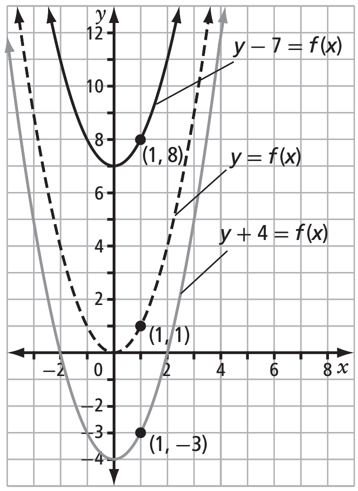
Chapter 1 Function Transformations

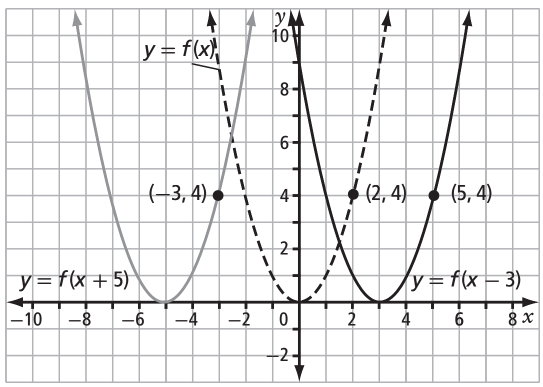
1.1 Horizontal and Vertical Translations

KEY IDEAS

- A translation can move the graph of a function up or down (vertical translation) and right or left (horizontal translation). A translation moves each point on the graph by the same fixed amount so that the location of the graph changes but its shape and orientation remain the same.



- A vertical translation of function $y = f(x)$ by k units is written $y - k = f(x)$. Each point (x, y) on the graph of the base function is mapped to $(x, y + k)$ on the transformed function. Note that the sign of k is opposite to the sign in the equation of the function.
  - If k is *positive*, the graph of the function moves *up*. Example: In $y - 7 = f(x)$, $k = 7$. Each point (x, y) on the graph of $y = f(x)$ is mapped to $(x, y + 7)$. If $f(x) = x^2$, as illustrated, (1, 1) maps to (1, 8).
  - If k is *negative*, the graph of the function moves *down*. Example: In $y + 4 = f(x)$, $k = -4$. Each point (x, y) on the graph of $y = f(x)$ is mapped to $(x, y - 4)$. If $f(x) = x^2$, (1, 1) maps to (1, −3).



- A horizontal translation of function $y = f(x)$ by h units is written $y = f(x - h)$. Each point (x, y) on the graph of the base function is mapped to $(x + h, y)$ on the transformed function. Note that the sign of h is opposite to the sign in the equation of the function.
  - If h is *positive*, the graph of the function shifts to the *right*. Example: In $y = f(x - 3)$, $h = 3$. Each point (x, y) on the graph of $y = f(x)$ is mapped to $(x + 3, y)$. If $f(x) = x^2$, (2, 4) maps to (5, 4).
  - If h is *negative*, the graph of the function shifts to the *left*. Example: In $y = f(x + 5)$, $h = -5$. Each point (x, y) on the graph of $y = f(x)$ is mapped to $(x - 5, y)$. If $f(x) = x^2$, (2, 4) maps to (−3, 4).

- Vertical and horizontal translations may be combined. The graph of $y - k = f(x - h)$ maps each point (x, y) in the base function to $(x + h, y + k)$ in the transformed function.

Working Example 1: Graph Translations of the Form $y - k = f(x - h)$

a) For $f(x) = |x|$, graph $y + 6 = f(x - 4)$ and give the equation of the transformed function.

b) For $f(x)$ as shown, graph $y + 5 = f(x + 2)$.

Solution

a) For $f(x) = |x|$, the transformed function $y + 6 = f(x - 4)$ is represented by $y + 6 = |x - 4|$.

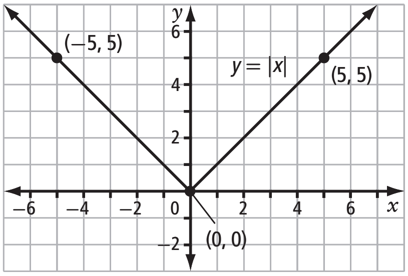
$h =$ ________ means a horizontal translation ________ units to the ________.
(*left* or *right*)

$k =$ ________ means a vertical translation ________ units ________.
(*up* or *down*)

The parameters *h* and *k* have the opposite signs to what appear in the equation.

Key points: (x, y) maps to $(x + h, y + k)$

(x, y)	→	$(x + h, y + k)$
(−5, 5)	→	
(0, 0)	→	
(5, 5)	→	



Add these points to your graph and draw in the lines. Be sure to continue the lines to the edge of the graph.

b) The function $y = f(x)$ shown in the graph below will be transformed as follows:
$y + 5 = f(x + 2)$.

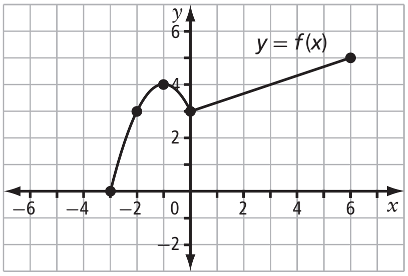
The translated function should be congruent to the base function.

$h =$ ________ means a horizontal translation ________ units to the ________.
(*left* or *right*)

$k =$ ________ means a vertical translation ________ units ________.
(*up* or *down*)

Choose key points from the graph (e.g., maximum and minimum values, endpoints) and map them to new coordinates under the transformation. Then, graph the new function.

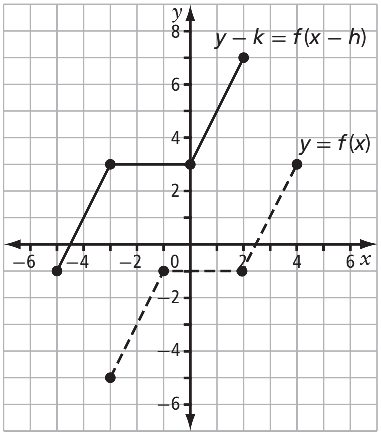
(x, y)	→	$(x + h, y + k)$
(−3, 0)	→	



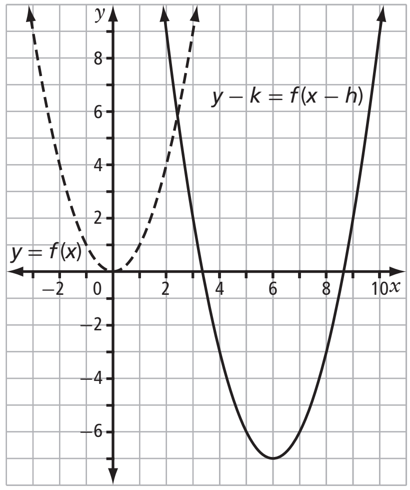
Working Example 2: Determine the Equation of a Translated Function

Determine an equation of the form $y - k = f(x - h)$ given the following graphs of $f(x)$ and of the transformed function.

a)



b)



Solution

a) Verify that the shapes are congruent by comparing slopes and lengths of line segments.

Identify key points in the base function and where they are mapped to in the translation.

Work backward from the graph to determine the parameters h and k.

$(x, y) \rightarrow$ ________________

$h =$ ________ $k =$ ________

This function is not easily described with an equation, so continue to call the base function $y = f(x)$. The equation describing the transformed function is:

b) Verify that the shapes are congruent by looking at the step pattern, starting at the vertex.

