Permutations, Combinations, and the Binomial Theorem

Opener

***Pre-Calculus 12*, pages 514–515**

Suggested Timing
30–45 min

Blackline Masters
BLM 11–1 Chapter 11 Prerequisite Skills
BLM U4–1 Unit 4 Project Checklist

Planning Notes

Permutations, combinations, and the binomial theorem introduce students to a branch of mathematics that covers topics including probability theory, topology, optimization, ergodic theory, and statistical physics.

Section 11.1 focuses on counting problems using the fundamental counting principle. It encourages students to explore a variety of strategies to determine the number of permutations of n elements taken r at a time.

As they work through Section 11.2, students acquire an understanding of the differences between permutations and combinations. Students must be able to determine whether a problem involves a permutation or a combination by the phrasing of the question. They must also develop strategies to solve a problem by developing different cases to satisfy any stated restrictions.

In Section 11.3, students relate binomial expansion to Pascal's triangle and to combinations. The ability to expand any binomial taken to any exponent (within reason) is an important skill for students to learn. It may be useful to have students expand a few basic binomials (such as $x + 1$) to powers up to 4 or 5, to recognize the value of this math skill.

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Students may have used different types of graphic organizers. Ask students which one(s) might be useful in this chapter.

Unit Project

In the Project Corner that follows section 11.3, students create a piece of art, by hand or using technology, to demonstrate a topic from this *Pre-Calculus 12* course. Students may use mathematics either to model a real-world object or create something using their imagination, working with any medium they like.

Meeting Student Needs

- Consider having students complete the questions on **BLM 11–1 Chapter 11 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Hand out **BLM U4–1 Unit 4 Project Checklist**, which provides a list of all the requirements for the Unit 4 Project.
- Provide students with a copy of the student learning outcomes for the unit. Discuss specific terms. Develop a sense of understanding of what students need to learn by the end of the unit.
- Students could research the Königsberg bridge problem or Kirkman's schoolgirl problem.

Enrichment

The introduction to the chapter describes the mathematics of the chapter as the art of counting. Counting is probably the earliest mathematics children learn. Challenge students to compare and contrast the childlike counting process with the sophisticated mathematics of counting as applied to permutations and combinations.

Gifted

Electronic passwords protect bank accounts, credit card information, and other personal data. Computers allow hackers to determine passwords by trying millions of possible sequences of symbols. Ask students to list what information about a person might lead to the creation of a password and, hence, the ability to predict it by narrowing the possibilities from billions to millions.

Career Link

Discuss with students what they know about actuarial science and the work that actuaries perform. Ask
- What do you know about actuarial science?
- What do you think actuaries study?
- How do you think actuaries might use mathematics? How might they use permutations, combinations, and the binomial theorem? What other branches of mathematics would be important to an actuary?

Encourage students to go online to research careers in actuarial science. Encourage them especially to explore the work of actuaries in helping governments and the private sector quantify and manage financial risk.

11.1 Permutations

***Pre-Calculus 12*, pages 516–527**

Suggested Timing

120–180 min

Materials

• index cards (optional)

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BLM 11–2 Section 11.1 Extra Practice

TM 11–1 How to Do Page 519 Example 2a) Using TI-83/84

Mathematical Processes for Specific Outcomes

PCBT1 Apply the fundamental counting principle to solve problems.

☑ Connections (CN)

☑ Problem Solving (PS)

☑ Reasoning (R)

☑ Visualization (V)

PCBT2 Determine the number of permutations of *n* elements taken *r* at a time to solve problems.

☑ Connections (CN)

☑ Problem Solving (PS)

☑ Reasoning (R)

☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–5, 7–9, 12, 14, 16, 17
Typical	#1, 2, 4–7, 9–16, one of 18–22, 24, 25, one of 26–29, C1–C4
Extension/Enrichment	#20, 21, 24, 25, 28–32, C1, C3–C5

Planning Notes

Discuss the outcomes and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found in this section.

Assign students to work in pairs. Have them try different activities to explore concepts involving permutations. For example, choose four 1-digit numbers. Make a list of the 4-digit numbers that could be created. Will repetition be allowed? Does repetition give more or fewer 4-digit numbers? Next, try working with 5 digits. What do students notice about the number of possible 5-digit numbers compared to the number having 4 digits?

For another activity, choose a location five blocks from the school. Have students determine how many ways they could leave the school and arrive at the location (without backtracking). Students could also investigate the number of handshakes that can be made between four people, five people, and six people. How is this exercise the same as creating 4-digit numbers? How is it different?

Some students may require a more concrete approach to completing the investigation on possible arrangements. Consider providing index cards that can be used to represent articles of clothing, and moved and rearranged as required in the investigation.

Investigate Possible Arrangements

This investigation requires students to determine the number of possible arrangements using clothing as a model. Organize the class into pairs to complete this investigation. Make sure that students record each arrangement using a sketch, list, or table to represent the placement of each card in their solution.

Ask coaching questions to help students visualize arrangements and lead them to a discussion of permutations. If you have one top and two pairs of pants, how many outfits can you make? How can you represent this arrangement using a tree diagram? If you add a second top, how many new outfits can you make? If each top can produce two different outfits and you have three tops, how many outfits are possible? What mathematical operation (addition, subtraction, multiplication, or division) did you use to determine the number of outfits? If you have two pairs of shoes, how does this affect the number of outfits that you can make? Given three tops, two pants, and two pairs of shoes, how can you determine the total number of outfits using only a mathematical operation? Can you sketch a diagram to represent the same set of arrangements? Discuss with your partner which technique you prefer and why.

Step 5 requires students to consider whether 1000 different outfits are possible using only tops, pants, and shoes. Have each group determine a solution that would lead to 1000 outfits and share it with the class. Discuss with the class if there is enough given information to determine the exact number of tops, pants, and shoes. Why or why not?

Meeting Student Needs

- Some students may prefer to work with index cards or similar manipulatives to complete the investigation using a more concrete model. Have students label each card to represent an article of clothing and use the cards to set up all possible arrangements.
- Students could bring clothes to school (or use paper cutouts) to create as many outfits as possible.
- Alternatively, create the tree diagram on the whiteboard and encourage different students to "create" an outfit.

Enrichment

Challenge students to devise and perform an experiment that involves predicting the actions of someone. For example, a classmate is selecting items from the school cafeteria. What is the most likely combination of items they will end up with? How do restrictions enter into the thinking? How does knowledge of the person affect the chances of a correct prediction?

Gifted

Good decision making is the ability to list and then evaluate possible outcomes in an appropriate way. Ask students to use a flowchart to model the decision-making process for a student deciding which universities to apply to. Assume each university has a non-refundable deposit policy, so the student does not want to waste money applying to universities that they might not want or be able to get into. Include facets such as cost, programs available, and competition for entrance places. How do the topics discussed in this section affect the way in which the student goes through the decision-making process? Finally, challenge students to apply their knowledge of permutations and combinations to creating a list of options as a first step to good decision making.

Common Errors

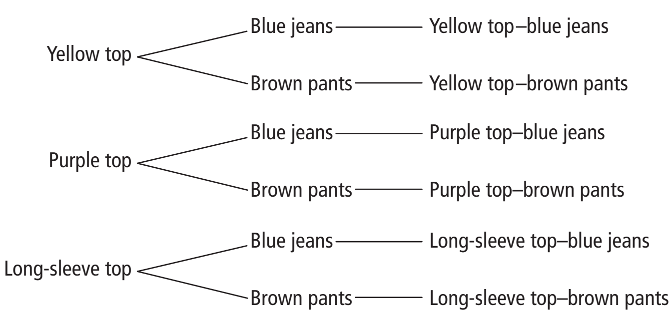
- When using the fundamental counting principle, students may add numbers of each item instead of multiply.

R_x Use index cards or similar visual aids to help students determine the answers to the Investigate. Have students match one top with each pair of pants. How many arrangements can you make? If you have two tops and two pairs of pants, how many different outfits can you create? If you have three tops and three pairs of pants, how many different outfits can you make? What do you get when you add 3 + 2? What is the product of 3 and 2? Which operation—adding or multiplying—gives the correct answer?

