Function Operations

Opener

Pre-Calculus 12, pages 472–473

Suggested Timing
30–45 min

Blackline Masters
BLM 10–1 Chapter 10 Prerequisite Skills
BLM U4–1 Unit 4 Project Checklist

Planning Notes

Prior to beginning this chapter, you may wish to review functions and function notation with students along with the vertical line test, domain and range, and graphing functions.

The first two sections of this chapter focus on operations with functions. Students add, subtract, multiply, and divide functions. In section 10.3, students learn how to work with composite functions.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter. Encourage students to use a summary method of their choice.

Unit Project

In this chapter, students revisit the Unit 4 Project introduced in Chapter 9. In section 10.2, they apply mathematical principles to music by writing lyrics and setting them to an appropriate melody.

Meeting Student Needs

- Consider having students complete the questions on **BLM 10–1 Chapter 10 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out **BLM U4–1 Unit 4 Project Checklist**, which provides a list of all the requirements for the Unit 4 Project.
- Review the key terms and have students review related terms and their definitions. Ensure students understand the terms before proceeding.
- Provide students with a checklist containing the learning outcomes for this unit. Discuss specific terms. Develop a sense of understanding of what students need to learn by the end of the unit.
- You may wish to provide students with a graph showing different operations on linear functions, such as a graph of $f(x) = 2x + 5$, $g(x) = 3x - 2$, and $h(x) = 5x + 3$. Have students describe the similarities and differences they notice.

Enrichment

Encourage students to explore the meaning of the statement "It is important to understand functional relationships between variables since they apply to the fields of engineering, business, physical sciences, and social sciences, to name a few." You may also wish to have students preview the chapter and look for ways in which combinations of functions can be used to model real-world phenomena.

Gifted

Have students investigate how function operations might be used to model wave interference, which is used to reduce sound energy in noise-cancelling headphones. Ask students to explore the differences between headphones that use insulation to block noise and headphones that use wave interference to reduce noise.

Career Link

Discuss with students what they know about the field of laser research and the applications of laser technology in biology, medicine, chemistry, and physics. Ask

- What do you know about the use of laser technology in science and medicine?
- What education do you think is required to work in this field?
- What other fields are related to laser research?
- How might mathematics be used in laser research?

Suggest that students go online to research careers in laser research. Are there any aspects of the career that surprise them?

10.1 Sums and Differences of Functions

***Pre-Calculus 12*, pages 474–487**

Suggested Timing
90–120 min

Materials
- grid paper
- graphing technology

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 10–2 Section 10.1 Extra Practice

Mathematical Processes for Specific Outcomes
RF1 Demonstrate an understanding of operations on functions.
- ✓ Connections (CN)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1a), b), 2a) b), 3–13
Typical	#1c), d), 2c), d), 3–11, one of 14–16, 17 or 18, C1–C3
Extension/Enrichment	#14, 17, 19–22, C1, C3

Planning Notes

Discuss the learning outcomes for this section. As a class, read the opening paragraph about wave functions.

Tell students that in this section they will add and subtract linear, quadratic, trigonometric, and exponential functions. They will also compare the domain and range of the combined function to the original functions.

Investigate Sums and Differences of Functions

For step 2, you may wish to remind students how to write the equation of a line in slope–intercept form given a graph of the line. For part b), students determine the equation $s(x)$ by finding the sum $(x - 1) + (-2x + 3)$.

For step 3, ask what are the domain and range of
- all non-horizontal and non-vertical linear functions?
- all horizontal linear functions?
- all vertical linear functions?

For step 8, students may wish to make a preliminary statement based on the results of the Investigate. They can then revisit their statement after they complete the examples, and again at the end of this section.

Meeting Student Needs

- You may wish to have students work in pairs or small groups to complete the Investigate.
- Some students may benefit from using graphing technology to check their work.
- For step 5, you may wish to have students use tape to make coordinate axes on the floor and use string to construct each linear function, including the x- and y-intercepts, as well as the points determined in the table of values.
- Some students may wish to make a presentation summarizing the material from the Investigate; others may wish to write a summary without making a presentation.

Common Errors

- Students often forget to distribute the negative sign when subtracting polynomials.

R_x For questions such as $(x^2 - 3x - 10) - (4x^2 + 7x - 3)$, encourage students to rewrite the question with a 1, and then show the distribution. For example,

$$(x^2 - 3x - 10) - (4x^2 + 7x - 3)$$
$$= (x^2 - 3x - 10) - \mathbf{1}(4x^2 + 7x - 3)$$
$$= x^2 - 3x - 10 - 4x^2 - 7x + 3$$

Answers

Investigate Sums and Differences of Functions

1. a)

x	$f(x)$	$g(x)$	$h(x)$
−2	−3	7	4
−1	−2	5	3
0	−1	3	2
1	0	1	1
2	1	−1	0
3	2	−3	−1
4	3	−5	−2

b) $h(x) = f(x) + g(x)$

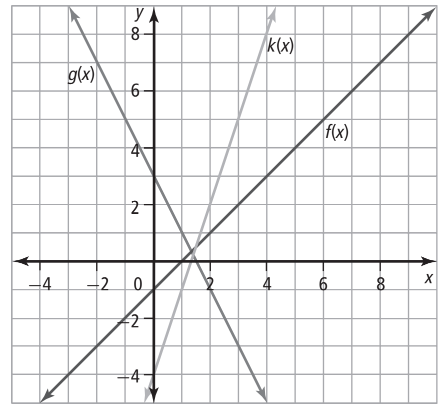
2. a) $f(x) = x - 1$; $g(x) = -2x + 3$; $h(x) = -x + 2$
b) $s(x) = -x + 2$; Yes: $s(x) = h(x)$

3. a) domain of $f(x)$: $\{x \mid x \in \mathrm{R}\}$; range of $f(x)$: $\{y \mid y \in \mathrm{R}\}$; domain of $g(x)$: $\{x \mid x \in \mathrm{R}\}$; range of $g(x)$: $\{y \mid y \in \mathrm{R}\}$
b) domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \in \mathrm{R}\}$; The domain and range are the same for all functions.