Identify key points (e.g., maximum and minimum values, intercepts).

$(x, y) \rightarrow$ ________________

$h =$ ________ $k =$ ________

What is the equation of the base function? (Hint: What kind of function is it?)

What is the equation of the transformed function?

Adding k to both sides of the equation $y - k = (x - h)^2$ will give the equation of a parabola in vertex form. Verify that this works.

Also see Example 3 on pages 10 and 11 of *Pre-Calculus 12*.

Check Your Understanding

Practise

1. Identify the values of the parameters h and k for each of the following functions.

a) $y = f(x - 10)$ $h =$ ________ $k =$ ________

b) $y - 3 = f(x + 2)$ $h =$ ________ $k =$ ________

c) $y = f(x - 17) + 13$ $h =$ ________ $k =$ ________

d) $y + 7 = (x + 1)^2$ $h =$ ________ $k =$ ________

e) $y - 4 = |x|$ $h =$ ________ $k =$ ________

You may need to rearrange the equation before answering.

2. Given $h = 2$ and $k = -5$, write an equation for each transformed function $y - k = f(x - h)$.

a) $f(x) = x^2$

b) $f(x) = |x|$

c) $f(x) = \frac{1}{x}$

3. Describe, using mapping notation, how the graphs of the following functions can be obtained from the graph of $y = f(x)$. Then, describe each transformation in words.

a) $y = f(x - 25)$ $(x, y) \rightarrow$ ________________

This represents a ________________ translation ________________ by ________ units.
(*horizontal* or *vertical*) (*right*/*left*/*up*/*down*)

b) $y + 50 = f(x)$ $(x, y) \rightarrow$ ________________

This represents a ________________ translation ________________ by ________ units.
(*horizontal* or *vertical*) (*right*/*left*/*up*/*down*)

c) $y - 10 = f(x + 20)$ $(x, y) \rightarrow$ ________________

This represents a ____.

See also #8 on page 13 of *Pre-Calculus 12*.

4. Given the graph of $y = f(x)$, graph the transformed function on the same set of axes. Write the transformation using mapping notation.

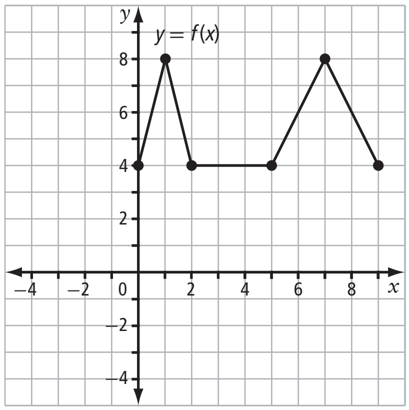
a) Graph $y + 7 = f(x + 2)$.

$h =$ ________ means a horizontal translation ________ units to the ________.
(*left* or *right*)

$k =$ ________ means a vertical translation ________ units ________.
(*up* or *down*)

Key points: (x, y) maps to $(x + h, y + k)$

(x, y)	$\rightarrow$	$(x + h, y + k)$
(0, 4)	$\rightarrow$	

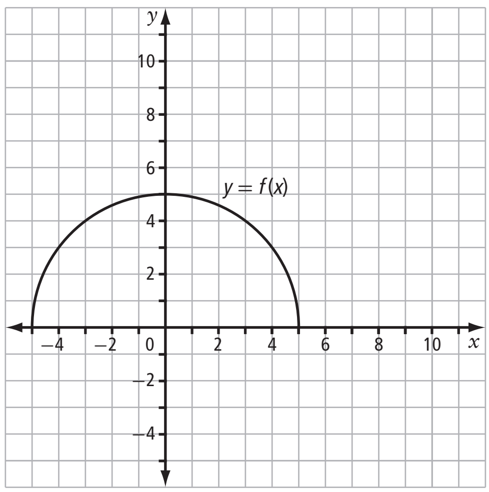


Verify that your mapping is correct by checking that the translated function is congruent to the base.

b) Graph $y + 2 = f(x - 5)$.

Key points:

(x, y)	$\rightarrow$	$(x + h, y + k)$



Apply

5. The graph of the function $f(x) = x^2$ is translated 6 units to the right and 4 units down to form the transformed function $y = g(x)$.

a) Identify the values of the parameters h and k. $h =$ ________ $k =$ ________

b) Write the transformation $f(x) \rightarrow g(x)$ using mapping notation.

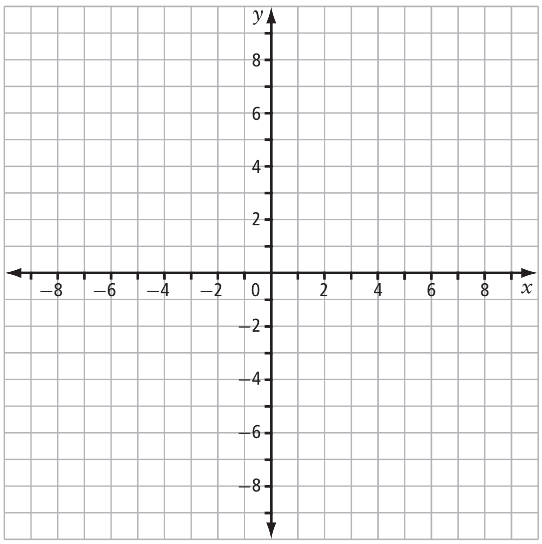
____

c) Determine the equation of the function $y = g(x)$. ____________________________

d) Graph $f(x)$ and $g(x)$ on the same set of axes.

Key points:

(x, y)	$\rightarrow$	$(x + h, y + k)$



e) Compare the vertex of $f(x)$ to that of $g(x)$. What do you notice?

Vertex of $f(x)$:

Vertex of $g(x)$:

f) Compare the domain and range of $f(x)$ to those of $g(x)$. What do you notice?

Domain of $f(x)$:

Domain of $g(x)$:

Range of $f(x)$:

Range of $g(x)$:

6. The graph of the function $f(x) = \frac{1}{x}$ is translated 5 units to the left and 2 units up to form the transformed function $y = g(x)$.

a) Identify the values of the parameters h and k. $h =$ ________ $k =$ ________

b) Write the transformation $f(x) \rightarrow g(x)$ using mapping notation.

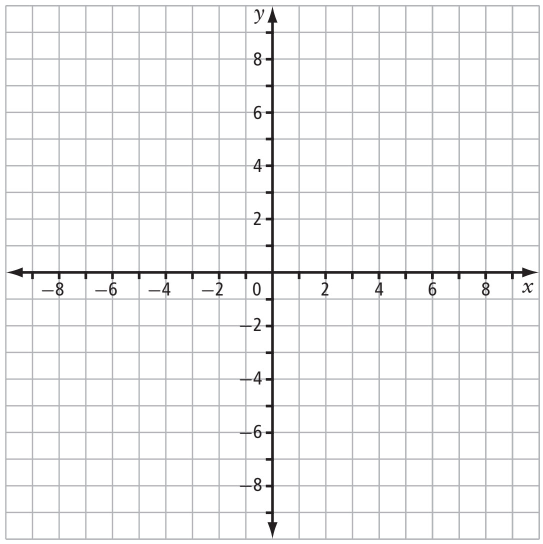
____

c) Determine the equation of the function $y = g(x)$. ____________________________

d) Graph $f(x)$ and $g(x)$ on the same set of axes.