Answers

Investigate Possible Arrangements

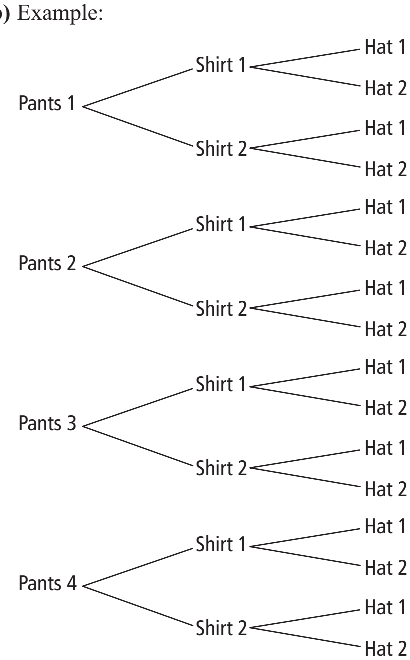
1. 6



2. 12

3. a) 16

b) Example:



Answers

4. In step 1, the number of outfits can be determined by multiplying the number of tops by the number of pairs of pants. In step 2, multiply the number of outfits from step 1 by the number of pairs of shoes. In step 3, the number of different outfits is determined by multiplying the number of hats by the number of tops by the number of pairs of pants.

5. Example:
Let t represent the number of tops, let p represent the number of pairs of pants, and let s represent the number of pairs of shoes.
So, $t \times p \times s = 1000$.

If the friend has four pairs of shoes, then

$$\begin{aligned} t \times p \times s &= 1000 \\ t \times p \times 4 &= 1000 \\ \frac{t \times p \times 4}{4} &= \frac{1000}{4} \\ t \times p &= 250 \end{aligned}$$

If the friend has ten pairs of pants, then

$$\begin{aligned} t \times p &= 250 \\ t \times 10 &= 250 \\ t &= 25 \end{aligned}$$

Therefore, if the friend has 25 tops, 10 pairs of pants, and 4 pairs of shoes, they will have $25 \times 10 \times 4 = 1000$ different outfits.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work with a partner for this activity. • If students are having difficulty with steps 1 and 2, have the items of clothing available for students to physically create the combinations. Encourage them to use multiplication to verify whether they have all the possible outfits. Similarly, #3 could be achieved by using different students for the pants, tops, and shoes. • Ensure that students are comfortable with and clear about using multiplication before moving on to #5. To assess the learning, provide an additional example with different clothing combinations, other objects in the class, or a menu from the school cafeteria.

Example 1

Have the class work in pairs and challenge students to develop at least two methods to solve part a). Have the groups share their strategies with the class, including anything that they discovered in the Investigate that helped them develop their solution. Have students change partners and then develop a strategy to solve part b). Again, discuss strategies as a class. How are the questions in parts a) and b) similar? How is part b) different from part a)? In part b), are there any restrictions on who can fill any position? Are there any restrictions for seating girls and boys? How many choices are there for seat 1 if a boy must sit in that seat? How many boys remain to sit in seat 7? How many choices of boys and girls remain to sit in each of seats 2 through 6? Which mathematical operation should you use to determine the total number of arrangements?

In part b) of the Your Turn, if any group is having difficulty, consider using coaching questions: How does the question in part b) differ from the question in part a)? How many choices are there for the first digit? Does the outcome of the first digit affect the other digits? How many choices are there for the second digit? How many choices are there for the third digit?

Example 2

Some students may need coaching to connect with permutation notation. Ask students what the variable n represents in the notation ${}_nP_r$. What does the variable r represent? What does the variable P represent? The notation ${}_nP_r$ indicates the number of arrangements that exist given n objects taken r objects at a time. In part a), given the expression ${}_9P_4$, how many objects are being arranged? How many objects are arranged together? How many arrangements are there of 9 objects arranged 4 at a time?

In Example 2a), how can you represent ${}_9P_4$ using factorial notation? How can you simplify the expression $\frac{9!}{5!}$? Is it possible to expand 9! to $9 \times 8 \times 7 \times 6 \times 5!$? How can you reduce $\frac{9 \times 8 \times 7 \times 6 \times 5!}{5!}$?

For Example 2b), discuss with the class the mathematical expectations suggested by the phrase "Show that …" Students may demonstrate that $100! + 99! = 101(99!)$ by evaluating each side of the equation and determining that both sides have equal value, or by following a strategy similar to the one in the student resource. Discuss how both approaches satisfy the "show that" direction.

In Example 2c), if students have difficulty recognizing how to begin solving the equation ${}_nP_3 = 60$, ask if there is another way to write the expression ${}_nP_3$ using factorial notation. How can you reduce the equation $\frac{n!}{(n-3)!}$? Is it possible to expand $n!$? In order to reduce the fraction (that is, cancel out the expression $(n - 3)!$ from the denominator), what is the last term in the expansion of $n!$? Can you solve the expression $n(n - 1)(n - 2) = 60$ in its present form? Can you determine the values of n, $n - 1$, and $n - 2$ by determining three consecutive numbers that have a product of 60? Is there an algebraic method to solve the equation $n(n - 1)(n - 2) = 60$? Can you expand the product of $n(n - 1)(n - 2)$? What is it? Can you solve the new polynomial equation by factoring? Can you solve $n(n - 1)(n - 2) = 60$ by graphing?

Example 3

Have students share their strategies and solutions with the class. Coaching questions for weaker students might include: How many different three-letter arrangements are there using the letters of the word *had*? How many different three-letter arrangements are there using the letters of the word *boo*? How many more different arrangements exist using the letters *h*, *a*, and *d* than the letters *b*, *o*, and *o*? How do identical letters affect the number of arrangements? How can we adjust our solution of arranging *n* items, when *a* of them are identical? How many letters in the work *aardvark* are identical? How can you determine the number of different arrangements given that there are 3 *a*'s and 2 *r*'s?

In part b), ask students what movements are possible in moving from point A to point B? Are all vertical movements identical? Are all movements to the right identical? How many total single grid movements are required to move from point A to point B? How many identical vertical movements are required? How many identical horizontal movements are required? How can you determine the number of arrangements of movements up and to the right that are required to move from point A to point B?

Example 4

Have students work through Example 4 individually and then discuss together as a class. To help students visualize the seating, allow them to arrange five different objects. Alternatively, you may wish to work through this example as a class and have five students demonstrate the seating options in front of the class. Encourage students to try to come up with another method of solving part c). Ask them:

- How many choices are there for seat 1?
- Does the outcome of seat 1 affect the choices for seat 2? In what way?
- For each choice for seat 1, how many choices are there for seat 2? How could you show this using a tree diagram?

This discussion could lead right into Example 5, which involves permutations involving cases. Have students read the notes on Arrangements Requiring Cases.

Example 5

Some students may not understand the need for identifying the restrictions that must be considered in order to solve problems involving cases. Initiate a discussion using the situation described under Arrangements Requiring Cases. Ask students why two cases need to be considered. What are the two cases if both end chairs must be occupied either by girls or boys? Given the case of the end seats occupied by girls, how many different girls could sit in the first seat? How many different girls can now be chosen for the last seat? Who can sit in the second seat (boy only, girl only, either)? How many choices do you have for the second seat? How many choices remain for the third, fourth, fifth, and sixth seats? How many different seating arrangements are possible if a girl must sit in the end seats?

In the second case of boys sitting in the end seats, how many different boys are possible for the first seat? How many boys can be chosen for the last seat? Who can sit in the second seat? How many choices are there for each of the remaining seats? How many arrangements are possible if a boy must sit in the end seats? How many arrangements are possible if either girls or boys occupy the end seats?

Meeting Student Needs

- For part b) of Example 1, have each pair of students create a set of place cards using the names of seven students. Encourage them to manipulate the cards for the given scenario. Emphasize that the numbers on the blanks refer to the number of *choices* available for each position.
- Emphasize that, in order to use factorial notation, the objects must be distinguishable. Have students determine the value of ${}_nP_r$ where $n = r$. Have them write a conclusion based on their findings.
- For Example 2, provide students with **TM 11–1 How to Do Page 519 Example 2a) Using TI-83/84**.
- For Example 3, ask students to list, in their notebook, the different arrangements of the letters in the word *pole*. (There are 24 arrangements.) Have them repeat the activity using the word *polo*. What do they notice about the different arrangements?

- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.
- Students could also create posters containing the Key Ideas information. Post them around the classroom.

Common Errors

- Students do not have an understanding of the values of factorials.

R_x Have students construct a table with a list of factorials from 1! to 15! using multiplication.

- Students do not understand permutation notation.

R_x Introduce the idea that a permutation infers that *n* objects are taken *r* objects at a time. Have students practise by stating the values of *n* and *r* from a list of examples.

- Students have difficulty reducing factorial fractions.

R_x Have students identify the larger factorial expression and write out its factors down to the lower factorial expression. By cancelling out the lower factorial expression from both the numerator and denominator, the expression will be reduced. Have students practise at least five examples.

- Students often look at factorial questions and misinterpret them as "like terms" creating, for example, 3! + 2! to result in 5!

R_x Have students work out both sides of the statement to compare.

Answers

Example 1: Your Turn

a) 60

b) If repetitions of the digits are allowed, there are more options for each digit; 125 numbers can be formed.

Example 2: Your Turn

a) 42

b)
$$\begin{aligned} 5! - 3! &= 19(3!) \\ 5 \times 4 \times (3!) - 3! &= 19(3!) \\ 20(3!) - 3! &= 19(3!) \\ 19(3!) &= 19(3!) \end{aligned} \qquad \text{or} \qquad \begin{aligned} 5! - 3! &= 19(3!) \\ 120 - 6 &= 19(6) \\ 114 &= 114 \end{aligned}$$

c) $n = 8$

Example 3: Your Turn

a) 10 different five-digit numbers

b) 56

Example 4: Your Turn

a) 8

b) 12

c) 48

Example 5: Your Turn

340 four-digit odd numbers

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Provide students with cards numbered from 1 to 5 so they may physically begin to create possible combinations of numbers. • Encourage students who are attempting to create number combinations physically to verbalize their thinking. Ask them how they could use the investigation to shorten the work they are attempting. Tree diagrams are a better approach for visual learners. You may also wish to put the numbers in a container and draw them out one at a time, asking students how many numbers there were in the container when you reached in for the first number, then for the second number, and so on. This may be a more familiar method or approach for students to start with. • Remind students from previous work that in probability, an *and* situation involved multiplication.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • It is important for students to explore at least two approaches to solving problems such as Example 2. It is not sufficient for them to be able to use a calculator only. • It is important that students realize the operational word *show* means to provide methods and show all steps, rather than simply verifying both sides of the equality sign.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • For students having difficulty with the concepts in this example, provide them the letters of the word *label* with one of the *l*'s labelled l_1 and the other labelled l_2. Tell them that an *l* is the same even though it may have a different subscript. Visually, this may help students see that the placement of the *l*'s does not change the word, which is the reason for dealing with repetitions.
Example 4 Have students do the Your Turn related to Example 4.	• You may wish to have students work in pairs. • Point out to students that in part a) the two mathematics posters are not just together, they are on an end. • Remind students that cases that involve more than one possible approach or an *or* statement involve addition.
Example 5 Have students do the Your Turn related to Example 5.	• Have students solve a simpler version of the YourTurn first, such as with repeating digits, before attempting the given problem. • Allow students to develop their own cases. For example, students may wish to use the cases where the last digit is a seven and the last digit is not a seven.

Check Your Understanding

It is preferable to have students work in pairs when completing these questions.

In #3, ask students what is expected in solving a mathematical problem when directed to "Show that …" In this case, are you asked to show equality or inequality? How can you show that the expression on the left side is not equal to the right side?

For #5, ask students to describe a common feature of the words. Since all words have at least one set of identical (repeating) letters, what strategy can you use to determine the number of possible arrangements?

In #6, how many names can be listed first? How many of the remaining names can be listed second? listed third? listed last? What mathematical operation do you use to answer this question?

In #7, can you replace the permutation notation with factorials? How can you reduce the factorial expressions? How can you solve the remaining polynomial functions for either n or r?

For each of #9 to 17, how many objects in total are being arranged? How many objects are being arranged together at the same time? What restrictions exist? Can you sketch a diagram, make a list, or create a table to help visualize the problem and identify the restrictions?

Approach #23 by asking students in how many ways 3 objects grouped 3 at a time can be arranged. How many different arrangements can you make with 4 different objects grouped 4 at a time? Since 3 objects grouped 3 at a time is 3! and 4 objects grouped 4 at a time is 4!, then how many arrangements exist for n objects grouped n at a time? Using permutation notation, how would you write the number of arrangements for n objects grouped n at a time? Can you write ${}_nP_n$ using factorials? What fraction do you get after reducing the factorial expression? If the number of arrangements of n objects grouped n together is $n!$, and the permutation notation to express n objects grouped n at a time is ${}_nP_n$, what does ${}_nP_n$ equal in factorial notation? Since $\frac{n!}{0!} = n!$, what is the value of 0!?

For #27, students should discover that the total possible arrangements minus those with repeated digits equals the arrangements of integers without repeated digits.

In #29, ask how many different ways the cube can be painted all one colour, such as all green or all blue. In how many unique ways can you paint the cube with only one green side and five blue sides? In how many unique ways can you paint the cube with exactly 2 green sides? exactly 3 green sides? exactly 4 green sides? exactly 5 green sides? What is the total number of different arrangements?

For #32, encourage students to volunteer to represent A, B, C, D, and E (and, eventually, F). Ask them to create arrangements, based on the requirements, at the front of the classroom.

Meeting Student Needs

- Students could work in pairs or small groups. Before completing any of the questions in the Check Your Understanding, have students identify each question as FCP (fundamental counting principle), P (permutation), PR (permutation with repetitions), or PC (permutation with cases). When finished, students can compare their classification of questions and then complete at least two of each type.

- Encourage students to use the formula to calculate a permutation. Then, check the answer with a calculator.
- You may wish to review vowels and consonants for ESL learners prior to having students work on #9 and 12.
- Provide **BLM 11–2 Section 11.1 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–5, 7–9, 12, 14, 16, 17. Students who have no problems with these questions can go on to the remaining questions.	• For #1, have students use multiplication to check their work. Allow students to choose their own method for #3. It is important that students grasp the concept in #3, since this is a common mistake. Ensure students do not treat them as *like terms*. • To assess students' understanding, you may wish to have them solve #2 in two different ways. Encourage students to show all their steps for #6 and 7. • Encourage students to use diagrams for the Apply questions if it makes visualizing the question easier. Alternately, if students are using the multiplication method, have them use spaces or dashes to represent each of the events or conditions, but have them label each space so it is clear that they understand what they are determining. Example: (Pants)(Shirts) = Outfits • In #23, $0! = 1$ is an important rule for students to remember. If they are having difficulty generalizing it, have them begin with specific values and then work toward generalizing the rule for n. • Have students complete #24 without a calculator first. Then, use technology to verify that it cannot be done.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• If students have difficulty with C1, have them replace a with n and b with r. You may wish to have them refer to #25 to assist them with part b). • For C2, you may wish to review a suggested activity for one of the examples before having students generalize the statement in C2. You may wish to give them the letters of the word label with one l labelled l_1 and the other l labelled l_2. Tell them that an l is the same even though it may have a subscript. Visually, this may help students see that the placement of the l's does not change the word, which is the reason for dealing with repetitions. • Encourage students to show all steps in C3. Remind them that they have been asked to simplify and not to determine a value. Some students will believe that they should be able to solve for n or r in all cases.

Combinations

11.2

***Pre-Calculus 12*, pages 528–536**

Suggested Timing

90–120 min

Materials

- standard deck of cards (optional)
- coloured markers, beads, or marbles (optional)

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BLM 11–3 Section 11.2 Extra Practice
TM 11–2 How to do Page 536 #C4 Using *The Geometer's Sketchpad*®
TM 11–3 How to Do Page 536 #C4 Using *GeoGebra*

Mathematical Processes for Specific Outcomes

PCBT3 Determine the number of combinations of *n* different elements taken *r* at a time to solve problems.

