work together on Example 3, applying what they learned in the first two examples.
- Students may wish to write the second point of the Key Ideas onto an index card to use for reference while working through the Check Your Understanding questions.

Enrichment

Encourage students to create a graph of a function that represents a real-world situation, and has either a point of discontinuity or an asymptote. Have them speculate about what is happening in the real-world scenario on the "margins" of an asymptote or a point of discontinuity (that is, as the graph approaches the non-permissible value(s)).

Gifted

Challenge students to work with a partner to create rational functions whose graphs have either an asymptote or point of discontinuity that are mathematically interesting to determine. For example, they might consider some trig functions as they approach zero.

Common Errors

- Students may miss the point of discontinuity in the graph by not stating the restrictions before simplifying the function.

R$_x$ Suggest that students get in the habit of listing any restrictions as soon as they factor the numerator and denominator. Present students with the following example, and tell them that they should fill in the restriction in step 2. Tell them that they should get in the habit of writing the question out in this form.

$$f(x) = \frac{x^2 - 5x + 6}{x - 3}$$

$$f(x) = \frac{(x - 3)(x - 2)}{x - 3}, x \neq 3$$

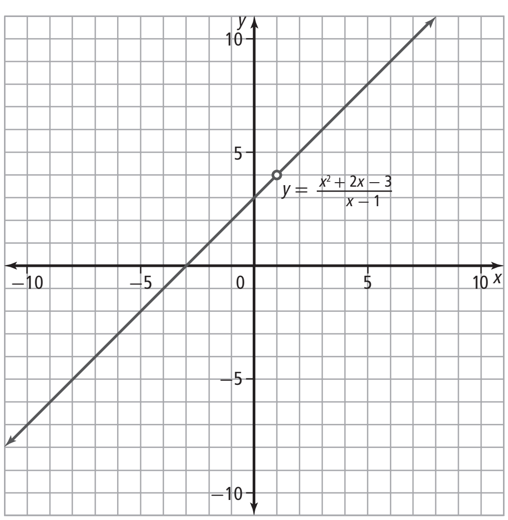
$$f(x) = x - 2$$

If $x \neq 3$, then $y \neq 1$ (by substitution, $3 - 2 = 1$). Therefore, there is a point of discontinuity in the graph at (3, 1).

Answers

Example 1: Your Turn

The function is linear, but has a discontinuity at (1, 4).



Example 2: Your Turn

The first function is a rational function. In the second function, the numerator and denominator have a common factor of $(x + 3)$. The common factor can be divided in the numerator and denominator, leaving a linear function with a point of discontinuity.

Example 3: Your Turn

Graph 1 is function $L(x)$: The numerator and denominator have a common factor of $(x - 1)$, which results in a point of discontinuity at $(1, \frac{1}{2})$.
$(x + 1)$ is a factor of the denominator and it accounts for the asymptote at $x = -1$.
Graph 2 is function $M(x)$: There is a common factor of $(3 - x)$, which results in a point of discontinuity at $(3, -1)$.
Graph 3 is function $K(x)$: There are no common factors, and the denominator factors to $(x - 2)(x + 1)$, meaning that there are vertical asymptotes at $x = -1$ and $x = 2$.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • You may want to review the meaning of *undefined* with some students. • If students are having difficulty factoring, assist them in setting up their factors. Have them verbalize the common factor before proceeding. Ask students whether they need to be concerned about the factor that has been cancelled.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Assist students who need help with factoring with the first rational expression. Suggest they set up a chart to help them identify and explain the characteristics. Check for understanding by having them complete the second rational expression independently, and then comparing their results with a partner.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Encourage students to choose the method (algebraic or graphing) that they feel most comfortable with to solve the question initially. Have them verify using the other approach to ensure they are able to solve in multiple approaches, and they can make the link between algebra and graphing.