Key points:

(x, y)	$\rightarrow$	$(x + h, y + k)$



e) Compare the domain, range, and asymptotes of $f(x)$ to those of $g(x)$. What do you notice?

Domain of $f(x)$:

Domain of $g(x)$:

Range of $f(x)$:

Range of $g(x)$:

Horizontal asymptote of $f(x)$:

Horizontal asymptote of $g(x)$:

Vertical asymptote of $f(x)$:

Vertical asymptote of $g(x)$:

Connect

7. Complete the table using equations, mapping notation, and diagrams. Be sure to include information on the location of key features (such as vertex and asymptotes) where applicable.

	Horizontal Translation		Vertical Translation	
Function	**to the right 1 unit**	**to the left 3 units**	**up 2 units**	**down 4 units**
Quadratic $y = x^2$				
Absolute value $y = \|x\|$				
Reciprocal $y = \frac{1}{x}$				
Any function $y = f(x)$				

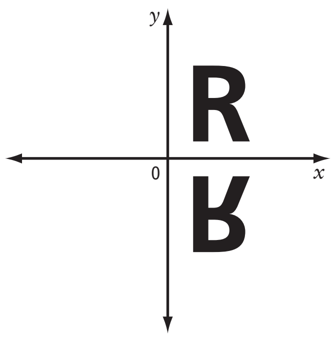
1.2 Reflections and Stretches

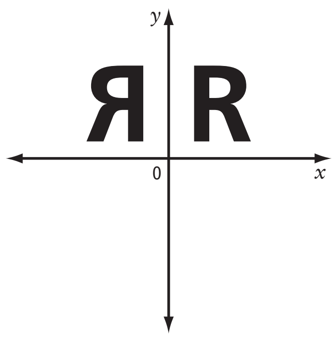
KEY IDEAS

- A reflection creates a mirror image of the graph of a function across a line of reflection. Any points where the function crosses the line of reflection do not move (invariant points). A reflection may change the orientation of the function but its shape remains the same.

Vertical reflection:
- $y = -f(x)$
- $(x, y) \rightarrow (x, -y)$
- line of reflection: x-axis
- also known as a reflection in the x-axis

Horizontal reflection:
- $y = f(-x)$
- $(x, y) \rightarrow (-x, y)$
- line of reflection: y-axis
- also known as a reflection in the y-axis

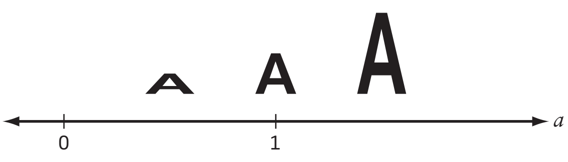


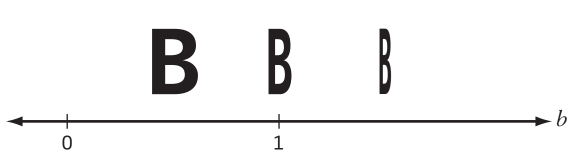


- A stretch changes the shape of a graph but not its orientation. A vertical stretch makes a function shorter (compression) or taller (expansion) because the stretch multiplies or divides each y-coordinate by a constant factor while leaving the x-coordinate unchanged. A horizontal stretch makes a function narrower (compression) or wider (expansion) because the stretch multiplies or divides each x-coordinate by a constant factor while leaving the y-coordinate unchanged.

Vertical stretch by a factor of $|a|$:
- $y = af(x)$ or $\frac{1}{a}y = f(x)$
- $(x, y) \rightarrow (x, ay)$
- shorter: $0 < |a| < 1$
- taller: $|a| > 1$

Horizontal stretch by a factor of $\frac{1}{|b|}$:
- $y = f(bx)$
- $(x, y) \rightarrow \left(\frac{1}{b}x, y\right)$
- wider: $0 < |b| < 1$
- narrower: $|b| > 1$





Working Example 1: Graph Reflections of a Function $y = f(x)$

Given $y = f(x)$, graph the indicated transformation on the same set of axes. Give the mapping notation representing the transformation. Identify any invariant points.

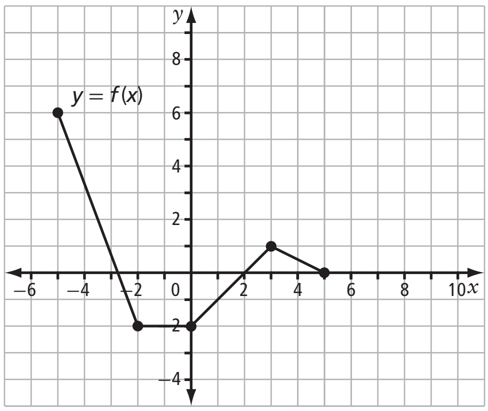
a) $y = f(-x)$ **b)** $y = -f(x)$

Solution

a) $y = f(-x)$ represents a ______________ reflection of the function in the ________ -axis.
(*horizontal* or *vertical*) (x or y)

Key points:

(x, y)	$\rightarrow$	
$(-6, 6)$	$\rightarrow$	
$(-2, -2)$	$\rightarrow$	
$(0, -2)$	$\rightarrow$	
$(3, 1)$	$\rightarrow$	
$(5, 0)$	$\rightarrow$	

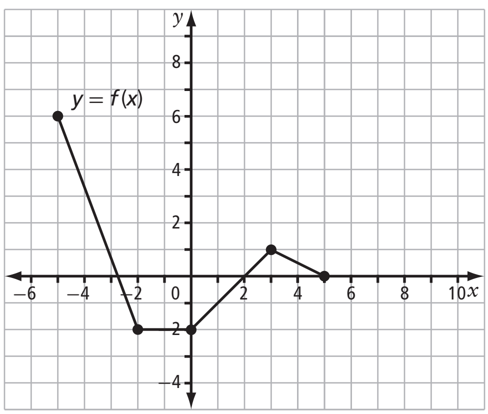


Invariant point(s): ______________

b) $y = -f(x)$ represents a ______________ reflection of the function in the ________-axis.
(*horizontal* or *vertical*) (x or y)

Key points:

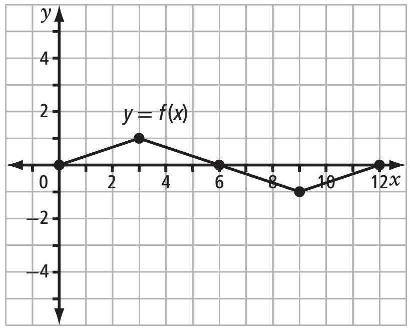
(x, y)	$\rightarrow$	
$(-6, 6)$	$\rightarrow$	
$(-2, -2)$	$\rightarrow$	
$(0, -2)$	$\rightarrow$	
$(3, 1)$	$\rightarrow$	
$(5, 0)$	$\rightarrow$	



Invariant point(s): ______________

Working Example 2: Graph Vertical and Horizontal Stretches of a Function *y* = *f*(*x*)

Given $y = f(x)$, graph $y = 5f(3x)$ on the same set of axes.
Give the mapping notation representing the transformation.



Solution

For $y = 5f(3x)$,

$a =$ ________ represents a vertical stretch by a factor of ________.
Will the new graph be shorter or taller than the graph of the base function?