- ✓ Connections (CN)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–5, 7, 9–12, and one of 17–19
Typical	#1–3, 6–14, one of 17–19, 21, C1–C3
Extension/Enrichment	#8, 14–16, 20–24, C2–C4

Planning Notes

Discuss the outcome and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.

It is important for students to be able to identify the difference between a permutation and a combination. The Investigate will help students recognize how permutations and combinations are similar and how they differ. Solving word problems requires students to first identify the arrangement as being either a permutation or a combination. Remind students to use the proper notation, write the number of arrangements using factorials, and reduce when possible.

Investigate Making Selections When Order Is Not Important

Organize the class into pairs or groups of three. Have each group answer the questions in step 1 and share their ideas and solutions with the class. Then, have the groups answer the questions in steps 2 and 3 and again share their solutions with the class.

Some groups may need some coaching. Ask leading questions such as the following:

In step 1, is there a difference in the positions being elected? What is the difference between the first and third positions? Does order matter? How many ways can you arrange 4 objects given 3 at a time when order matters?

In step 2, if all the committee members are the same, does it matter if you are selected first or third? Do you think there are more or fewer arrangements when order is not important? To help students answer this question, take them through the following investigation: Consider the four letters A, B, C, and D. Write down all the arrangements of any 3 of these 4. Circle all arrangements involving A, B, and C. How many did you circle? If order *matters*, how many arrangements are there for the letters ABC? If order *does not* matter, how many arrangements are there? Put a round bracket around any arrangements of the letters A, B, and D. How many arrangements did you bracket? If order *matters*, how many arrangements are there for the letters ABD? If order *does not matter*, how many arrangements are there? How many more arrangements exist when order matters versus when order does not matter? Is there a way to express 6 using factorial notation? How many ways are there to arrange 3 distinct objects using factorial notation? In step 2, how many ways can you arrange the three students? If there are 24 distinct arrangements of 4 objects taken 3 at a time when order is important, and 3! or 6 arrangements of the three students, then how many different arrangements are there when order is not important?

For step 3, encourage each group to develop at least two strategies. How many students shake hands at one time? Does the order of handshakes matter? If order matters, how many arrangements exist for 6 objects taken 2 at a time? How many ways can you order a handshake involving 2 people? If order does not matter, how many arrangements exist for 6 objects taken 2 at a time?

Can you think of a more concrete method to solve step 3? Can you make a table to indicate the number of handshakes for each student? How many different handshakes does the first student make? Does this list include the handshake between the first and second student? How many new handshakes does the second student make? How many new handshakes for the third student? Can you continue this list for all 6 students? Given the table, how can you determine the total number of handshakes?

In step 4, how many people shake hands at a time? Have students use permutation notation to express the number of ways you can arrange n objects taken 2 at a time when order matters. What operation would you use to change the number of arrangements where order matters into the number of arrangements where order does not matter? If the objects are taken 2 at a time, how many arrangements of 2 objects exist? By what do you divide the number of n objects taken 2 at a time?

How can you simplify the expression $\dfrac{\frac{n!}{(n-2)!}}{\frac{2!}{1}}$? What formula can you use to determine the number of handshakes for n people?

Meeting Student Needs

- If possible, have students act out the handshake question. Have them problem solve the most efficient method to count the handshakes. One student may act as a recorder, writing down the names in each pair as they shake hands.
- Create a short relay race in which an object is passed from hand to hand along a row of standing students. Does the order in which the students stand make a difference to the final outcome of the relay?

Answers

Investigate Making Selections When Order Is Not Important

1. a) Yes, since each position elected is for a different executive position.
b) 24

2. a) No. Since each position on the committee the same, the order of being selected does not matter.
b) 4
c) The number of selections of winners when order is important is 6 or 3! times greater than when order is not important.

3. a) 15
b) Example:
Method 1: Consider each student separately and determine the total number of handshakes for all 6 students. The first student shakes 5 hands, the second student shakes 4 hands, the third student shakes 3 hands, and so on. The total number of handshakes is $1 + 2 + 3 + 4 = 15$.
Method 2: Consider each handshake as a different arrangement of 6 objects taken 2 at a time. If order matters, then there are ${}_6P_2$, or 30, handshakes. Since the order of shaking hands between two people does not matter, divide ${}_6P_2$ by 2! or 2. The number of handshakes is $\frac{30}{2!} = \frac{30}{2} = 15$.

4. $\frac{{}_nP_2}{2!}$, or $\frac{n!}{(n-2)!2!}$

5. In both cases there is an understanding of n objects taken r at a time. Permutations require that the order of the grouped objects is important. Combinations involve arrangements where order of the grouped objects is not important.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • To check for understanding, you may wish to give students five cards from a standard deck of cards and ask them to arrange as many three-card hands as possible. You could then have them complete the exercise algebraically.

Example 1

For part a), how many of the total number of students in the class are to see the principal at the same time? Does the order of selection matter when students are chosen in groups to see the principal? How many different selections are possible? For part b), how many females are to see the principal at a time? How many selections are possible from the 12 females arranged 2 at a time? How many males are to see the principal at the same time? How many selections are there for the 18 males seeing the principal 3 at a time? Given the number of possible selections of females and the number of possible selections of males, what is the total number of different selections? What mathematical operation did you use to determine the number of selections including both females *and* males? In part c), since Brooklyn is female, how many selections are possible for the remaining 11 females taken 1 at a time? How many selections are possible for the 18 males to be chosen 3 at a time? How can you determine the total number of possible selections involving Brooklyn *and* 1 other female *and* 3 other males?

Example 2

How many ways can Rianna choose to answer 2 of the 4 questions in part A of the exam? How many ways can she choose to solve 3 of the 5 questions in part B? What mathematical operation is implied when using the word *and* between combinations? If Rianna has 6 choices in part A *and* 10 choices in part B, how many choices does she have for parts A and B together?

If Rianna must choose *at least* 4 of the 5 questions in part B, what two cases exist for her selections in Example 2b)? Given the case where Rianna chooses 2 of 4 questions in part A and 4 of 5 questions in part B, how many different selections does she have for this case? If Rianna chooses 2 of 4 questions in part A and 5 of 5 questions in part B, how many different selections does she have for this case? If Rianna can satisfy either the first case *or* the second case, what mathematical operation do you use to determine the total number of combinations for case 1 or case 2?

Example 3

For this example, it is important that students recognize that factorials are a product of a list of diminishing values. To reduce a factorial fraction, students need to write the largest factorial as a product of values down to the lowest factorial.

Some students may require coaching to express abstract combinations as factorials involving variables. In part a), how do you write the combination expression ${}_nC_5$ using factorials? What are the first four terms of the expansion of $(n - 1)!$? Can you write the combination ${}_{n-1}C_3$ using factorial notation? Can you write the expression $\dfrac{{}_nC_5}{{}_{n-1}C_3}$ in factorial notation? What strategy would you use to reduce the expression $\dfrac{\frac{n!}{(n-5)!5!}}{\frac{(n-1)!}{(n-4)!3!}}$?

Describe how to reduce the expressions $\dfrac{n!}{(n-1)!}$, $\dfrac{(n-4)!}{(n-5)!}$, and $\dfrac{n!}{(n-5)!5!} \times \dfrac{(n-4)!3!}{(n-1)!}$.

In part b), how do you use factorials to write the expressions $2({}_nC_2)$ and ${}_{n+1}C_3$? What strategy would you use to simplify the expression $2\left(\dfrac{n!}{(n-2)!2!}\right) = \dfrac{(n+1)!}{(n-2)!3!}$? Describe how you would rewrite the expression $n! = \dfrac{(n+1)!}{3!}$ with the variables on only one side of the equation. Can you reduce the expression $\dfrac{(n+1)!}{n!}$? How? What is the value of n?

Meeting Student Needs

- Develop an understanding that ${}_nC_k = {}_nC_{n-k}$.
- Some students will benefit from using the Combination feature on a calculator.
- In a question involving the fundamental counting principle, a key word is *and*. If event A *and* event B take place, multiply the results.
- For Example 3, students may benefit from first solving a similar problem that does not contain variables. Students may then make the transition to the more theoretical questions presented in parts a) and b).
- Have students write a short statement that compares and contrasts permutations with combinations.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Students use a permutation instead of a combination to solve a problem.