4.

x	$f(x)$	$g(x)$	$h(x)$	$k(x) = f(x) - g(x)$
−2	−3	7	4	−10
−1	−2	5	3	−7
0	−1	3	2	−4
1	0	1	1	−1
2	1	−1	0	2
3	2	−3	−1	5
4	3	−5	−2	8

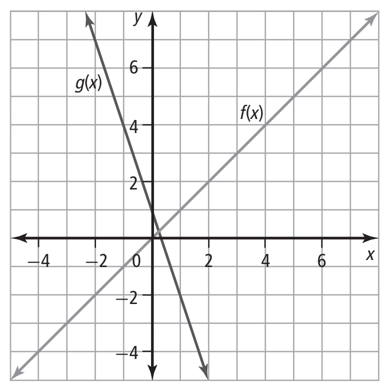
5. a)



b) domain of $k(x)$: $\{x \mid x \in \mathrm{R}\}$; range of $k(x)$: $\{y \mid y \in \mathrm{R}\}$; The domain and range are the same for all functions.

6. $d(x) = -3x + 4$; $d(x)$ is the opposite of $k(x)$; it is the difference $g(x) - f(x)$.

7. a) Example: $f(x) = x$ and $g(x) = -3x + 1$



b) I can find the equation of the sum, $h(x)$, of $f(x)$ and $g(x)$ by adding $x + (-3x + 1)$ and simplifying. The sum of the functions is $h(x) = -2x + 1$. Then I can graph $h(x)$.
c) I can find the difference function, $q(x) = g(x) - f(x)$ by subtracting $(-3x + 1) - x$ and simplifying. The difference of the functions is $q(x) = 1 - 4x$. Then I can graph $q(x)$.

8. Yes, unless there are non-permissible values.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • Review #2b) and #4 with the class before assigning the Reflect and Respond questions. • Encourage students to use a table and complete it for $f(x)$, $g(x)$, and the sum and difference.

Example 1

You may wish to have students determine the domain and range of $f(x)$ and $g(x)$, and then compare them with the domain and range of $h(x)$.

Have students compare the shape of the graph of $h(x)$ to the graphs of $f(x)$ and $g(x)$. Ask students to describe the transformations that map $g(x)$ onto $h(x)$.

For part d), you may wish to show students alternative methods to determine the values of the functions when $x = 4$. For example, they could read the values from the graph or use the Calculate function on a graphing calculator.

Example 2

For the table in part b) Method 1, ensure students understand why $f(x)$ is undefined when $x < 1$ and how the undefined values for $f(x)$ relate to the undefined values of $h(x)$.

For Your Turn part d), students compare $(f - g)(x)$ with $(g - f)(x)$. Ask

- What do the graphs of $(f - g)(x)$ and $(g - f)(x)$ look like?
- Can you use the words *inverse*, *reflection*, *opposite*, or *mirror-image* to describe and compare the graphs?
- How do the domain and range of the functions compare?

Example 3

Ensure students understand both methods for determining the sum of functions given a graph—from the graph by adding corresponding y-values, or algebraically by finding the sum of the equations. Ask

- In this case, the sum $(f + g)(x)$ is a constant, which represents a horizontal line. What features of the graphs of $f(x)$ and $g(x)$ indicate that $(f + g)(x)$ will be a horizontal line?
  [The lines are reflections of each other].
- Is $(f + g)(x) = (g + f)(x)$ in every case? Explain.

For Your Turn, students find the difference $(f - g)(x)$ using the same graphs as in the example. Ask:

- Does the combined function produce a horizontal line?
- Would $(f - g)(x) = (g - f)(x)$? Explain.
- How will the graphs of $y = (f - g)(x)$ and $y = (g - f)(x)$ compare?

Example 4

It may be helpful for some students to discuss how total cost of production, revenue from sales, and profit are related. Ask

- What is a simple equation relating total cost (C) and revenue (R) to profit (P)?
- What are fixed costs?
- What is the break-even point in a business?
- Is it possible to have a negative profit? If so, what does it mean?

Meeting Student Needs

- For Example 2 Your Turn, students work with an absolute value function and a linear function. Suggest that they graph the functions using a different method from those shown in the example.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Students may square binomials incorrectly. For example, they may expand $(3x - 5)^2$ and get $9x^2 - 25$.

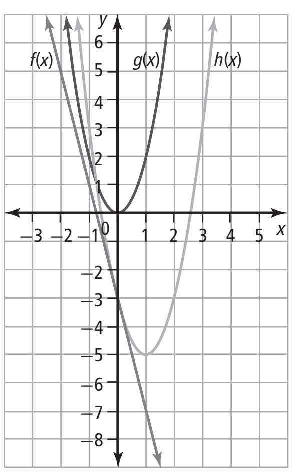
R_x Suggest students write $(3x - 5)^2$ as $(3x - 5)(3x - 5)$, and then use the distributive property to multiply.