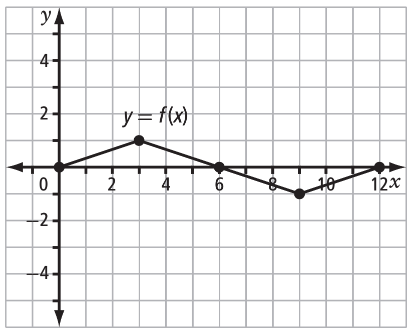
$b =$ ________ represents a horizontal stretch by a factor of ________.
Will the new graph be wider or narrower than the graph of the base function?

Apply the transformations in two stages: vertical stretch first, followed by the horizontal stretch. Graph using key points at the end of each stage. Use a different colour for each stage.

Does the order matter?

Vertical stretch by a factor of 5, followed by a horizontal stretch by a factor of $\frac{1}{3}$:

(x, y)	$\rightarrow$		$\rightarrow$	
$(0, 0)$				
$(3, 1)$				

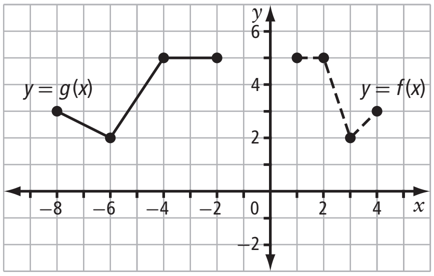


Compare the domain and range of the base function to those of the image. How many patterns (width-wise) could fit in the width of the base function?

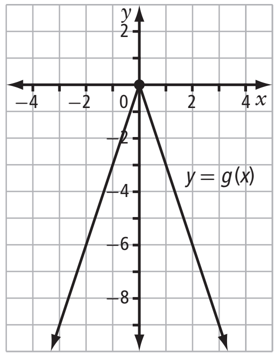
Working Example 3: Write the Equation of a Transformed Function

The graph of the function $y = f(x)$ has been transformed by a series of stretches and/or reflections. Write the equation of the transformed function $g(x)$.

a)



b)



Solution

a) Key points:

$f(x)$	→	$g(x)$
(1, 5)	→	(−2, 5)
(2, 5)	→	
(3, 2)	→	
(4, 3)	→	
(x, y)	→	

Has the orientation changed (reflection)?

In which direction?

Has the shape changed (stretch)?

In which direction?

By how much?

Equation: ______________________________

b) The base function $f(x)$ is not shown. What must it be? Add it to the graph.

Key points:

$f(x)$	→	$g(x)$
(−3, 3)	→	
(0, 0)	→	
(3, 3)	→	
(x, y)	→	

Has the orientation changed (reflection)?

In which direction?

Has the shape changed (stretch)?

In which direction?

By how much?

Equation: ______________________________

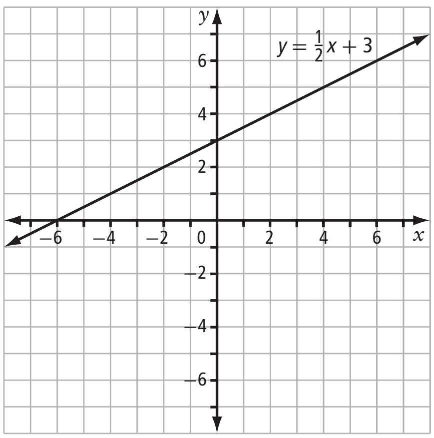
How can you tell whether $g(x)$ is narrower or taller than $f(x)$? Does it matter? What other common function has this property?

Check Your Understanding

Practise

1. Graph the horizontal reflection (reflection in the y-axis) of each function. State the equation of the reflected function in simplified form. Note any features of the function that change and any that stay the same.

a)



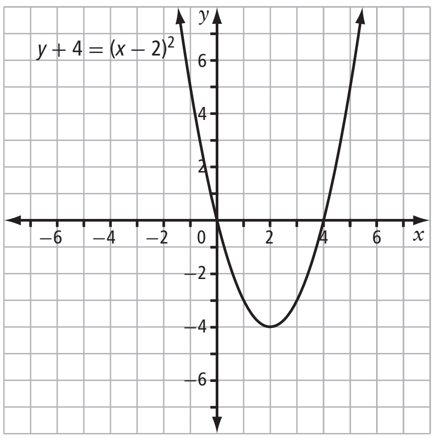
Equation of function: $y = \frac{1}{2}x + 3$

Equation of reflected function:

____

Notes:

b)



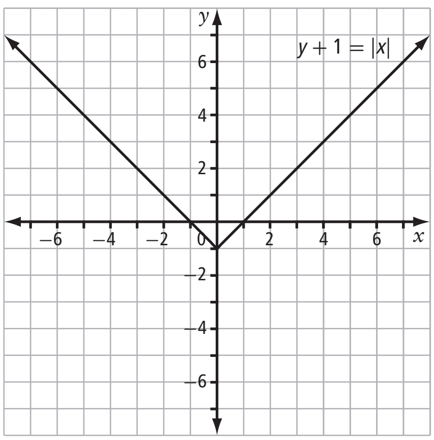
Equation of function: $y + 4 = (x - 2)^2$

Equation of reflected function:

____

Notes:

c)



Equation of function: $y + 1 = |x|$

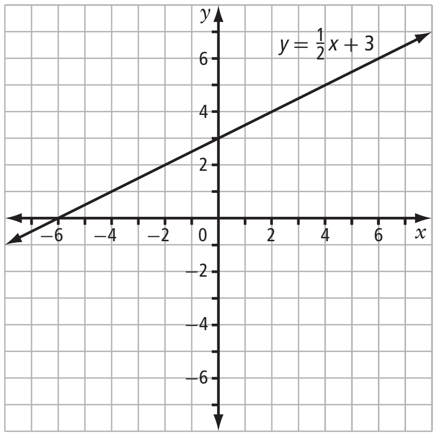
Equation of reflected function:

____

Notes:

2. Graph the vertical reflection (reflection in the x-axis) of each function. State the equation of the reflected function in simplified form. Note any features of the function that change and any that stay the same.

a)



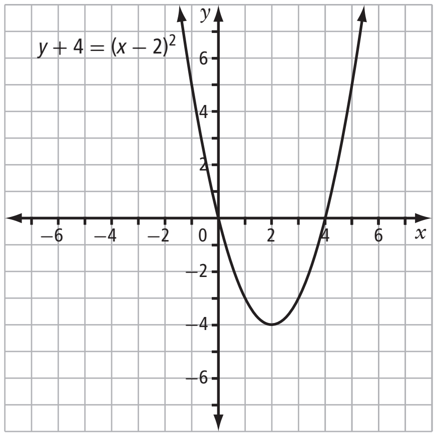
Equation of function: $y = \frac{1}{2}x + 3$

Equation of reflected function:

____

Notes:

b)



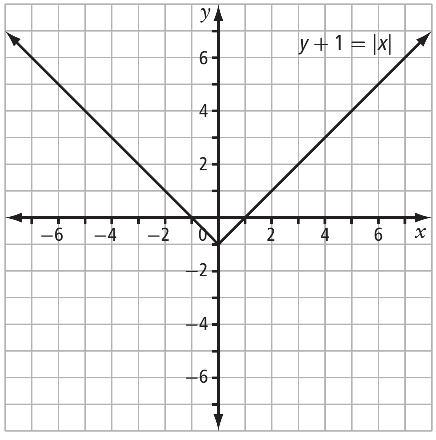
Equation of function: $y + 4 = (x - 2)^2$

Equation of reflected function:

____

Notes:

c)



Equation of function: $y + 1 = |x|$

Equation of reflected function:

____

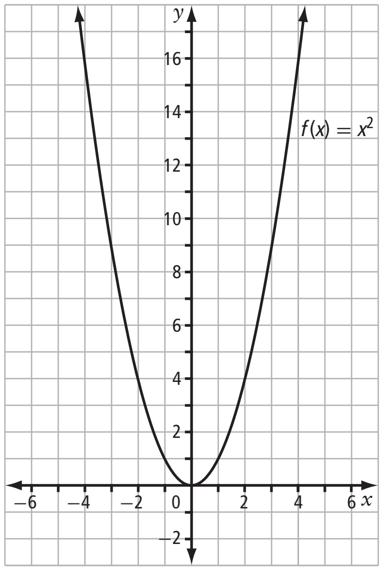
Notes:

3. Given $f(x) = x^2$, graph the following transformations. Give the equation and mapping notation for each transformation.

a) vertical stretch by a factor of $\frac{1}{4}$

Key points: (x, y) maps to (x, ay)

(x, y)	$\rightarrow$	
$(0, 0)$	$\rightarrow$	
$(\pm 1, 1)$	$\rightarrow$	
$(\pm 2, 4)$	$\rightarrow$	
$(\pm 3, 9)$	$\rightarrow$	
$(\pm 4, 16)$	$\rightarrow$	

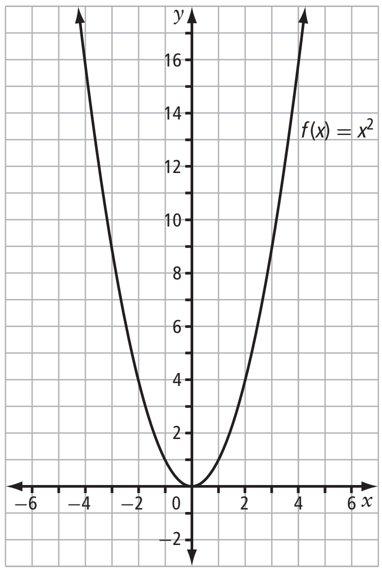


Equation: ____________________

b) horizontal stretch by a factor of 2 (b = reciprocal of the stretch factor)

Key points: (x, y) maps to $\left(\frac{1}{b}x, y\right)$

(x, y)	$\rightarrow$	
$(0, 0)$	$\rightarrow$	
$(\pm 1, 1)$	$\rightarrow$	
$(\pm 2, 4)$	$\rightarrow$	
$(\pm 3, 9)$	$\rightarrow$	
$(\pm 4, 16)$	$\rightarrow$	



Equation: ____________________

4. Compare your answers in parts a) and b) of #3.

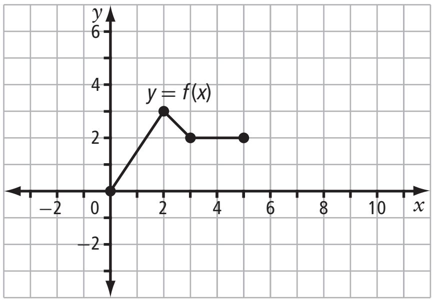
a) Show algebraically why both transformations result in the same transformed function.

b) Give another example of a pair of horizontal and vertical stretches that would result in the same transformed function.

Apply

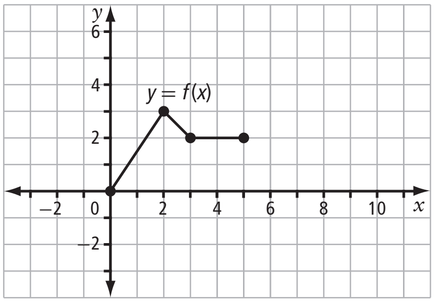
5. Write an equation representing each of the following transformations of $y = f(x)$. Then, graph each transformation.

a) vertical stretch by a factor of 2



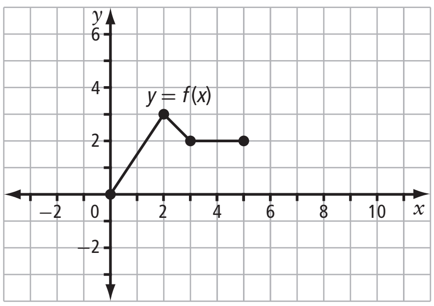
Equation of transformed function:

b) reflection in the x-axis and horizontal stretch by a factor of 2



Equation of transformed function:

c) reflection in the y-axis and horizontal stretch by a factor of $\frac{1}{2}$



Equation of transformed function:

Recall that $y = f(bx)$ results in a horizontal stretch of $\frac{1}{|b|}$.

Connect

6. Use sketches, graphs, equations, mapping notation, and words to describe how to achieve the following transformations to the graph of $y = x^2$.

Shorter	Taller

Narrower	Wider

1.3 Combining Transformations

KEY IDEAS

- Types of transformations include stretches, reflections, and translations.
- Multiple transformations can be applied to the same function. The same order of operations followed when you work with numbers (sometimes called BEDMAS) applies to transformations: first multiplication and division (stretches, reflections), and then addition/subtraction (translations).

  $y - k = af(b(x - h))$

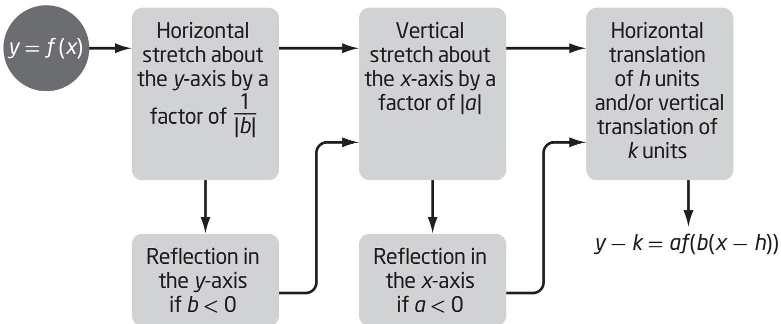
- The following three-step process will help you to keep organized.

  Step 1: horizontal stretch by a factor of $\frac{1}{|b|}$ followed by reflection in the y-axis if $b < 0$

  Step 2: vertical stretch by a factor of $|a|$ followed by reflection in the x-axis if $a < 0$

  Step 3: horizontal and/or vertical translations (h and k)

$$(x, y) \rightarrow \left(\frac{1}{b}x, y\right) \rightarrow \left(\frac{1}{b}x, ay\right) \rightarrow \left(\frac{1}{b}x + h, ay + k\right)$$



Working Example 1: Combinations of Transformations

Graph each of the following transformed functions. Show each stage of the transformation in a different colour and label each stage.

a) $y + 2 = -\left|\frac{1}{3}x - \frac{4}{3}\right|$ **b)** $y - 5 = \frac{1}{2}f(-x)$