R

$\mathbf{R_x}$ Have students recall the definitions of *permutation* and *combination*. Have students write the definitions in their own words and create an example of each.

- Students do not recognize that restrictions require different cases to be developed.

R_x Have students always list the objects or items to be arranged and list any restrictions on those objects for each specific case. Then, determine the arrangements for each case. Students should recognize that questions such as "How many arrangements of case A *and* case B … ?" direct them to multiply the arrangements to get the total number of arrangements, while questions such as "How many arrangements of case A *or* case B … ?" direct them to add the number of arrangements for each case to get the total number of arrangements.

Answers

Example 1: Your Turn

a) 715 ways **b)** 210 ways

Example 2: Your Turn

531 ways

Example 3: Your Turn

a) $\frac{n-1}{n-4}$

b) $n = 5$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Students may find it easier to equate the grade 11 student with one of the females in the example. • Have students label each of the calculations, such as one grade 11 student *and* two grade 12 students. • Remind students that the process of *and* means to multiply. • It is important to assess that students understand when order is important and when it is not.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • Point out to students that *at least* usually means that there is more than one possible combination, and it implies that the solution will be read as a possible solution *or* another possible solution *or* …. The process of *or* implies adding. • If students are having difficulty determining whether to add or multiply, have them verbalize what *at least three* reds could be (3 reds *or* 4 reds *or* 5 reds).
Example 3 Have students do the Your Turn related to Example 3.	• You may wish to have students work in pairs. • Remind students that these questions are best solved by expanding each notation and looking for common factors to cancel. Notations should be simplified following the same rules as those used for rational expressions.

Check Your Understanding

Give students an opportunity to work in pairs when working on these questions.

For #1, 2, 4, and 5, encourage students to recall the difference between a permutation and a combination. How do you know whether a problem involves a permutation or a combination?

In #3, ask: Can you use the definition of ${}_nP_r$ and ${}_nC_r$ to evaluate the given expressions? What is the first step that you would use to evaluate a permutation or a combination expression? How do you reduce or simplify factorial expressions?

In #7, what restrictions exist in each part? Can you use the restrictions to determine cases you can use to solve the problem? In part a), the first number less than 1000 has how many digits? How many different numbers can you make using one digit? using two digits? using three digits? What cases can you list to solve this problem? In part b), are there any five-member teams with no grade 12 students? Does the number of grade 12 students chosen for the team change how you would determine the number of possible teams?

For #8 and 9, encourage students to rewrite each combination expression using factorials. For #9, ask students to justify their answers in their own words.

For #10 and 11, ask students whether the order of selection is important. Does the problem involve a permutation or a combination? How many objects do you have to arrange? How many objects are arranged together at the same time? Can you express the situation in combination notation? Does the problem require cases in order to solve it? If it does, what cases are involved? What mathematical operation is implied by the word *and*? What mathematical operation is implied by the word *or*?

In #12, can you rewrite the given expressions using factorial notation? What is the simplified version of the left side of the equation? Can you simplify the right side of the equation? What is required to add fractions with different denominators? What is the common denominator for the sum on the left side of the equation? How do you change each fraction so that you end up with a common denominator? Once you have rewritten the two fractions with a common denominator, can you factor the numerator of the expression? Can you simplify the remaining factorial expression? Is it the same as the mathematical expression on the right side of the equation?

For #15a), can you solve this problem by sketching a diagram? Can you solve it using combinations? How many points are required to draw a line? How many points do you have? Given that you know how many points need to be joined, and how many points are needed to create a line, can you summarize the information using combination notation? How many different lines can you draw? For #15b), how many points need to be joined to form a triangle? In #15c), does the number of points given change the strategy you would use to solve this problem? Is this question best solved by sketching a diagram or by using combinations?

In #17, does the situation involve a permutation or a combination? Are there any restrictions in part a)? How many people are available in total? How many are required for the jury? How many ways can you select a 12-person jury given 20 possible jurors? What are the restrictions in part b)? How many juries can you select if a jury must contain 7 women *and* 5 men? Does part c) involve any restrictions? What different cases do you need to solve this problem?

For #20b), how many ways can you choose 4 paintings from the 20 selected for hanging? How many ways can you arrange the 4 pictures hanging near the entrance? How can you combine this information to solve the problem?

In #21a), how many ways can you select 13 cards from a standard deck of 52 cards? How many ways can you choose the cards for the next hand? and the next? What about the last hand, how many ways can you select the cards for this hand?

For #24, does the expression *pa* represent the product of p and a, or does it represent a number with the digits p and a? What is the value of the expression $\frac{{}_5C_2}{3}$? What is the value of the expression $\frac{{}_{3(5)}C_{3(2)}}{3}$ or $\frac{{}_{15}C_6}{3}$? Do these expressions have the same remainder? Can you write the remainders as fractions?

Meeting Student Needs

- For #1, the Aboriginal tradition of a welcome circle consists of each person shaking the hand of every other person in a clockwise manner, first from inside of the circle then the outside of the circle. Each person receives and gives a handshake to everyone else.
- Once students complete #3, have them check the answers with a calculator.
- In the Apply section, students should continue to determine whether a question is a permutation or a combination.
- Provide **TM 11–2 How to Do Page 536 #C4 Using *The Geometer's Sketchpad®*** and **TM 11–3 How to Do Page 536 #C4 Using *GeoGebra*** to help students answer the question.
- Provide **BLM 11–3 Section 11.2 Extra Practice** to students who would benefit from more practice.

Enrichment

Challenge students to create an interesting, challenging, and unique problem that uses an equation involving ${}_nC_r$ notation. Have them solve the problem and check their answer with a peer.

Gifted

Encourage students to show their understanding of the similarities and differences between combinations and permutations by creating a foldable of their choice that highlights the similarities and differences, and the pros and cons of the applications of each. Finally, have students give an excellent example of each that clearly shows a sophisticated understanding of the distinction between the two terms.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1–5, 7, 9–12, and one of #17–19. Students who have no problems with these questions can go on to the remaining questions.	• Question 1 is an important assessment question for all levels of students. It is core to the chapter, since students must know when to use the permutations or combinations. Question 3 is similarly important; encourage students to evaluate the expressions without using a calculator. • Ensure that students can do #5 before moving on to any other question. If they cannot, spend time reviewing the differences between permutations and combinations. • For #8, help students see the pattern that is presented, and discuss the results as a class. This will help prepare students for the next section. • For the Apply questions, some students may find it helpful to work with coloured markers, beads, or marbles. Many students have a difficult time framing the questions, and coloured shapes or markers can assist with this. Note that #17 to 19 are similar. Letting students choose the question they complete may allow them to choose something they are familiar with. Having manipulatives available for all questions would be beneficial.
Assessment *as* Learning	
Create Connections Have all students complete C1–C3.	• Suggest to students that they reflect on their own locker combination to answer C1. Most schools provide combination locks that have three or four numbers in the combination. • If the use of a and b confuses students, have them replace the variables with n and r. In the end, however, they should generalize the rule for a and b. • You may wish to have students work in pairs for C3 and discuss their thinking.

The Binomial Theorem

11.3

***Pre-Calculus 12*, pages 537–545**

Suggested Timing

90–120 min

Materials

- counters
- copies of Pascal's triangle
- grid paper (optional)

Blackline Masters

Master 3 Centimetre Grid Paper
BLM 11–4 Section 11.3 Extra Practice

Mathematical Processes for Specific Outcomes

PCBT4 Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

☑ Connections (CN)
☑ Reasoning (R)
☑ Visualization (V)

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 3–6, 7b), d), 8, 11, 12, 16, 17a), b)
Typical	#2–6, 7a), c), e), 9–13, one of 14–16, 17c), d), 18, C1–C4
Extension/Enrichment	#7c), e), 13, 17c), 19–24, C1–C4

Planning Notes

Discuss the outcome and achievement indicators for this section. Encourage students to develop a list of the terms and definitions found within this section.

Have students do research about Blaise Pascal to learn about his significant contributions to mathematics and the development of technology.