Answers

Example 1: Your Turn

a) $h(x) = 2x^2 - 4x - 3$

b)

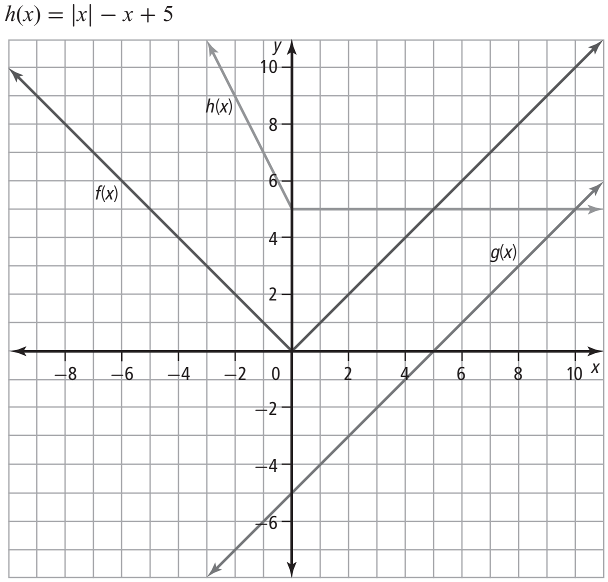


c) domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \geq -5\ y \in \mathrm{R}\}$

Example 2: Your Turn

a) $h(x) = |x| - x + 5$

b)

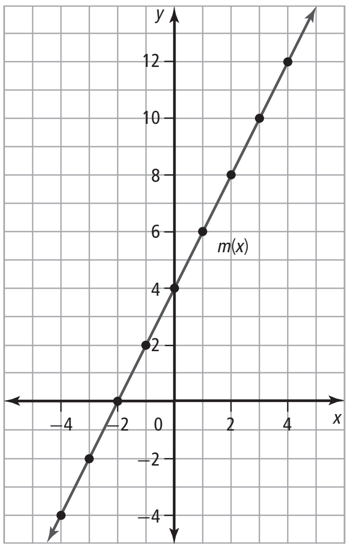


Answers

c) domain: $\{x \mid x \in \mathrm{R}\}$; range: $\{y \mid y \geq 5\ y \in \mathrm{R}\}$

d) No; $(f - g)(x) = -(g - f)(x)$

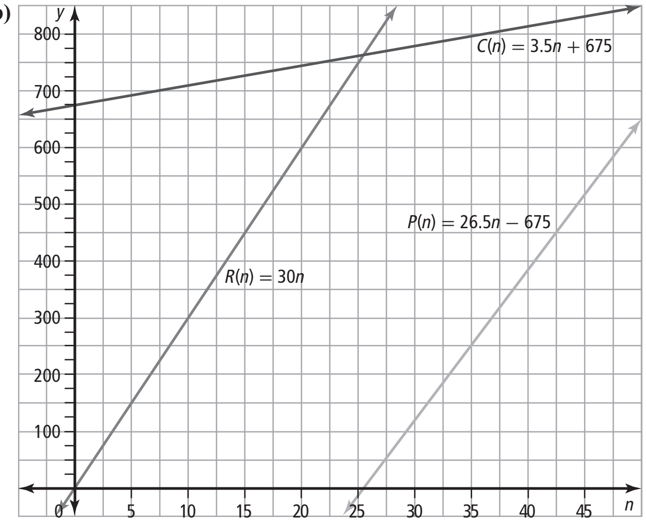
Example 3: Your Turn



Example 4: Your Turn

a) $C(n) = 3.5n + 675$; $R(n) = 30n$; $P(n) = 26.5n - 675$

b)



26 booklets

c)

Function	Domain
$C(n)$	$\{n \mid n \geq 0, n \in \mathrm{R}\}$
$R(n)$	$\{n \mid n \geq 0, n \in \mathrm{R}\}$
$P(n)$	$\{n \mid n \geq 0, n \in \mathrm{R}\}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Encourage students to use a table to help organize the values and make it more visual. • You may wish to complete $h(x)$ algebraically with the entire class.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Discuss the answers to the questions in green text. • Encourage students to record the values in a three-column table. • Discuss the answer to part d) as a class. Make a list of similarities and differences on the board so students can place the information into their organizer.
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Students may find it easier to solve for the difference algebraically, or using a table of values or a spreadsheet, first, and then use this information to graph. Encourage students to use a method of their choice to solve and then graph.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Ask students to identify the constant or fixed value. • It may be necessary to review the profit formula with students and ensure they understand it.

Check Your Understanding

For #1–5, encourage students to use graphing technology to reinforce the visual representation of the sum or difference of functions.

For #3, suggest students use two methods for finding the values: finding the equation of the combined function and then substituting; and substituting into each function, and then adding or subtracting the result.

For #6, it is not necessary for students to find the equation of the functions.

For #7, students may find it helps to use a table of values like the one shown, choosing three or four values for x.

x	$f(x)$	$g(x)$	$(f+g)(x)$	$(f-g)(x)$	$(g-f)(x)$

For #9–11 remind students to be careful when subtracting polynomials.

For #11, use a simpler example, such as $10 = 12 - x$, to help students see how to work backward.

For #13, ensure students understand the meaning of the term *break-even point*.

Note that #14–16, 18, and 19 involve trigonometric functions. Remind students to set their calculators to radians. Students may benefit from reviewing the trigonometric functions and their transformations.

For #15b), refer students to the Did You Know? for information about destructive and constructive interference.

For #16, students may wish to do further research about AC and DC power supplies.

For #22, ensure students recognize that non-permissible values result in holes in the domain and/or range.

C1 is an excellent journal question. Students could discuss the answers and include examples to support their argument.

For C3, some students may benefit from reviewing exponential and trigonometric functions. Ensure students understand which axis represents time and which axis represents height. A simple oscillation is represented by a sinusoidal function. A damped oscillation is represented by a sinusoidal function that exhibits exponential decay. Neither the sum nor the difference of an exponential function and a sinusoidal function will produce the damped oscillations of this model. However, a product of a sinusoidal and an exponential function will.

Meeting Student Needs

- Provide **BLM 10–2 Section 10.1 Extra Practice** to students who would benefit from more practice.
- Encourage students to summarize the Key Ideas of this section. Some may sketch graphs, some may show tables of values of two functions and the sum or difference of the two, and some may choose to write a response. Some students may create a poem or short rap video that would outline how to find the sum or difference of two functions.

Enrichment

Ask students to consider how the sum or difference of two functions might be zero. Students should list several examples. Challenge students to develop a rule for creating pairs of opposite functions.