Check Your Understanding

For #2, ask students how they would decide which points to include in their table of values. Ask

- How can a table of values show that there is an asymptote?
- What does it mean to have points converge on a value?
- What does it mean if points are divergent?
- How does a table of values show that there might be a point of discontinuity in the graph?

For #3, some students may need help to develop a strategy to analyse the characteristics of the graphs of the functions. Have a discussion about characteristics, such as intercepts, asymptotes, end behaviour of the arms, and domain and range. Ask them to identify ways in which the graphs reflect each of these characteristics. How are the functions the same? How are they different?

In #7, students need to identify which points are necessary to determine the equation of the function. Ask students what they know about the function if its graph has a point of discontinuity in it. What do they know about it if there is an asymptote? Ask

- What do the x-intercepts tell you about the graph? What do the y-intercepts tell you?
- What is the meaning of the point of discontinuity in the graph?
- Is there more than one possible equation for each graph?

Encourage students to compare their equations with those of other students to see how many possible equations they come up with.

For #8, ask students what part of the function they can determine from the vertical asymptote. Discuss with them what they know about the numerator and denominator of the function if there is a point of discontinuity in the graph. Help them understand that a point of discontinuity in the graph means that the numerator and denominator share a common factor. Then discuss whether there is more than one possible rational function for each set of characteristics. Ask students how they can check the reasonableness of their answers. Have them compare answers with another classmate.

For #10, ask students what type of equation produces a horizontal line. Is this the simplified equation or the original equation? If the equation simplifies to a horizontal line, students should recognize that this means that the numerator and denominator have common factors and only the constant is left. They should recognize that if there are two points of discontinuity in the graph, then there are two common factors in the numerator and denominator.

For #12, you may want to have a brief discussion about small planes and the effects of the wind. Discuss with students why a tail wind is represented by $+w$, and a head wind is represented by $-w$. Ask students how they would determine the domain for this function. How would they account for the domain in the graph? What range of values would be reasonable to show speed of the wind?

Ask students what values are possible for p in #14, and what values are not possible. They should recognize that p cannot be negative and that the highest value p could be is 100. Once they have determined the highest and lowest values for p, they can determine the range of values for the cost, C. Knowing these values gives students an idea of possible window settings for p and C when working with their calculators.

For #16, ask students what parts of the graph are needed to determine the equation. They should recognize that the two vertical asymptotes will be the two factors in the denominator of the function. Ask, "What do the x-intercepts tell you about the function?" The x-intercepts help students find the factors of the numerator.

If students are struggling with #18, encourage them to substitute values for a, b, and c to see if they can generalize a pattern.

To start #20, students could verbally describe the transformation. Ask what effect each transformation has on the original function. If students are struggling, encourage them to work one step at a time. Once they have described the transformation, they could match each part of their description with a parameter in the equation.

For #23a), ask students if there is something they can do to the function to make it easier to work with. They should realize that they need to factor the denominator of the first term and then add the two using a common denominator. Once they have done that, ask them what the asymptotes are. How do they determine if there is a point of discontinuity in the graph? Could they determine these points from the original equation before they simplified it? Then, for part b), ask students why they need to factor the numerator and denominator to answer this question. After factoring the numerator and denominator, what do they notice? Will there be an asymptote? a point of discontinuity?

When discussing C2 with students, ask them if they can provide an example that would support the statement, or provide a counterexample that would disprove the statement. Also, ask them if all rational functions are polynomials, and ask them to provide proof for their response.