Provide **Master 3 Centimetre Grid Paper** for students to construct a 20-row Pascal's triangle. Students should work in pairs and use a pencil to complete the diagram, since it is easy to make small errors. Have students orient their grid paper to landscape view. Then, start in the middle square of the top row to write the first value of Pascal's triangle. Make sure that students leave one square between any numbers written. It may be helpful for students to write the row numbers down the left side of the page and sums of each row down the right side of the page.

Though students may find building a Pascal's triangle to be tedious work, the activity in step 3 of the investigation is of value to help them see the relationships between one row and the next row down. If you decide not to have students create their own Pascal's triangle, build a master copy and provide each group of students with two copies. Students can use one copy to determine patterns for multiples of 7 and multiples of 5, and use the second copy to look for multiples of 3 and even numbers. Have students use either highlighters or counters to cover numbers to identify related values.

Investigate Patterns in Pascal's Triangle

This investigation helps students explore and recognize numerical patterns that exist in Pascal's triangle. Have students work in pairs.

For step 3, one student can write down all the information while the other uses a calculator to determine the sums. To identify multiples of 7, 5, 3, and even numbers, some students may require some coaching. For example, have them list the first four multiples of 7. How can you determine if a large number is a multiple of 7? What shapes occur when you identify multiples of 7? How many of these shapes did you discover? Repeat the questions for each set of multiples.

For step 4, can you write the sum of each row using exponents or powers? Can you write the number 4 as a power? What is the base of the exponent? Can you write all of the sums as powers of 2? How do you write 1 as a power of 2? What relationship do you recognize between the row number and the exponent of each power of 2?

In step 5, using combinations notation ($_nC_r$), the first term in the third row is $_2C_0$. What is the relationship between the row number and the value of n? What is the value of $_2C_0$? of $_2C_1$? of $_2C_2$? What are the values of the third row in Pascal's triangle?

In step 6, can you use the expansion of $(x + y)^2$ to expand $(x + y)^3$ and $(x + y)^4$?

Meeting Student Needs

- Organize students into small groups to work through steps 4 through 6. As a class, answer the questions in steps 7, 8, and 9.

Enrichment

Encourage students to explore the work of Pascal and others that led to the development of Pascal's triangle. What is it about the visual nature of the triangle that allows numerical patterns to be seen? What other examples of the use of visual patterns have mathematicians used over time to analyse and solve problems? Ask students to create examples that highlight this methodology as applied to mathematics.

Gifted

Encourage students to explore the questions, "What other examples are there of patterns similar to Pascal's work with regard to binomial expansion? How might technology be used to expand such patterns, and what new learning might come from this?"

Common Errors

- Students do not recognize that the sums of each row are a power of 2.

R_x Once students have determined their sums for each row, ask them to write out each sum as a product of prime factors. What do they notice about the prime factors? Ask students if they can simplify the product of factors using exponents. What do they notice about the base of each power? Can the number 1 be written as a power of 2?

Answers

Investigate Patterns In Pascal's Triangle

1. Example: All outside values are equal to 1. The middle numbers are determined by adding the two numbers above them. The number of terms in each row increases by one.

2. 1 5 10 10 5 1

3. Example: The multiples of 7 form three descending triangles, beginning with a triangle having a base in row 8 and its vertex in row 13. Two similar inverted triangles are reflected by the centre line of Pascal's triangle, beginning in row 15 and ending in row 20. Six inverted triangles are formed by the multiples of 5, with the first inverted triangle starting in row 6 and ending in row 9. Multiples of 3 form seven small inverted triangles with the first beginning in row 4. They also produce one large triangle and what appear to be two other large inverted triangles beginning in row 19. Even numbers also form many defined shapes. All shapes formed by any set of multiples are reflected by a centre line drawn down the middle of Pascal's triangle.

4. All sums of each row are powers of base 2. An expression to summarize each row's sum is 2^{r-1}, where r is the row number.

5. ${}_4C_0$, ${}_4C_1$, ${}_4C_2$, ${}_4C_3$, ${}_4C_4$

6. $(x + y)^2 = x^2 + 2xy + y^2$; $(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3$; $(x + y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$

7. Example: All rows have outside values of 1. All of the middle values are determined by adding the two values directly above each new value. For example, the sixth row of Pascal's triangle begins with 1. The next term is determined by adding the two values above it: $(1 + 4) = 5$. The third term is the sum of the two values above it: $(4 + 6) = 10$, and so on.

8. The value of the exponent for the sum of terms of a particular row is defined as 2^{r-1}, where r is the row number. The terms for each row written in the form nCr have values of $n = r - 1$, where r is the row number.

9. The coefficients of the expansion of $(x + y)^n$ are equal to the terms of the $(n + 1)^{th}$ row of Pascal's triangle.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students work in pairs for this activity. • As a class, discuss the patterns that are determined in #4 and 5 before assigning #7 and 8. • Have students compare answers to their expansions for #9. As a class, have them suggest the value of the coefficients in the expansion of $(x + y)^6$.

Example 1

This example helps students recognize the value of using Pascal's triangle or combinations to make binomial expansions much easier to determine than expanding by multiplication.

In part a), if you use Pascal's triangle, which row will produce the coefficients for $(p + q)^6$? If you use combination notation, what is the value of n for the binomial expansion of $(p + q)^6$? What relationship exists between the value of n and the exponent of the binomial? If ${}_6C_0$ and ${}_6C_1$ are the first two terms of row 7 of Pascal's triangle, what is the relationship between r and the term number? Using combinations notation, what are the seven coefficients of the expansion of $(p + q)^6$?

For part b), what relationship exists between the exponent of the binomial and the row of Pascal's triangle that determines the coefficients of the expansion? Once you have written the expansion, what do you notice about the exponents of p moving from left to right? What do you notice about the exponents of q moving left to right? What is the sum of the exponents for each term in the expansion? What do you notice about the values on either side of the number 20 in row 7 of Pascal's triangle? Is this true for every row in the triangle?

Example 2

In this example, students are expected to come up with strategies to fully expand any binomial taken to any natural number exponent, and to determine a particular term of a binomial expansion. To coach students needing help using the binomial theorem in part a), ask questions such as

- What is the full expansion of the binomial $(x + y)^4$?
- The variable x in the expansion of $(x + y)^4$ represents what expression from $(2a - 3b)^4$?
- Can you substitute $2a$ for x and $-3b$ for y in the expansion of $(x + y)^4$ to determine the expansion of $(2a - 3b)^4$?
- What relationship exists between the value of r $({}_nC_r)$ and the value of the exponent of $-3b$ in the expansion?
- What is the product of ${}_4C_0$, $(2a)^4$, and $(-3b)^0$?
- What is the simplified version of the first term ${}_4C_0(2a)^4(-3b)^0$?
- What is the second term of the expansion of $(2a - 3b)^4$ using combination notation?
- If ${}_4C_1(2a)^3(-3b)^1$ is the second term of the expansion, what is the relationship between the value of r $({}_nC_r)$ and the exponent of $-3b$? What is the relationship between the value of n $({}_nC_r)$ and the sum of the exponents of $(2a)^3$ and $(-3b)^1$?
- What strategy have you discovered to determine each term of the expansion of $(2a - 3b)^4$?

For part b), ask students

- For the binomial $(4b - 5)^6$, what is the value of n $({}_nC_r)$ for a sixth-degree expansion?
- What is the relationship between the value of r $({}_nC_r)$ and the term number?
- What is the value of r for the third term?
- Using combination notation, what is the first factor of the third term?
- What is the value of ${}_6C_2$?
- What are the values of the exponents for the factors $4b$ and -5 in the third term?
- Can you simplify the expression ${}_6C_2(4b)^2(-5)^2$?

In part c),

- If you use the expression ${}_nC_k(a^2)^{n-k}\left(\frac{-1}{a}\right)^k$ to expand $\left(a^2 - \frac{1}{a}\right)^5$, what is the value of n?
- Since the simplified form of the expansion has the variable a, what is the value of the product of $(a^2)^{5-k}$ and $\left(\frac{-1}{a}\right)^k$?
- What value of k satisfies the equation $(a^2)^{5-k}\left(\frac{-1}{a}\right)^k = a^1$?
- Can you use exponent laws to simplify the product of $(a^2)^{5-k}\left(\frac{-1}{a}\right)^k$?
- Can you simplify the expression $a^{10-2k}\left(\frac{-1}{a}\right)$? How?
- What is the simplified form of $\frac{a^{10-2k}}{a^k}$? Can you solve the equation $a^{10-3k} = a^1$ for k?
- If the value of k is 3, what is the simplified form of the expression ${}_5C_3(a^2)^2\left(\frac{-1}{a}\right)^3$?