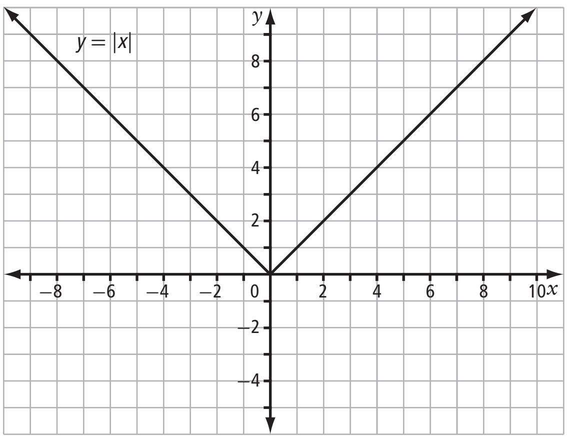
Solution

a) The transformations applied to $y = |x|$ to obtain $y + 2 = -\left|\frac{1}{3}x - \frac{4}{3}\right|$ are, in order,

i) __________________________

ii) reflection in the x-axis

iii) __________________________



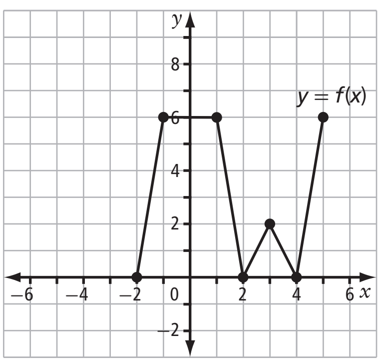
Be sure to factor the input of the function so that it is in the form $f[b(x - h)]$.

b) The transformations applied to $y = f(x)$ to obtain $y - 5 = \frac{1}{2}f(-x)$ are, in order,

i) reflection in the y-axis

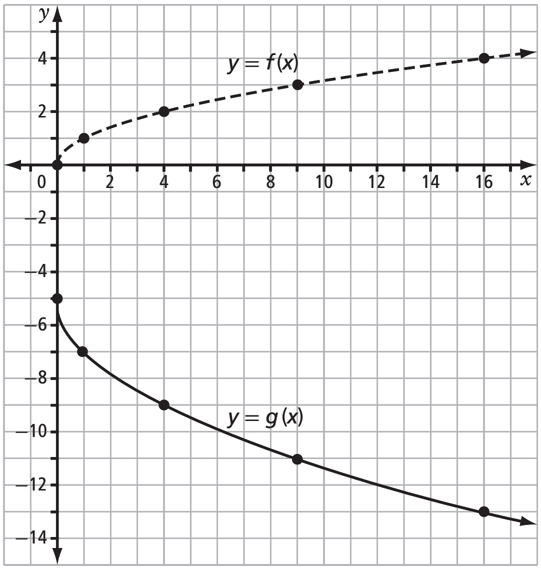
ii) vertical stretch by a factor of ________

iii) __________________________



Working Example 2: Determine the Equation of a Translated Function

Determine an equation for $g(x)$ of the form $y - k = af(b(x - h))$ given the graphs of $y = f(x)$ and of the transformed function $y = g(x)$.



Solution

Work backward through the three stages of transformations.

Horizontal and vertical translations:
Any points $(x, 0)$ on the graph of $f(x)$ are unaffected by vertical stretches and reflections.

Any points $(0, y)$ on the graph of $f(x)$ are unaffected by horizontal stretches and reflections.

So, the key point $(0, 0)$ on the graph of $f(x)$ can be used to determine the horizontal and vertical translations.

In the equation for $g(x)$, $h =$ ________ and $k =$ ________.

Before proceeding, add a sketch to the graph above that shows $g(x)$ without translations.

Vertical stretches and reflections:
- Is the transformed function reflected across the x-axis? (Y/N)
- Is the transformed function the same shape as the base function? (Y/N)

If the answer to the second question is no, measure the vertical distance between the x-axis and key points on $f(x)$. Then, measure the vertical distance between the x-axis and the image points on $g(x)$ and subtract the value of k. Compare the vertical distances.

In $g(x)$, $a =$ ________.

Horizontal stretches and reflections:
- Is the transformed function reflected in the y-axis? (Y/N)
- Is the transformed function the same shape as the base function? (Y/N)

If the answer to the second question is no, measure the horizontal distance between the y-axis and key points on $f(x)$. Then, measure the horizontal distance between the y-axis and the image points on $g(x)$ and subtract the value of h. Compare the horizontal distances.

In $g(x)$, $b =$ ________.

Now, put the transformations together.

Equation representing $g(x)$: ____________________________

Also see Example 3 on pages 37 and 38 of *Pre-Calculus 12*.

Check Your Understanding

Practise

You may need to factor the equation before answering.

1. Describe, in order, the transformations represented by each equation.

   a) $y + 5 = 4f(-x)$

   i)

   ii)

   iii)

   b) $y = -f(2x + 14)$

   i)

   ii)

   iii)

   c) $y = 1.75f[0.25(x - 1.5)]$

   i)

   ii)

   iii)

   d) $y - 3 = -\frac{1}{2}f(-3x - 3)$

   i)

   ii)

   iii)

2. Determine the equation of each transformed function.

   a) $y = f(x)$ is stretched horizontally by a factor of 6, reflected in the x-axis, and translated 7 units down.

   b) $y = |x|$ is reflected in the y-axis, stretched vertically by a factor of $\frac{1}{2}$, and translated 3 units to the right.

   c) $y = x^2$ is reflected in the x-axis, stretched horizontally by a factor of 3, and translated so that the vertex is at $(10, -4)$.

3. The key point $(1, 10)$ is on the graph of $y = f(x)$. Determine the coordinates of its image point under each transformation.

   a) $y + 4 = f(x - 5)$

   $(x, y) \rightarrow$

   $(1, 10) \rightarrow$

   b) $y = -f(x + 12)$

   $(x, y) \rightarrow$

   $(1, 10) \rightarrow$

   c) $y = 3f(-0.5x + 10)$

   $(x, y) \rightarrow$

   $(1, 10) \rightarrow$

This is similar to #6 on page 39 of *Pre-Calculus 12*.

4. If the key point $(-2, -8)$ is on the graph of $y = f(x)$, determine the coordinates of its image point under each of the transformations in #3.

Apply

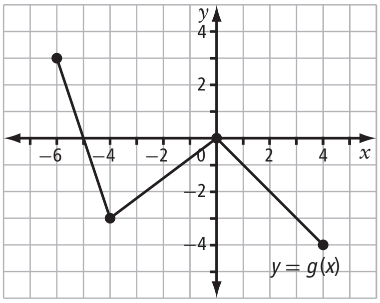
5. The graph of the function $y = g(x)$ is given. Graph each of the following transformations of the function. Show each stage of the transformation in a different colour.

a) $y + 2 = -g(2x)$

i)

ii)

iii)

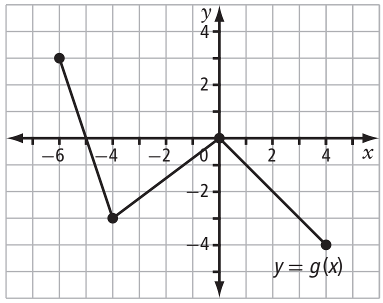


b) $y = g(-4x + 12)$

i)

ii)

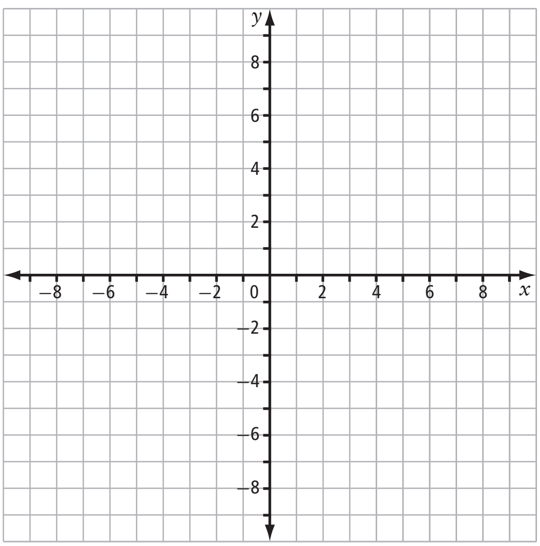
iii)



6. The graph of the function $f(x) = |x|$ is stretched vertically by a factor of 2, reflected in the x-axis, and translated 6 units to the left and 3 units down to form the transformed function $y = g(x)$.

a) Determine the equation of the function $y = g(x)$.

b) Graph $y = g(x)$.