Meeting Student Needs

- If students wrote out the Key Ideas on an index card as suggested earlier in this section, they might use this card to assist them with #2.
- Students may choose to use graphing calculators to assist them with #5 and 6. The visual representation may help students develop an understanding of the concepts presented in the section.
- Once again, if you were able to have a pilot come to the class, students could connect the concepts of piloting a small plane to #12. A discussion about various types of small planes could possibly be part of the presentation.
- Discuss #15 and 17 with a physics teacher to determine whether the concept can be taught concurrently in both classes. Students would benefit from the two perspectives.
- Provide **BLM 9–3 Section 9.2 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2a), b), 3, 4a), b), 5, 6a), b), 7, and 13, 14 or 15. Students who have no problems with these questions can go on to the remaining questions.	• Discuss with students what a non-permissible value is and how it appears on a graph. Ask students to identify where they would look for the non-permissible value and why knowing the factors of the quadratic would be necessary. Having this discussion will help them in solving #1–6. • Some students may benefit from revisiting the meaning of an open circle and how to take a value from the graph and write it as a factor. Having this understanding may help students in solving #7 and 8. Encourage students to sketch a graph of the information for #8. • For #13, you may wish to suggest that students work with a partner or compare their equation with that of a partner and discuss the differences.
Assessment *as* Learning	
Create Connections Have all students complete C1 and C3.	• Students might benefit in working with a partner for C1. Suggest they generate three or four examples as a team. Have partners share their examples on the board and discuss their thinking with the class. Have students offer explanations of C1b) before having them write their own response in their organizer. • Students who are having difficulty with C3 may refer back to the examples and generate similar models of their own for graphing. C3 offers students a way to test their own understanding of the variety of ways in which rational expressions can be expressed. Have them sketch samples in their graphic organizer for future reference.

9.3 Connecting Graphs and Rational Equations

***Pre-Calculus 12*, pages 457–467**

Suggested Timing
90–120 min

Materials
- graphing technology

Blackline Masters
BLM 9–4 Section 9.3 Extra Practice

Mathematical Processes for Specific Outcomes
RF14 Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials).

- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2, 3a), b), 4a), b), 5, 7, 11
Typical	#1, 2, 3c), d), 4c), d), 5–7, 8 or 9, 10, 12 or 13, C1–C3
Extension/Enrichment	#6, 8 or 9, 10, 12 or 13, 14–17, C1–C3

Planning Notes

Before beginning the section, discuss the opener. Ask students if they have ever considered how medicine dosages work considering that they do not leave the system "all at once," but rather diminish gradually. How do doctors determine the dosage needed to fight off infections? How does a doctor know how much of a drug is in your system at any one time? Does medication ever fully leave your system?

Before students begin their work on the Investigate, take the opportunity to reactivate and strengthen their understanding of solving equations. Ask

- What does it mean to solve an equation algebraically?
- What does it mean to solve an equation graphically?
- What is the connection between an algebraic and a graphical solution?

In this discussion, stress the importance of stating restrictions and checking the solution(s) against those restrictions.

You may also want to revisit how to solve systems of equations. Students might use algebraic approaches, such as substitution or elimination, and/or graphical approaches, such as calculating the point of intersection or finding x-intercepts.

Investigate Solving Rational Equations

Some students may be confused about what is meant by "a reasonable domain." Have a class discussion about what considerations might define reasonableness. Ask

- Based on the function, what values can t not equal?
- Based on the context of the question, what values can t not equal?

After students have graphed the function, ask them to verbalize some of the characteristics of the graph. Ask

- Where is the graph increasing or decreasing? What does this mean?
- Will the graph ever cross the x-axis? Will it ever touch the x-axis?
- How can you algebraically determine when the blood concentration reaches 10 mg/dL? How can you determine this graphically?

You may need to remind students about why they must state restrictions before attempting a solution. You may also need to lead them through the solving process, for both an algebraic and graphical approach. Ask

- How do you eliminate a fraction?
- When solving a quadratic equation algebraically, why must the equation be equated to zero?
- How many solutions are there?
- How could you solve this equation graphically using systems of equations?
- How could you solve this equation graphically using x-intercepts?
- Which solving method do you prefer? Why?

Meeting Student Needs

- Discuss the outcome(s) and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found in this section.