Gifted

Waves transmit energy. A water wave passes through water, moving water particles up and down as the energy passes through them. Occasionally, ocean waves can sum to produce larger waves or standing waves, or they can even cancel each other out. Challenge students to use sine wave functions as the basis for modelling the behaviour of converging ocean waves. Students should show their thinking in whatever manner they think is best.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1a), b), 2a) b), and 3–13. Students who have no problems with these questions can go on to the remaining questions.	• Remind students that for #10 and 11, they must work backwards and find one of the original functions. • Some students may benefit from using a table in #13. Students may need help to identify the fixed cost and the variable cost.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• If students are having difficulty with C1, have them examine the results from any of the questions in which they used a table. This may help students to prove or disprove the statements in #1a) and b). • For C2, students may benefit from completing a table first, and then graphing. • For C3, discuss as a class or in groups what happens to a bungee jumper from the time he leaves the platform and describe his motion. Review with students the meanings of sinusoidal and exponential features.

Products and Quotients of Functions

10.2

***Pre-Calculus 12*, pages 488–498**

Suggested Timing
90–120 min

Materials
- grid paper
- graphing technology

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 10–3 Section 10.2 Extra Practice

Mathematical Processes for Specific Outcomes
RF1 Demonstrate an understanding of operations on, and compositions of, functions.
- ☑ Connections (CN)
- ☑ Reasoning (R)
- ☑ Technology (T)
- ☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–4, 5a), b), 7
Typical	#1–4, 5c), d), 6–8, 9 or 10, C2–C3
Extension/Enrichment	#8, 10, 11, 13, C2–C3

Planning Notes

Discuss the learning outcomes for this section. As a class, read the opening paragraph about the applications of products and quotients of functions.

Prior to beginning the Investigate, you may wish to review multiplying polynomials, factoring polynomials, and operations with rational expressions.

Investigate Products and Quotients of Functions

For step 1, ask students to describe any patterns they see in the $p(x)$ values in the table. Ask students to think about the reasons why the x-intercepts are the same for all three functions.

For steps 1 and 4, students may wish to use grid paper transparencies and draw each function on a separate transparency. The transparencies can be overlaid to show the changes taking place. You may then wish to project the transparencies onto the whiteboard for class discussion.

For step 2, ask
- Does order matter when multiplying the polynomials? Explain.

For step 3, have students discuss with a partner or in a small group why the range of $p(x)$ is not the combined ranges of $f(x)$ and $g(x)$.

In step 4, ensure students understand why there is an undefined value for $x = 2$ and know how to show an undefined value on a graph. Ask students why there is no x-intercept for $q(x)$.

For step 5, students may need to be reminded to factor in order to simplify the quotient $\left(\frac{f}{g}\right)(x)$. Ensure students check for non-permissible values.

Meeting Student Needs

- Ask students to summarize their findings from the Investigate and compare them to the Link the Ideas that follows.

Common Errors

- Students may neglect to find non-permissible values before simplifying a rational expression.

R_x Have students simplify the expression $\frac{x - 1}{x^2 + 3x - 4}$ and show all their steps. Then, have them graph the original expression using a graphing calculator. Have students use the Calculate, Trace, or Table feature to determine the values of y when $x = 1$ and $x = -4$. Ensure students understand that since the factor $x - 1$ was part of the original denominator, $x = 1$ is non-permissible, since it makes the original denominator 0.

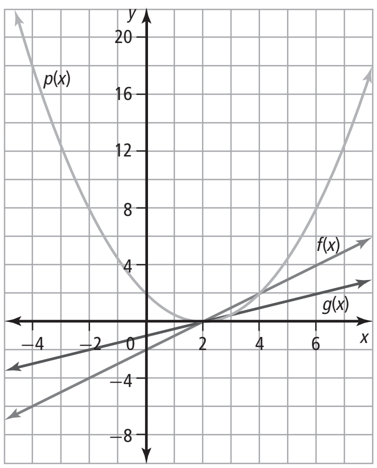
Answers

Investigate Products and Quotients of Functions

1. a)

x	$f(x)$	$g(x)$	$p(x)$
−4	−6	−3	18
−2	−4	−2	8
0	−2	−1	2
2	0	0	0
4	2	1	2
6	4	2	8

b) The graph will be a parabola.



c) The x-intercepts of $f(x)$, $g(x)$, and $p(x)$ are all 2.

2. a) $f(x) = x - 2$; $g(x) = \frac{1}{2}x - 1$

b) Multiply $f(x)$ by $g(x)$ and simplify; $p(x) = \frac{1}{2}x^2 - 2x + 2$

3. a)

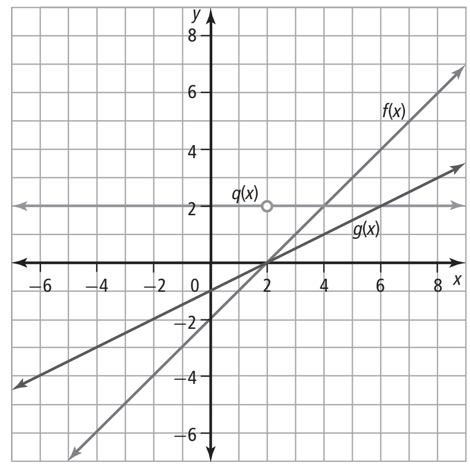
Function	Domain	Range
$f(x)$	$\{x \mid x \in \mathrm{R}\}$	$\{y \mid y \in \mathrm{R}\}$
$g(x)$	$\{x \mid x \in \mathrm{R}\}$	$\{y \mid y \in \mathrm{R}\}$
$p(x)$	$\{x \mid x \in \mathrm{R}\}$	$\{y \mid y \geq 0, y \in \mathrm{R}\}$

b) Yes, the domain of the product is the domain common to the individual functions, as it is for a sum or a difference.

4. a)

x	$f(x)$	$g(x)$	$p(x)$	$q(x)$
−4	−6	−3	18	2
−2	−4	−2	8	2
0	−2	−1	2	2
2	0	0	0	does not exist
4	2	1	2	2
6	4	2	8	2

b) The graph will be a horizontal line through $y = 2$.



c) The x-intercept of $g(x)$ is a non-permissible value for $q(x)$.