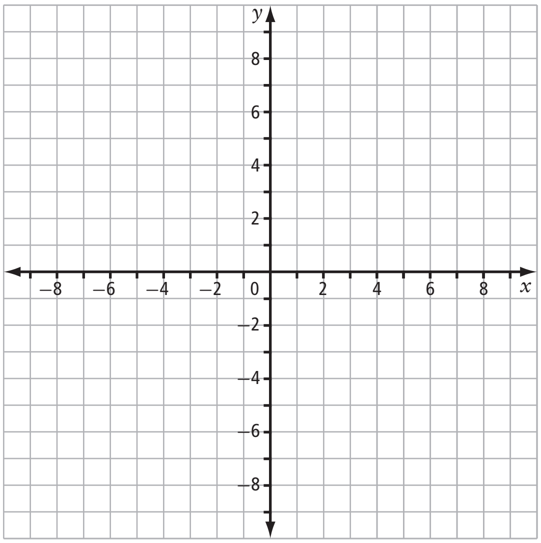


Start by graphing the base function $y = f(x)$.

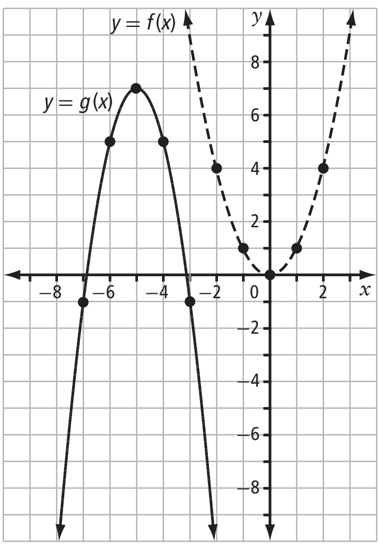
7. The graph of the function $f(x) = \frac{1}{x}$ is stretched horizontally by a factor of 4, reflected in the x-axis, and translated 4 units to the right and 1 unit down to form the transformed function $y = g(x)$.

a) Determine the equation of the function $y = g(x)$.

b) Graph $y = g(x)$.

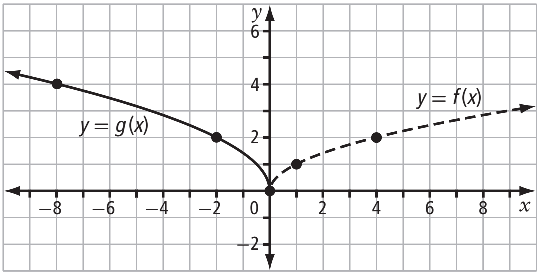


8. Determine an equation for $g(x)$ of the form $y - k = af(b(x - h))$ given the graphs of $y = f(x)$ and the transformed function $y = g(x)$.



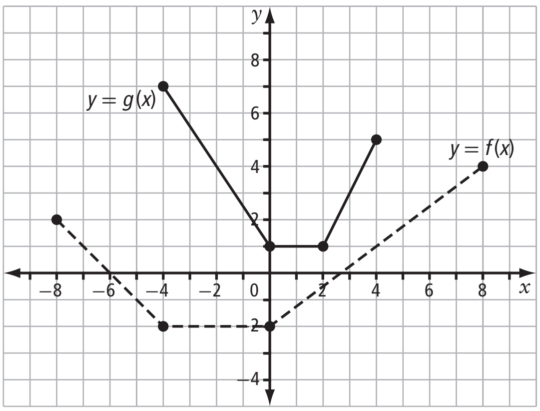
Equation:

9. Determine an equation for $g(x)$ of the form $y - k = af(b(x - h))$ given the graphs of $y = f(x)$ and the transformed function $y = g(x)$.



Equation:

10. Determine an equation of the form $y - k = af(b(x - h))$ given the following graphs of $y = f(x)$ and of the transformed function $y = g(x)$.



Consider each of the possible types of transformations in reverse order: translations, vertical stretches and reflections, and horizontal stretches and reflections.

For additional similar questions, see #10 on page 40 of *Pre-Calculus 12*.

Connect

11. Choose your transformations:

a) horizontal stretch by a factor of ________ (a = ________); horizontal reflection? (Y/N)

b) vertical stretch by a factor of ________ (b = ________); vertical reflection?(Y/N)

c) translations by ____

Write equations representing the transformed function after each stage of the transformation. Simplify each equation if necessary.

Function	Horizontal Stretch and/or Reflection	Vertical Stretch and/or Reflection	Translations
$y = f(x)$			
$y = x$			
$y = \|x\|$			
$y = x^2$			
$y = \frac{1}{x}$			

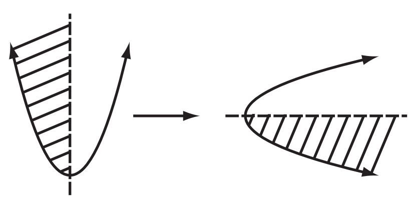
1.4 Inverse of a Relation

KEY IDEAS

- The inverse of a function $y = f(x)$ is denoted $y = f^{-1}(x)$ if the inverse is a function. The -1 is not an exponent because f represents a function, not a variable. You have already seen this notation with trigonometric functions. Example: $\sin^{-1}(\theta)$, where $f(\theta) = \sin(\theta)$ and the variable is θ.
- The inverse of a function reverses the processes represented by that function. For example, the process of squaring a number is reversed by taking the square root. The process of taking the reciprocal of a number is reversed by taking the reciprocal again.
- To determine the inverse of a function, interchange the x- and y-coordinates.

$(x, y) \rightarrow (y, x)$
or
$y = f(x) \rightarrow x = f(y)$
or
reflect in the line $y = x$

- When working with an equation of a function $y = f(x)$, interchange x for y. Then, solve for y to get an equation for the inverse. If the inverse is a function, then $y = f^{-1}(x)$.
- If the inverse of a function is not a function (recall the vertical line test), restrict the domain of the base function so that the inverse becomes a function. You will see this frequently with quadratic functions. For example, the inverse of $f(x) = x^2$, $x \geq 0$, is $f^{-1}(x) = \sqrt{x}$. The inverse will be a function only if the domain of the base function is restricted.
- Restricting the domain is necessary for any function that changes direction (increasing to decreasing, or vice versa) at some point in the domain of the function.



Working Example 1: Determine the Inverse of a Relation

Determine the inverse of the given relation when it is described as

a) an equation **b)** a table of values **c)** a graph

Solution

a) $f(x) = -\frac{2}{3}x + 5$

Steps:

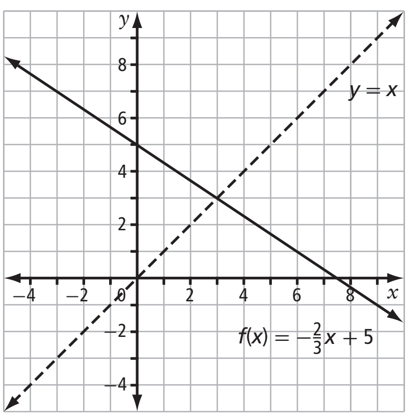
1. Substitute y for $f(x)$. $y = -\frac{2}{3}x + 5$
2. Interchange x and y. ________ $= -\frac{2}{3}$ ________ $+ 5$
3. Solve for y.