Enrichment

In reference to the Did You Know? information on bioavailability on page 457, ask students to list factors that might create a rational equation that produces a graph that is inverse in nature. For example, perhaps the amount of food consumed with a medicine is inversely proportional to the amount of medicine absorbed in the bloodstream. Ask students to comment on the implications to pharmacists advising patients on such medicines. How might a pharmacist use a graph of the function to explain to patients how the amount of food consumed affects bioavailability?

Students who are interested in pursuing a career in the medical field may wish to further research bioavailability and the factors that may influence the administration of medicines. Other students may want to research other real-world applications involving medicine. For example, they may wish to research a particular antibiotic or other medicine and its uses. Or, they might research what immunization(s) are required for a volunteer mission to Mexico or another part of the world to assist with a community development project there? What considerations must be taken into account? Students could then present their findings to the class.

Gifted

Encourage students to reflect on how to determine approximate solutions to rational equations without the use of technology. Have them consider Newton's method for approximating square roots. They could compare methods of approximating square roots with approximating solutions to rational equations.

Common Errors

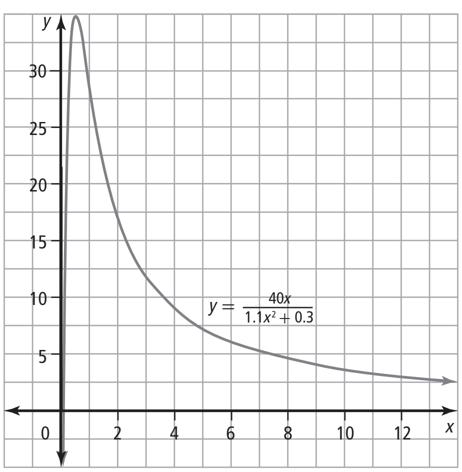
- Some students incorrectly cancel terms from a rational function.

R$_x$ Present students with the following example and ask them to identify the error: $\frac{x-2}{x+3} = \frac{-2}{3}$.
Since only common terms can be cancelled, and $x - 2$ is the entire numerator, only a denominator of $x - 2$ cancels it. If students are having difficulty, suggest that they substitute for x to see if both sides of the equation are equal.

Answers

Investigate Solving Rational Equations

1. a)



b) The concentration increases rapidly for a short time, and then decreases rapidly for a short period before beginning to diminish at a much slower rate, moving toward zero.

2. a) Example: The doctor might need to re-administer the drug.
b) Example: Graphically, you could check to see when the dose is below 10 mg/dL. It is also possible to solve the problem algebraically.
c) $\approx$ 3.6 h (Some students may say 3.5 h since at 3.6 h, the bloodstream concentration will no longer be 10 mg/dL.)

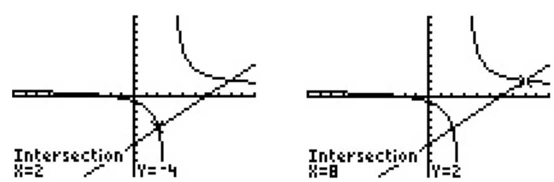
3. Example: Algebraic methods give you an exact answer; a graphical method is faster, but not exact.

4. $x \neq 3$

$x + 2 = x^2 - 9x + 18$	Multiply to get rid of denominator.
$0 = x^2 - 10x + 16$	Set left side to 0.
$0 = (x - 2)(x - 8)$	Factor.
$x = 2, 8$	Solve.

Factoring gives you the answer of $x = 2, 8$.

5. Graph the two sides of the equation on a graphing calculator and find where the two graphs intersect.



6. Example: I prefer the graphical approach because it helps me visualize what is happening to the values. Or, I prefer the algebraic method because it enables me to find the solution without going through the process of graphing, and it is often more accurate.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs. • List the methods students used for #3 and 6 on the board and discuss them as a class. Discuss the strengths and weaknesses of each approach. Encourage students to write the list and a brief descriptor of each in their notebook or graphic organizer for future reference.