Meeting Student Needs

- In the opening expansion, have students identify patterns. For example, what pattern is found in the coefficients? What is the degree of each term?
- In Example 1, use the knowledge that ${}_nC_k = {}_nC_{n-k}$ to help develop the patterns. What do students notice about the value of r and the second exponent in the term?
- Students could determine a pattern to follow to create an expansion:
  - Write the coefficients.
  - Write the two parts of each term.
  - Write the exponents of the first part from the value of n down to zero. Then, write the exponents of the second part from zero to the value of n.
  - Simplify.
- Ask each student to create one question, either an expansion or determining a specific term, and write a solution. Students could exchange problems with another student, complete, and then return to be checked.
- Have students record their own summary of the Key Ideas, including examples. Have them store the summary for each section in the same location for review purposes.

Common Errors

- Students do not understand the application of the general term of the binomial theorem.

R_x Ask what the relationship is between the value of n in the expression ${}_nC_k(x)^{n-k}(y)^k$ and the value of the exponent in the binomial expansion. What is the relationship between the sum of the exponents and the value of n? How are the term number of the expansion and the value of k related? Describe a strategy to determine a particular term of a binomial expansion. Have students practise their understanding by describing a specific term in the form ${}_nC_k(x)^{n-k}(y)^k$ from a list of five or six binomial expansions. Do not require them to simplify the expression, but use the experience to solidify their application of the notation ${}_nC_k(x)^{n-k}(y)^k$.

Answers

Example 1: Your Turn

a) 1 5 10 10 5 1

b) When expanding an expression of the form $(x + y)^n$, it is easier to use the terms of the $(n + 1)$ row of Pascal's triangle. When determining the value of a particular term in the expansion, using combinations seems easier.

c) 6

d) $5c^4d$

Example 2: Your Turn

a) 9 **b)** $-614\ 656a^5$

c) $2187a^7 + 10\ 206a^6b + 20\ 412a^5b^2 + 22\ 680a^4b^3 + \ldots$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish to have students work in pairs. • Refer students to the chart in the Link the Ideas to assist them in determining the coefficients of the expansion. • Remind students of the $n + 1$ rule in determining the number of terms in their expansion and hence a specific term.
Example 2 Have students do the Your Turn related to Example 2.	• You may wish to have students work in pairs. • If students are having difficulty determining the number of terms, have them expand the binomial but not evaluate each term. Help them see the pattern in the expansion and link it to the fourth term. • You may wish to review the pattern determined by the signs of each term when there is a subtraction sign in the original binomial.

Check Your Understanding

For #1, encourage students to recall how values are produced in subsequent rows of Pascal's triangle. What are the values of the end numbers in the triangle? What operation do you use to determine the value of middle numbers in any row of Pascal's triangle?

In #2, what pattern do you discover about expressing terms of Pascal's triangle using combination notation? Which value (n or r) is equal to the second term of any row in Pascal's triangle? What relationship exists between the value of r and the term number in a row of Pascal's triangle? What are the two possible values of r in the equation ${}_nC_r = 1$?

For #3, can you express each term as a combination if you know the value of the second term and the term number in a row of Pascal's triangle?

In #4, ask students what relationship exists between the exponent of a binomial and the number of terms in the expansion.

For #5 and 6, how many terms will be in the expansion of each given binomial expression? Ask students whether they prefer to use the values from Pascal's triangle, combinations, or the general term of the binomial theorem to complete binomial expansions. Ask them if their strategy changes when dealing with binomial terms such as $(3a - 4b)^3$ instead of $(a + 1)^3$.

In #10, what relationship do you notice between the value of the numbers on the handle of each hockey stick pattern and the value of the number on each stick's blade?

For #12, how do the first and last terms help you determine the values of a and b in the binomial $(a + b)^n$?

For #13, what is the value of 11^2? 11^3? 11^4? What do you notice about the digits in each of these values and the terms in a row of Pascal's triangle? What relationship exists between the value of m in 11^m and the row number in Pascal's triangle?

In #15, encourage students to recall the basic differences between a permutation and a combination. Is it possible to have no one attend the dinner? What other possible cases exist for the number of people who may attend the dinner?

In #16, what are the first two terms of the tree diagram? Describe how to draw a tree diagram for the given situation. How would you write the expression HHH and HTH as exponents?

For #19a), consider the expression $(x^2)^K\left(-\frac{2}{x}\right)^M$. What relationship must exist between K and M for the variable x to cancel out? What values of K and M have a sum of 12 where M is twice the value of K? In the product $(y)^K\left(-\frac{1}{y^2}\right)^M$, what relationship must exist between K and M for the variable y to cancel out? What values of K and M have a sum of 12 where K is twice the value of M?

For #20, can you use the general term of the binomial theorem to solve the question?

In #21, have students state by what value you multiply the first term to get the second term. By what value do you multiply the second term to get the third term? Can you use this question strategy to determine all the multipliers? What do you notice about the values of the multipliers once you get past the middle term or terms?

In #22, can you produce any value by adding the two numbers directly above a middle number? Can you produce any value by adding the two numbers on each side of a middle number? Can you produce any value by adding the two numbers below a middle number? What relationships exist between the values of the denominators of the outside numbers on the triangle?

In #23, ask students whether the expression $(a + b + c)^2$ can be expanded algebraically. What patterns do you notice between the coefficients of your expansion? Can you expand the expression $(a + b + c)^3$? What patterns do you notice between the coefficients of your expansion? Can you predict a pattern for the expansion of $(a + b + c)^4$?

For #24, can you model this situation by sketching diagrams using the number of given points? What patterns do you notice?

Meeting Student Needs

- Students should complete the checklist of understanding provided in the opener of this section. This will guide them when deciding which questions they need to practise to gain a deeper understanding of the necessary outcomes for this section.
- Consider posting questions around the classroom. Students could rotate from one station to the next, comparing solutions with other students at the same station.
- Have students create an art presentation as the Project Corner activity that follows section 11.3
- Provide **BLM 11–4 Section 11.3 Extra Practice** to students who would benefit from more practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 3–6, 7b)–d), 8, 11, 12, 16, and 17 a) and b). Students who have no problems with these questions can go on to the remaining questions.	• Students should be successful in #1 before moving on. Have them verbalize how they will determine the next row before beginning. • In #3, students must demonstrate that they know the original power of the binomial before determining the combination. • Allow students to expand one of the binomials in #4 if they are unable to determine the number of terms. Remind them of the $n + 1$ rule. This will also assist them in answering #5–7, 11, and 12.
Assessment *as* Learning	
Create Connections Have all students complete C1, C3, and C4.	• Students may wish to refer to one of #5–7 to answer C1. Allow them to work from their example back to the general terms. • Students may best verbalize their preferences to a partner for C3. Challenge them to list the similarities and differences of their responses. • Students may also wish to share their examples and problems for C4 with a partner.

11 Chapter 11 Review and Practice Test

***Pre-Calculus 12*, pages 546–548**

Suggested Timing
75–105 min

Blackline Masters
BLM 11–2 Section 11.1 Extra Practice
BLM 11–3 Section 11.2 Extra Practice
BLM 11–4 Section 11.3 Extra Practice
BLM 11–5 Chapter 11 Study Guide
BLM 11–6 Chapter 11 Test

Planning Notes

Have students who are not confident discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test. You may wish to provide students with **BLM 11–5 Chapter 11 Study Guide**, which links the achievement indicators to the questions on the Chapter 11 Practice Test in the form of self-assessment. This master also provides locations in the student resource where students can review specific concepts in the chapter.

The practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum that will meet the related curriculum outcomes: #1–13.