5. a) Divide $f(x)$ by $g(x)$ and simplify; $q(x) = 2, x \neq 2$

b) domain of $q(x)$: $\{x \mid x \neq 2, x \in \mathrm{R}\}$;
range of $q(x)$: $\{y \mid y = 2, y \in \mathrm{R}\}$
The relationships between the domains and ranges of $f(x)$, $g(x)$, and $q(x)$ is much different when $q(x)$ is a quotient of $f(x)$ and $g(x)$ than when $q(x)$ is a sum or difference of $f(x)$ and $g(x)$.

6. a) I can multiply $f(x)$ by $g(x)$, then simplify, and then sketch the resulting function.

b) I can divide $f(x)$ by $g(x)$, then simplify, and then sketch the resulting function.

c) For the product, $p(x)$, of functions $f(x)$ and $g(x)$, the x-intercepts of $f(x)$ and $g(x)$ are the x-intercepts of $p(x)$. For the quotient, $q(x)$, of functions $f(x)$ and $g(x)$, the x-intercept of the divisor, $f(x)$ or $g(x)$, results in non-permissible values of $q(x)$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • You may wish to remind students about non-permissible values and ask students to explain what role they play in the domain and range of a quotient of functions. • Ask students how they could use technology in two ways to identify the non-permissible values (graphing and table of values). • Prompt students to consider products and quotients of functions by asking whether they must algebraically multiply or divide $f(x)$ and $g(x)$.

Example 1

You may wish to review how to multiply a binomial by a trinomial and how to square a binomial. Ask students if they can think of another method to determine the domain and range of $h(x)$.

For Your Turn, ensure students check for restrictions on the domain of $h(x)$.

Example 2

Ensure students determine the non-permissible values before simplifying. Ask students how they will show the non-permissible values on the graph.

Example 3

In this application, the variable g is used in place of x. Some students may find this confusing, since in previous examples g has been used as a function name.

There are five functions in this example: $T(g)$, $N(g)$, $A(g)$, $r(g)$, and $p(g)$. Discuss with students what each function represents, and how the functions $r(g)$ and $p(g)$ are combined functions of the other three. Ask

- Which domain values (if any) are not reasonable for functions $T(g)$ and $N(g)$?
- Which functions have non-permissible values? What are they?

For Your Turn, ensure students understand the meaning of the term *amplitude*.

Meeting Student Needs

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Students may wish to make a video containing examples of how to create the product or quotient of two functions and the resulting graph.

Common Errors

- Some students may have difficulty multiplying a binomial and trinomial.

R$_x$ Students should use the distributive property to expand. Before the product is simplified, it will have six terms. For example,

$$\begin{aligned}&(3x - 5)(2x^2 - x + 8)\\&= 3x(2x^2 - x + 8) - 5(2x^2 - x + 8)\\&= 6x^3 - 3x^2 + 24x - 10x^2 + 5x - 40\end{aligned}$$

Answers

Example 1: Your Turn

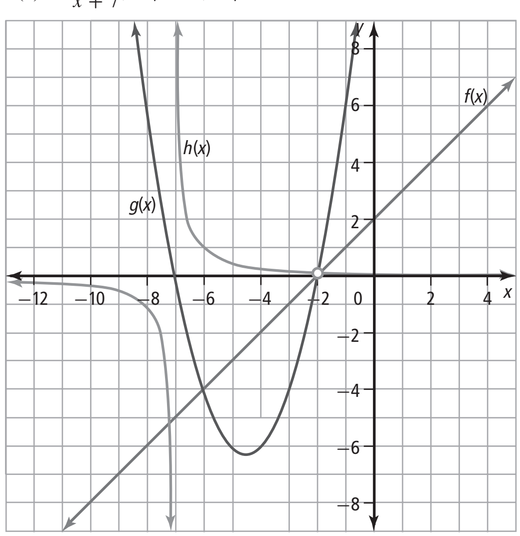
$h(x) = x^2\sqrt{4x - 5}$

domain: $\left\{x \mid x > \frac{5}{4}, x \in \mathrm{R}\right\}$; range: $\{y \mid y > 0, y \in \mathrm{R}\}$

Example 2: Your Turn

a) $h(x) = \frac{1}{x + 7}, x \neq -2, x \neq -7$

b)



c) domain: $\{x \mid x \neq -2, x \neq -7, x \in \mathrm{R}\}$; range: $\{y \mid y \neq 0, y \neq \frac{1}{5}, y \in \mathrm{R}\}$

Example 3: Your Turn

a) $y = 0.95^t\ 10 \cos 2t$

Since p is the horizontal distance of the pendulum from its resting position and q is the portion of the original amplitude, the product $(p \cdot q)(x)$ represents the actual distance of the pendulum from its resting position as a function of time.

b) The function decreases from a maximum value at time $t = 0$ to a minimum value where the distance to the resting position is zero. The function decreases because the pendulum is a damped harmonic oscillator and the distance of the pendulum gradually decreases until it stops.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Ask students how they would identify any non-permissible values for $f(x)$ and $g(x)$. • Some students may find it easier to identify the domain and range by graphing $f(x)$, $g(x)$, and $h(x)$.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Remind students to identify any non-permissible values *before* they cancel any common factors. Encourage students to use a table to record values for each step of the process. • Some students may need help to factor the expression for $g(x)$. • Some students may prefer to use a table rather than a graph to complete part c).
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Ask students to explain what the product represents. This may help them put their work into context.

Check Your Understanding

For #1d), some students may benefit from reviewing how to multiply and divide radicals.

For #6a), students must find the product of three binomials. Some students may find it helps to first multiply one pair of binomials, and then multiply the result by the third binomial.

For #7 and 8, students must work backward to determine $g(x)$.