Example 1

Help students reactivate their knowledge of the relationship between roots and x-intercepts. Ask
- What does it mean to find the roots of an equation?
- Why is it necessary to multiply each term of the equation by the LCD?
- What are the restrictions of the rational equation?
- What is the y-value of the equation when finding the roots?
- When finding the x-intercepts, what is the y-coordinate?
- What is the connection between the y-coordinate of the x-intercepts and the y-value in the original equation?

You may need to revisit the concept of finding the LCD. Also, when students are working on their own, be sure that they are applying the distributive property correctly when multiplying by the LCD.

Example 2

For part a), have a discussion with students about the two graphical methods shown in the example. For the first method, ask
- Why do you make the equation equal to zero?
- How is the fact that you set the equation to zero reflected in the first graphical method?

For the second method, ask
- Why can you enter each side of the equation into the graphing calculator separately?
- Why are you finding the point of intersection using this method?
- Is the x-coordinate or the y-coordinate the solution?

For part b), you might want to discuss why it is important to state the restrictions before eliminating the fractions. You might also ask students if there is an algebraic method to solve the equation other than using the quadratic formula. Suggest that students try to solve by completing the square. Ask them which method they prefer and why.

Example 3

You may want to revisit the importance of stating the restrictions, or non-permissible values, before solving algebraically. Have students recall the word *extraneous*, and ensure that all students know what it means. Ask
- What is an extraneous root?
- One type of extraneous root is a root that is a restriction. Can you think of another type?
- Once you have solved the equation, checking the solution in the original equation is very important. Will solving an equation graphically show extraneous roots? Explain.
- Considering what we have discussed, which method for solving rational equations is more accurate?

Example 4

Ask students why the equation equals 0.8 and not 80. You might also have a discussion about implied and functional restrictions. What are these in the context of this question? Discuss with students how they can check their solution.

Meeting Student Needs

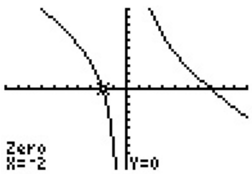
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Before working through Example 1, encourage students to solve and graph a polynomial function to review the connection between the solution and the x-intercepts. Again, revisit this connection in the example.
- Have students work in pairs to discuss Methods 1 and 2 in Example 2. Have them identify which method they prefer and why.
- If students are working with index cards from earlier sections, they could add a note to their card from section 9.2, indicating that the non-permissible value from the equation is an extraneous root.
- For Example 4, if a student in the class has personal sports or fitness statistics, use these to make an equation to illustrate a personal goal. Solve the equation to provide information for the student.
- Have student work in pairs. Have one partner explain the key ideas in Examples 1 and 3 to the other partner, and then switch roles for Examples 2 and 4.

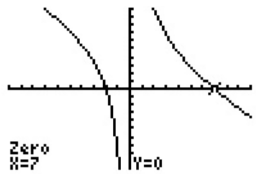
Answers

Example 1: Your Turn

a) $x \neq 0, x = -2, 7$

b)

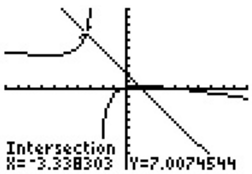


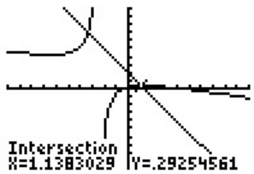


c) The roots of the equation are the same as the x-coordinates of the x-intercepts.

Example 2: Your Turn

a)

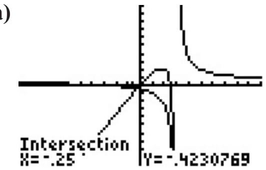




b) $x \approx -3.34, 1.14$

Example 3: Your Turn

a)



$x = -0.25$. The second point of intersection at $x = 3$ is dismissed because this a non-permissible value.

b) The solutions are the same.