Meeting Student Needs

- Students should select questions in the Review depending on the outcomes that require further study.
- Cut out squares of three different colours. Number squares of the first colour 1 to 7, squares of the second colour 8 to 14, and squares of the third colour 15 to 20. Have pairs of students draw three questions from each colour (replacing the squares) to complete. Other questions can then be attempted when students review them at home.
- You may wish to have students complete the practice test prior to working through the Review questions. Once students mark the practice test, they can then focus on completing just the necessary questions in the Review.
- Students who require more practice on a particular topic may refer to **BLM 11–2 Section 11.1 Extra Practice**, **BLM 11–3 Section 11.2 Extra Practice**, and **BLM 11–4 Section 11.3 Extra Practice**.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 11 Review The Chapter 11 Review provides an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource. **Minimum:** #1–5, 7–9, 11, 12, 14, and 16–18	• You may wish to have students work in small groups of similar ability. • Encourage students to use **BLM 11–5 Chapter 11 Study Guide** to identify areas they may need some extra work in before starting the practice test.
Chapter 11 Study Guide This master will help students identify and locate reinforcement for skills that are developed in this chapter.	• Encourage students to use the practice test as a guide for any areas in which they require further assistance. The minimum questions suggested are questions that students should be able to confidently answer. Encourage students to try additional questions. • Consider allowing students to use any summative charts, concept maps, or graphic organizers when completing the practice test.
Assessment *of* Learning	
Chapter 11 Test After students complete the practice test, you may wish to use **BLM 11–6 Chapter 11 Test** as a summative assessment. **Minimum:** #1–13	• Before the test, coach students in areas in which they are having difficulty. • You may wish to have students refer to **BLM 11–5 Chapter 11 Study Guide** and identify areas they need reinforcement in before beginning the chapter test.

Unit 4 Project Wrap-Up

UNIT 4

***Pre-Calculus 12*, page 549**

Suggested Timing
60–90 min

Blackline Masters
Master 1 Holistic Project Rubric
Master 2 Ana-Holistic Project Rubric
BLM U4–1 Unit 4 Project Checklist

Mathematical Processes
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

General Outcome
Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes
RF1 Demonstrate an understanding of operations on, and compositions of, functions.
RF14 Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials or trinomials).

General Outcome
Develop algebraic and numeric reasoning that involves combinatorics.

Specific Outcomes
PCBT1 Apply the fundamental counting principle to solve problems.
PCBT2 Determine the number of permutations of *n* elements taken *r* at a time to solve problems.
PCBT3 Determine the number of combinations of *n* different elements taken *r* at a time to solve problems.
PCBT4 Expand powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers).

Planning Notes

Ensure students are aware of the Unit 4 project information provided on page 427, the Project Corners on pages 456, 498, and 545, and the Unit 4 Project Wrap-Up on page 549.

Have students use **BLM U4–1 Unit 4 Project Checklist** to make sure that all parts of their project have been completed. As a class, brainstorm different ways students can do their presentations. You may wish to limit the time each student is allowed to present.

You may wish to work with the class to create a specific rubric for the project using either **Master 1 Holistic Project Rubric** or **Master 2 Ana-Holistic Project Rubric** as a template. Review the general holistic points within the 1–5 scoring levels. Discuss with students how they might achieve each of these levels in the Unit 4 Project. A completed rubric in each style for this project is available at www.mcgrawhill.ca/school/learningcentres by following the links. Note that these are just samples; your class rubric may have more detail.

Ask questions such as the following:
- What are the big ideas in the unit? (For example, one big idea is that Pascal's triangle and combinations can be used to determine the coefficients of a binomial expansion.)
- Which of the big ideas are involved in the project?
- What part of the project could you complete or get partially correct to indicate that you have a basic understanding of what was learned in the three chapters? (Should you get a pass mark if you can perform operations on rational functions having numerators and denominators that are monomials, binomials, or trinomials?)
- What would be on a level 1 project? What might you start on correctly? What could be considered a significant start?
- What would be expected for a level 5 project? What should it include? Try to help students realize that a level 5 project may have a minor error or omission that does not affect the final result.

- Knowing the expectations of levels 1, 3, and 5 projects, what would be expected for a level 4? Help students understand that this is still an honours level and therefore the work should be reflective of this. However, even an honours project may have a minor error or omission. Discuss the difference between a major conceptual error and a minor miscalculation or omission. Understanding this point will help clarify for students the expectations and differences between a pass and an above-average result, and may encourage some students to work toward the highest level. Repeat the process for level 2.

Use the rubric to ensure that students understand the criteria for an acceptable level, as well as what would warrant either an unacceptable or an honours grading.

Assessment	Supporting Learning
Assessment *of* Learning	
Unit 4 Project This unit project gives students an opportunity to apply and demonstrate their knowledge of the following: • sketching the graph of a function that is the sum, difference, product, or quotient of two functions, given their graphs • writing the equation of a function that is the sum, difference, product, or quotient of two or more functions, given their equations • writing a function as the sum, difference, product or quotient of two or more functions • determining, given the equations of two functions, the equations of related composite functions and explaining any restrictions • graphing and analyzing rational functions (limited to numerators and denominators that are monomials, binomials or trinomials) • solving problems by applying the fundamental counting principle, determining the number of permutations of *n* elements taken *r* at a time, or determine the number of combinations of *n* different elements taken *r* at a time • expanding powers of a binomial in a variety of ways, including using the binomial theorem (restricted to exponents that are natural numbers) Work with students to develop assessment criteria for this project. **Master 1 Holistic Project Rubric** and **Master 2 Ana-Holistic Project Rubric** provide descriptors that will assist you in assessing students' work on the Unit 4 Project.	• You may wish to have students use **BLM U4–1 Unit 4 Project Checklist**, which provides a list of the required components for the Unit 4 Project. • Reviewing the Project Corner boxes at the end of some sections of Chapters 9 to 11 will assist students in developing appropriate data presentations. • Make sure students recognize what is expected for the minimum requirements for an acceptable project as well as the difference between level 5 and level 4. • Clarify the expectations and the scoring with students using **Master 1 Holistic Project Rubric**, **Master 2 Ana-Holistic Project Rubric**, or the rubric you develop as a class. It is recommended that you review the scoring rubric at the beginning of the project, as well as intermittently throughout the project to refresh student understanding of the project assessment.

UNIT 4

Cumulative Review and Test

***Pre-Calculus 12*, pages 550–553**

Suggested Timing
60–90 min

Blackline Masters
BLM U4–2 Unit 4 Test

Planning Notes

Have students work independently to complete the review, and then compare their solutions with those of a classmate. Alternatively, you may wish to assign the cumulative review to reinforce the concepts, skills, and processes learned so far. If students encounter difficulties, provide an opportunity for them to share strategies with other students. Encourage them to refer to their notes, and then to the specific section in the student resource. Once they have determined a suitable strategy, have students add it to their notes. Consider having students make a list of questions they found difficult. They can then use the list to help them prepare for the unit test.

Meeting Student Needs

- Have students review the checklist containing the learning outcomes for Unit 4. Students who require more practice on a particular topic may refer to the extra practice BLM for the relevant chapter and section.
- Encourage students to review their own summary of the key ideas, including examples, presented in each of Chapters 9, 10, and 11.

Assessment	Supporting Learning
Assessment *for* Learning	
Cumulative Review, Chapters 9–11 The cumulative review provides an opportunity for students to assess themselves by completing selected questions pertaining to each chapter and checking their answers against the answers in the back of the student resource.	• Have students review their notes from each chapter to identify topics they had problems with, and do the questions related to those topics. Have students do at least one question that tests skills from each chapter. • Have students revisit any chapter section they are having difficulty with. • You may wish to have students review the study guide blackline masters from Chapters 9, 10, and 11 as well as practice tests and chapter tests they completed to help identify any skill areas that still require reinforcement.
Assessment *of* Learning	
Unit 4 Test After students complete the cumulative review, you may wish to use the unit test on pages 552 and 553 as a summative assessment.	• Consider allowing students to use their graphic organizers. • You may wish to have students complete **BLM U4–2 Unit 4 Test**, which provides a sample unit test. You may wish to use it as written or adapt it to meet the needs of your students. The answers to this unit test are on **BLM 11–7 Chapter 11 BLM Answers**.