For #9–11 and 13–15, remind students to set their calculators to radians, since these questions involve trigonometric functions. You may also wish to review these trigonometric identities: $\cos^2 x + \sin^2 x = 1$, and $\frac{\sin x}{\cos x} = \tan x$.

For #10, have students determine the values for which $\cos x = 0$ and $\tan x$ are undefined in order to determine the non-permissible values for $\frac{f(x)}{g(x)}$.

For #17, some students may need to be reminded to use the Pythagorean theorem.

For C3a), some students may need to review the factor theorem.

Meeting Student Needs

- Have students work individually or in pairs to complete the Project Corner on page 498.
- Remind students to always consider non-permissible values.
- Encourage students to graph combined functions to help determine the domain and range.
- For #4, ask students to explain if there can be non-permissible values when the combined function is formed from the product of two functions. If yes, ask students to give an example.
- Students can work in small groups to complete the Check Your Understanding questions. Each student completes a different question then shares the solution with the others in the group. This allows students to talk about the key ideas.
- Have students work in groups to create an application question and its solution. Have each group present their question to the class.
- Students should identify the questions that require more work before the chapter assessment.
- Provide **BLM 10–3 Section 10.2 Extra Practice** to students who would benefit from more practice.

Enrichment

Suggest that students explore the quotient of two functions from the perspective of non-permissible values. Have students create pairs of functions that when multiplied or divided produce non-permissible values, and identify those values.

Gifted

Ask students to describe in their own words the role and importance of domain when considering the quotient of functions. Further, have them create an example that supports their thinking.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–4, 5a), b), and 7. Students who have no problems with these questions can go on to the remaining questions.	• Review the Key Ideas with students who are having difficulty before going on to #1. Clarify any misunderstandings students may have. • Do #2a) as a class as it is important for students to complete questions in this form from a graph. Once students are proficient with #2, assign #3–5. • Remind students to identify non-permissible values *before* they cancel factors.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• For C1, encourage students to select two functions and multiply them to check whether their product is commutative. Have students discuss their results with a peer or in a small group. • Since C2 is a good understanding question that addresses the key concepts for this section, it is a good example for students to include in their graphic organizer. • For C3, some students may benefit from a review of rearranging formulas.

10.3 Composite Functions

***Pre-Calculus 12*, pages 499–509**

Suggested Timing

120–150 min

Materials

- grid paper
- graphing technology

Blackline Masters

Master 3 Centimetre Grid Paper
BLM 10–4 Section 10.3 Extra Practice

Mathematical Processes for Specific Outcomes

RF1 Demonstrate an understanding of operations on, and compositions of, functions.

✓ Connections (CN)
✓ Reasoning (R)
✓ Technology (T)
✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–6, 8, 9, 11, 12, 15, 16
Typical	#1–5, 7–10, 13, 14, 16–19, C1, C3, C4
Extension/Enrichment	#7, 10, 14, 17, 19–23, C1, C4

Planning Notes

Discuss with students the real-world examples of how costs are dependent on other changeable values. As a class, discuss other examples in which a cost depends on a value that in turn depends on another value. Then, as a class, brainstorm other scenarios in which this type of dependency occurs.

Investigate Composition of Functions

For the diagram in step 1, it may be helpful for some students to label the first oval in the diagram x. In the diagram, $f(x)$ becomes the input for $g(x)$. Thus, the final oval can be labelled $g(f(x))$.

For step 2, students may find it helpful to substitute an x-value from the diagram in step 1 into each of the function options to see which produces the desired output.

For step 3, students may wish to label the first curved arrow above the diagram $2x$ and label the second curved arrow $x^2 + 2$. This may help students to see how to use $2x$ and $x^2 + 2$ to write the composite function.

Meeting Student Needs

- You may wish to have students work in pairs to complete the Investigate.

Common Errors

- Some students may become confused when using $f(x)$ and $g(x)$ to determine the composite function algebraically.

R$_x$ Use a symbol to represent $f(x)$ to help simplify the substitution. For example,

Let $f(x) = ♦$; find the value of $g(x)$ when $x = ♦$.

$g(x) = x^2 + 2$

$g(♦) = ♦^2 + 2$

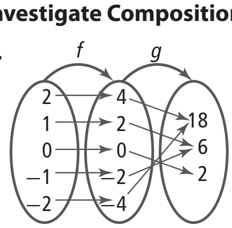
Since $f(x) = 2x$, $♦ = 2x$; replace ♦ on the right side with $2x$.

$$\begin{aligned} g(f(x)) &= (2x)^2 + 2 \\ &= 4x^2 + 2 \end{aligned}$$

Answers

Investigate Composition of Functions

1.

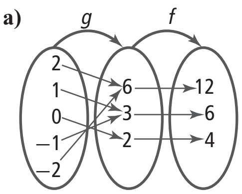


2. D

3.
$$\begin{aligned} h(x) &= g(f(x)) \\ &= (f(x))^2 + 2 \\ &= (2x)^2 + 2 \\ &= 4x^2 + 2 \end{aligned}$$

Answers

4. a)



b) $y = 2x^2 + 4$

5. Yes; Given $f(x)$ and $g(x)$ in step 1, $g(f(x)) \neq f(g(x))$.

6. Replace x in $g(x)$ with $f(x)$ and then simplify.

$$\begin{aligned} g(f(x)) &= -3(4x + 2) \\ &= -12x - 6 \end{aligned}$$

7. Example: The surface area of a cone that is 10 cm high depends on the area of the circular base. The area of the circular base depends on its radius.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • Discuss as a class the answers to step 4 before going on to the examples. • You may wish to have students do step 6 and use the results as a starting point to discuss the answer for step 5. If students are unsure, have them redo step 6 and use *g* as the input for *f*.

Example 1

Encourage students to work with both methods to evaluate composite functions. Students having difficulty should evaluate by determining the value of the inner function and substituting the result first.

You may wish to show the solution on a whiteboard using coloured markers to show the substitutions. The visual connection will be helpful for visual learners.