Example 4: Your Turn

$x = 23$; The students need to approach 23 more businesses.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Assist students who are having difficulty by writing a simplified version of the quadratic. Ask them to verbalize the steps to ensure that they understand how to proceed. • Discuss responses to part c) as a class. This may help students who can solve algebraically, but who are not making the connection to the graph of the function.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Have students identify which method they wish to use. Some students may need assistance in rewriting the equation in terms of *y*. Have students verbalize the steps before starting. • If students are using Method 2, using two equations, have them identify what the equations are before entering them into their calculator. Ask students what they are looking for on the graph to help solve the problem. • Using either method, students should be able to identify non-permissible values and verify their values. If technology is available in the classroom to present this to the class, it would be a helpful discussion to compare the two methods.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Have students identify which method they wish to use, a single equation or two equations. Some students may need help in rewriting the equation in terms of *y*. • If students are using two equations, have them identify what the equations are before entering them into their calculator. Ask them what they are looking for on the graph to help with the solution to the problem. • Using either method, students should be able to identify non-permissible values and verify their values. Discuss the results of part b) as a class.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Some students may need assistance in writing the equation. Have them define their variables and write them on the board. Then, ask them to verbally describe a general formula that represents what they are asked to find, and then write out the formula. Some students may also need to be reminded about the decimal format of percents. • Have students choose a method of their choice to solve, and then compare their response with that of a partner.

Check Your Understanding

Before students start #1, ask them to describe the steps they can take to rewrite each equation. Help student recognize that all terms need to be on one side of the equal sign so that they equate to zero. You might also help them see the relationship between solving for $y = 0$ and the x-intercept, $(x, 0)$.

Remind students to state their restrictions in #3. They should always remember to check their solution against the restrictions.

Watch how students are solving the equations in #4. They can use one of two methods:

- graph two equations (left side and right side) and find the point of intersection
- equate the equation to zero and find the x-intercept

Discuss the two methods and the pros and cons of each. Once students have a solution, ask them to describe how they will check it.

For #8, ask students how to eliminate fractions with different denominators. Students will need to identify the LCD of these fractions. Suggest that students multiply each term by the LCD, rather than multiplying each side. Ensure that students use the distributive property correctly. They sometimes forget to distribute to each term inside the brackets when multiplying by the LCD.

Students may need help setting up #15.

$$C(x) = \frac{10\text{ g} + 0.01x\text{ g}}{200\text{ mL} + x\text{ mL}}$$

This might be a good question to discuss as a class.

For #17, ask students what the difference is between solving an equation and solving an inequality. Students should remember that whenever they multiply or divide both sides by a negative number, they must switch the inequality sign. Discuss with students how many possible solutions there are when they solve an inequality compared to when they solve an equation. Discuss how to solve these inequalities graphically. You might also compare solving an equation graphically by identifying point(s) of intersection to solving an inequality graphically by determining the area of intersection.

For C2, discuss with students the difference in restrictions between rational and radical equations. Students should know that the restrictions for rational equations come from division by zero, whereas the restrictions for radical equations come from taking the square root of a negative number.

Meeting Student Needs

- For #1, some students should determine the solutions without making the calculations.
- You may want to have a physics teacher come to class for #10 to provide insight into the concepts in this question, and to provide some examples of this type of question for practice.
- Provide **BLM 9–4 Section 9.3 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2, 3a), b), 4a), b), 5, 7, and 11. Students who have no problems with these questions can go on to the remaining questions.	• Students are asked to match equations to a single function in #1. In each case, they are moving all values to one side of the equation. Be sure that students are successful with this question before moving on to #2 to 5. These later questions require students to find a single equation to either solve or verify. Assist students in getting started on #2 by having them verbalize the steps to write a single equation. It is important that they can explain how to do this because they are asked to identify an error of this type in #7. • Students choose the method they prefer to solve a real-world problem in #11. Ask students who choose to graph two equations to identify what the two equations are before beginning. Ask what they will look for on a graph. Ask students who choose to use one equation what value they will substitute into $C(t)$.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• C1 provides an opportunity for students to clarify their thinking and explain their understanding of solutions to rational equations. Have them compare their responses with a partner or take the question up as a class so that several approaches and lines of thinking can be explored. • For C2, have students explain what an extraneous root is and how one can be identified. Have them work on the compare and contrast individually, and then share their thinking with the class. • Have students record their response to C3 in their graphic organizer or notebook. Have them exchange their responses with a partner to see if their partner can identify any additional points for their method of choice.

Chapter 9 Review and Practice Test

9

***Pre-Calculus 12*, pages 468–471**

Suggested Timing
60–90 min each

Materials
- graphing technology
- poster board

Blackline Masters
BLM 9–2 Section 9.1 Extra Practice
BLM 9–3 Section 9.2 Extra Practice
BLM 9–4 Section 9.3 Extra Practice
BLM 9–5 Chapter 9 Study Guide
BLM 9–6 Chapter 9 Test

Planning Notes

Have students who are not confident discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test. You may wish to provide students with **BLM 9–5 Chapter 9 Study Guide**, which links the achievement indicators to the questions on the Chapter 9 Practice Test in the form of a self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum that will meet the related curriculum outcomes: #1–12

Meeting Student Needs

- Students should choose questions in the review based on the outcomes that they feel require further study.
- Students could sketch a graph of each Key Idea presented in the chapter. Suggest that they indicate what the equation of each sketch would include.
- You could put a selection of the review questions on poster board and arrange the posters around the classroom. Student pairs could move from poster to poster and solve the questions together. Encourage students to make notes about which questions were difficult or for which they required assistance. This will provide a focus for further study.
- Some students may choose to write notes on the process required to solve the questions without actually completing the questions. Focus on the key ideas developed in the chapter.
- You may wish to have students complete the practice test prior to working through the review questions. Their performance on the test will provide them with a sense of where they may want to focus further study.
- Once students complete the practice test, suggest that they compare their solutions with another student. They might want to discuss the questions in which they used different methods. Why do they prefer one method over the other?
- Students who require more practice on a particular topic may refer to **BLM 9–2 Section 9.1 Extra Practice**, **BLM 9–3 Section 9.2 Extra Practice**, and **BLM 9–4 Section 9.3 Extra Practice**.

Enrichment

In Salmon Arm, British Columbia, there are high-tech companies that create hardware and software solutions to reduce waste during the processing of trees into wood products. Lasers precisely measure the dimensions of each tree, and software determines exactly how to saw the tree in order to minimize waste and maximize the value of the tree. Ask students to develop a presentation that the companies could use to explain this process to the public. Their presentation should highlight how rational expressions and their graphs could be used by the software, and how these processes help the health of our forests.

Gifted

In a cyclotron, such as the one at the University of British Columbia, particles are accelerated to near speed-of-light velocities. Some particles created in collision experiments in cyclotrons are extremely short-lived, but because those particles are travelling so fast, they appear to "live" longer. Challenge students to make the connections between the rational expressions that describe Einstein's work on time dilation and the apparent longevity of such particles.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 9 Review The Chapter 9 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #1–3, 4–9, and 11.	• You may wish to have students work in small groups of similar ability. • Encourage students to use **BLM 9–5 Chapter 9 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 9 Study Guide This master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. The minimum questions suggested are questions that students should be able to confidently answer. Encourage students to try additional questions. • Consider allowing students to use any summative charts, concept maps, or graphic organizers when completing the practice test.
Assessment *of* Learning	
Chapter 9 Test After students complete the practice test, you may wish to use **BLM 9–6 Chapter 9 Test** as a summative assessment.	• Before the test, coach students in areas in which they are having difficulty. • You may wish to have students refer to **BLM 9–5 Chapter 9 Study Guide** and identify areas they need reinforcement in before beginning the chapter test.