Example 2

When finding the domain of composition of functions, some students may question why the domain of $g(f(x))$ is $\{x \mid x \geq 1, x \in \mathrm{R}\}$ when the final form of $g(f(x))$ is $x - 1$. Ask

- What happens when you evaluate $g(f(x))$ for $x = -2$ using Method 1 from Example 1?
- What happens when you evaluate $g(f(x))$ for $x < 1$ using the same method?
- What does this suggest about finding the domain of a composite function?

In Your Turn, $f(x)$ is the same as in Example 2, but $g(x)$ is the additive inverse of $g(x)$ from Example 2. Ask

- How do the composite functions from Example 2 and Your Turn compare?
- How do the domains compare?

Example 3

Before discussing this example, ask students to come up with as many compositions as they can using $f(x)$ and $g(x)$. Ask

- How many different compositions did you find?
- How do the compositions that you found compare to those listed in Example 3?
- How does graphing the composition function help you find the domain and range?
- How does graphing the composition function prevent you from finding restrictions on the domain and range?

Students may find it helps to plot both of the original functions and the resulting composite function. Ask students to describe how the composite function compares to the original functions.

Example 4

Ensure students understand that they are working backward to determine the original functions given a composite function. There are many possible solutions.

Example 5

Some students may need a review of rational exponents. Ask

- How do you write a radical as a fractional exponent, i.e., $\sqrt{x} = x^{\square}$?
- How do you write a fractional exponent as a radical, i.e., $x^{\frac{3}{2}} = \square$?

For Your Turn, remind students that the formula for the surface area of a sphere is $SA = 4\pi r$.

Meeting Student Needs

- Ask students to think of two functions in which the composition would be commutative. Discuss the characteristics of such functions.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Some students may confuse the composition symbol, $\circ$, with the multiplication symbol, $\bullet$.

R_x Have students avoid using the composition symbol and instead use the double set of brackets. For example, instead of $(f \circ g)(x)$, use $f(g(x))$.

- Some students may not substitute the inner function into all values of x in the outer function.

R_x To simplify the substitution, encourage students to use a symbol, such as ♦, to represent the inner function.

Answers

Example 1: Your Turn

$f(g(-11)) = 10$; Example: In this case, I preferred evaluating $g(-11)$ first, and then using the result (10) to evaluate $f(x)$.

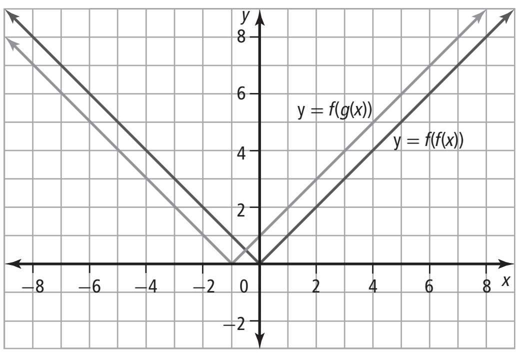
Example 2: Your Turn

$(g \circ f)(x) = 1 - x$

Function	Domain
$f(x)$	$\{x \mid x \geq 1, x \in R\}$
$g(x)$	$\{x \mid x \in R\}$
$(g \circ f)(x)$	$\{x \mid x \in R\}$

Example 3: Your Turn

$f(g(x)) = |x + 1|$; $f(f(x)) = |x|$



Function	Domain	Range
$f(g(x))$	$\{x \mid x \in R\}$	$\{y \mid y \geq 0, y \in R\}$
$f(f(x))$	$\{x \mid x \in R\}$	$\{y \mid y \geq 0, y \in R\}$

Example 4: Your Turn

$f(x) = x + \frac{3}{3 + x}$

$g(x) = \sqrt[3]{x}$

Example 5: Your Turn

a) $SA = 4\pi t$

b) 14.32 min

<table>
<tr><th>Assessment</th><th>Supporting Learning</th></tr>
<tr><td colspan="2">Assessment for Learning</td></tr>
<tr><td>Example 1
Have students do the Your Turn related to Example 1.</td><td>• Ensure students read the information on composite functions in the margin on page 500 of Pre-Calculus 12. Ensure they understand the different notations.
• Remind students that they can either evaluate one function first and then input this value into the second function, or determine the equation of the composite function first and then evaluate.
• Ask students which method they prefer and why. Do they think there might be a time when one method would be a better choice than the other? Explain.</td></tr>
<tr><td>Example 2
Have students do the Your Turn related to Example 2.</td><td>• Direct students to the margin definition on page 500 of Pre-Calculus 12 if they need to review the notation for composite functions.
• Ensure students are clear about the order of input of the functions for (f into g). Students may wish to graph each function to help them determine the domain and range.</td></tr>
<tr><td>Example 3
Have students do the Your Turn related to Example 3.</td><td>• Have students determine the equations of the composite functions, and then compare their equations with a classmate.
• Review absolute value and its effect on the equation, the domain, and the range.</td></tr>
<tr><td>Example 4
Have students do the Your Turn related to Example 4.</td><td>• Remind students to look for functions that are common to more than one term in the composite function. Ask what they would write in the place of these common functions to create a new function $f(x)$. Point out that the common function becomes $g(x)$. If students are uncertain, use an example such as $h(x) = 2x + \frac{4}{2x}$. Since $2x$ appears in both terms of $h(x)$, $g(x) = 2x$; thus $f(x) = x + \frac{4}{x}$.</td></tr>
</table>

Assessment	Supporting Learning
Assessment *for* Learning	
Example 5 Have students do the Your Turn related to Example 5.	• As a class, discuss the component functions that are needed to create the composite function. You may need to review the formula for surface area of a sphere with students.

Check Your Understanding

For #4, students may wonder how part a) differs from part c) and how part b) differs from part d). The only difference is the use of the variable a in parts a) and b) and the use of the variable x in parts c) and d).

For #6, remind students to consider restrictions when determining the domain and range of the composite function.

For #7, students are given the composite function and one of the component functions. They must work backward to determine $g(x)$.

For #9, students may need to review the power of a power property, $(x^n)^m = x^{n \times m}$.

For #11, students should consider which values of the domain are reasonable in this context.

For #12c), 13c), and 13d), challenge students to find the answer using a method other than the composition of two functions.

One method of finding the answer to #16b) is to use logarithms. You may wish to review how to use logarithms.

Students may find more than one solution to #17. Have students share their answers.

For #18, ask students to find another pair of functions, $f(x)$ and $g(x)$, in which $g(f(x)) = \frac{1}{g(x)}$. Have students show and explain why part b) is false.

For #20, you may wish to introduce inverse notation, $f^{-1}(x)$ since it is used in #23b). Ask students to find $f^{-1}(x)$ using the following method. They can then compare the answer to $g(x)$. In the function $f(x) = 5x + 10$, replace $f(x)$ with x and x with $f^{-1}(x)$, and solve for $f^{-1}(x)$. Example:

$$f(x) = 5x + 10$$
$$x = 5(f^{-1}(x)) + 10$$
$$x - 10 = 5(f^{-1}(x)) + 10 - 10$$
$$\frac{x - 10}{5} = \frac{5(f^{-1}(x))}{5}$$
$$\frac{1}{5}x - 2 = f^{-1}(x)$$

For #23, students who have not completed #20 will need to refer to the explanation of inverse functions. In part b) the inverse function notation is used, which some students may not be familiar with.

For C1, make sure students look at the symbol closely. The question is asking if $f(g(x)) = (f \cdot g)(x)$, not $f(g(x)) = (f \circ g)(x)$.

C2 may be confusing for some students. Remind them that a function does not always need to be represented as an equation—it can be expressed as ordered pairs. So, the x-value for $g(x)$ is the y-value from $f(x)$.

Refer students back to #23 when they have completed C3. Have students explain how $f(x)$ and $g(x)$ are related, and how inverse function notation can be used to show the relationship.

For C4, have students determine the slope of each linear function in steps 1 to 3, and then compare the slopes to their answers for steps 1 to 3. Have students explain whether the combined function $\frac{f(x + h) - f(x)}{h}$ will always equal the slope of $f(x)$ if $f(x)$ is a linear function. Mention to students that this will be the basis for calculating slope in Calculus.

Meeting Student Needs

- You may wish to post questions around the classroom and have students rotate from one station to the next. Students can compare their solutions with other students at the same station.
- Have students work in pairs, to complete two parts of #3 or 4. Each student then explains how to complete their chosen question.
- Have students look at #11–13 and 15–16. These questions relate to practical areas in which composite functions may be used.
- #8 highlights a common student error. Have students write a journal entry about this question.
- Provide **BLM 10–4 Section 10.3 Extra Practice** to students who would benefit from more practice.

Enrichment

Ask students to describe three real-life situations (different from those given in the student resource) that could be modelled by composite functions.

Gifted

Challenge students to develop the component and composite functions that model the situations they described in the Enrichment activity.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–6, 8, 9, 11, 12, 15, and 16. Students who have no problems with these questions can go on to the remaining questions.	• Students should successfully complete #1–3 before moving on. • For #4–6, students substitute an equation into another equation to find the composite. Once they have determined the composite equations, students may wish to compare their equations with those of a partner. • You may wish to complete #12a) as a class. • Students should complete #11, 15, and 16 independently. • For #15, ensure students understand the meaning of *period*.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• For C1, some students may find it helps to use specific equations for $f(x)$ and $g(x)$ before they respond more generally. Ask students to record their responses to this question in their notes. It is an excellent assessment as learning question. • For C2, remind students to work from the inside out using the value for $f(1)$ as input into $g(x)$. • For C3, have students evaluate both $g(f(x))$ and $f(g(x))$. Remind students to identify any non-permissible values.

Chapter 10 Review and Practice Test

10

***Pre-Calculus 12*, pages 510–513**

Suggested Timing
60–90 min each

Materials
- grid paper
- graphing technology

Blackline Masters
Master 3 Centimetre Grid Paper
BLM 10–2 Section 10.1 Extra Practice
BLM 10–3 Section 10.2 Extra Practice
BLM 10–4 Section 10.3 Extra Practice
BLM 10–5 Chapter 10 Study Guide
BLM 10–6 Chapter 10 Test

Planning Notes

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test. You may wish to provide students with **BLM 10–5 Chapter 10 Study Guide**, which links the achievement indicators to the questions on the Chapter 10 Practice Test in the form of self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

Have students who are not confident discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum that will meet the related curriculum outcomes: #1–10, 13.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 10–2 Section 10.1 Extra Practice**, **BLM 10–3 Section 10.2 Extra Practice**, and **BLM 10–4 Section 10.3 Extra Practice**.

Enrichment
Have students create a Venn diagram that shows the relationships among sums and differences, products and quotients, and composite functions.

Gifted
Have students add to the Venn diagram created in the Enrichment activity. Ensure they address the finer points of the chapter, including the role of mathematical thinking, division by zero, non-permissible values, and function notation. Other previously learned but relevant mathematics, such as factoring, should be included.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 10 Review The Chapter 10 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #2–4, 6–13	• Ask students to revisit any section from the review that they are having difficulty with before they begin working on the practice test. • Encourage students to use **BLM 10–5 Chapter 10 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 10 Study Guide This master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. The minimum questions suggested are questions that students should be able to confidently answer. Encourage students to try additional questions. • Consider allowing students to use any summative charts, concept maps, or graphic organizers when completing the practice test.
Assessment *of* Learning	
Chapter 10 Test After students complete the practice test, you may wish to use **BLM 10–6 Chapter 10 Test** as a summative assessment.	• Before the test, coach students in areas in which they are having difficulty. • You may wish to have students refer to **BLM 10–5 Chapter 10 Study Guide** and identify areas they need reinforcement in before beginning the chapter test.