4. If the inverse is a function, substitute $f^{-1}(x)$ for y.

b) Key points: x- and y-coordinates are interchanged

(x, y)	→	(y, x)
$(-3, 7)$	→	
$(0, 5)$	→	
$(3, 3)$	→	
$(6, 1)$	→	

c) The inverse of $f(x)$ is the reflection in the line $y = x$. Choose key points and interchange the x- and y-coordinates.



Identify any invariant point(s): ________________

Note that the equation, table of values, and graph all represent the same function. It is a good idea when working with inverses to verify your algebra using a graph.

What happens to the y-intercept?
What happens to the x-intercept?

Is the inverse of a linear function always a function?

Working Example 2: Determine the Equation of the Inverse of a Quadratic Function

Determine algebraically the equation of the inverse of the function $f(x) = (x + 3)^2 - 1$. Verify graphically that the relations are inverses of each other.

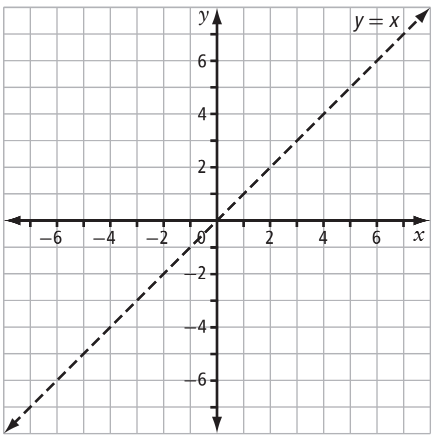
Solution

Let $y = f(x)$. To determine the inverse, interchange x and y (everywhere it says x, write y, and vice versa).

Solve for y.

When taking the square root of both sides, be sure to take the square root of the whole expression on each side.

Next, create the graph of $f(x) = (x + 3)^2 - 1$ and its inverse. Compare the graph to your solution above to verify that your algebra is correct (compare, for example, coordinates of the vertex and direction of opening).



To graph the inverse, choose some key points from the base function and interchange the *x*- and *y*-coordinates.

Is the inverse a function? (Y/N)

Divide the inverse into two branches (+ and −) at the vertex. Do the same for the base function.

In the base function, the equation of the axis of symmetry is

$x =$ ________

Restrict the domain to $\{x \mid x \geq$ ________$, x \in \mathrm{R}\}$.

Restricting the domain of $f(x)$ to the positive branch of the original parabola ($x \geq -3$) gives only the positive root from the equation of the inverse relation $y = \pm\sqrt{x + 1} - 3$.

Therefore, for the function $f(x) = (x + 3)^2 - 1$, $x \geq -3$, the inverse is

$f^{-1}(x) =$ ____________________________.

Alternatively, you could choose the negative branch and write that the inverse of the function:

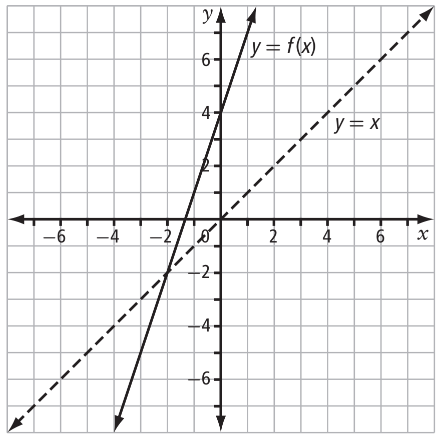
$f(x) = (x + 3)^2 - 1$, $x \leq -3$, is $f^{-1}(x) =$ ____________________________.

Check Your Understanding

Practise

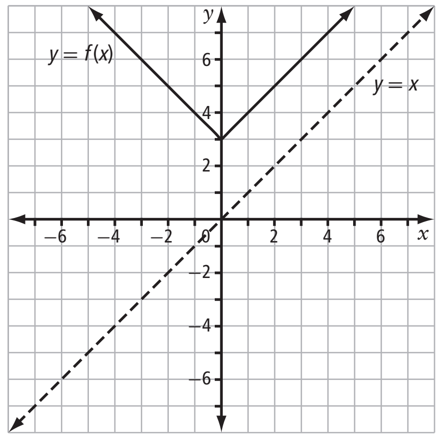
1. Graph the inverse relation of each function below. Determine whether the inverse is a function. Identify any invariant points.

a)



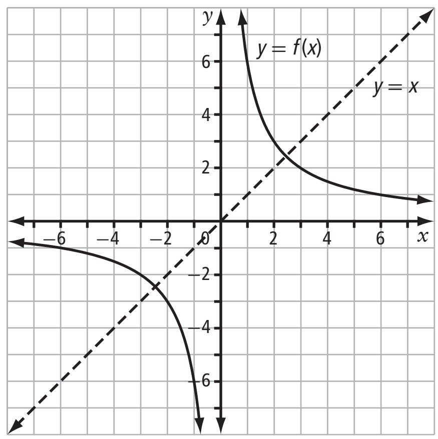
The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

b)



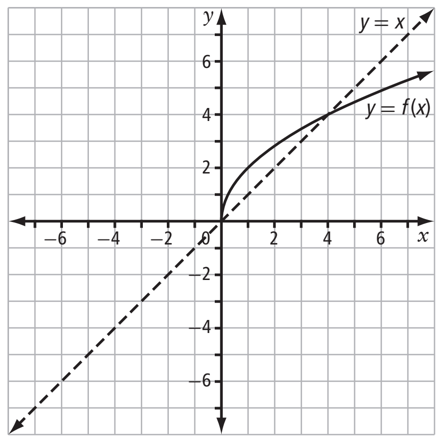
The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

c)



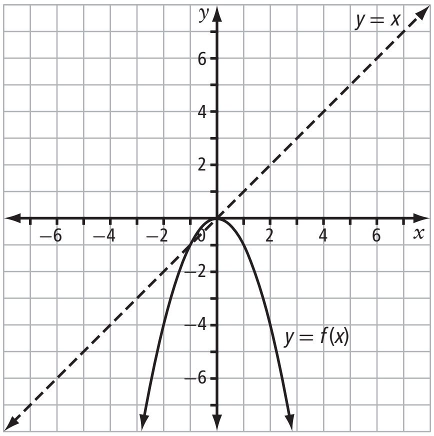
The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

d)



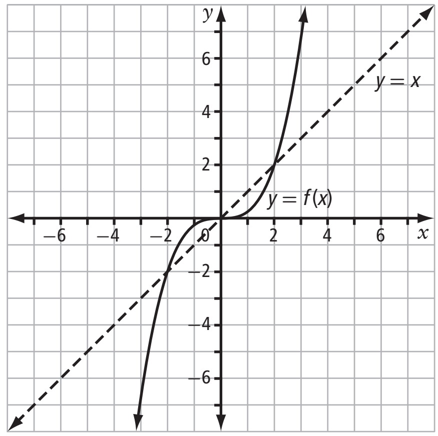
The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

e)



The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

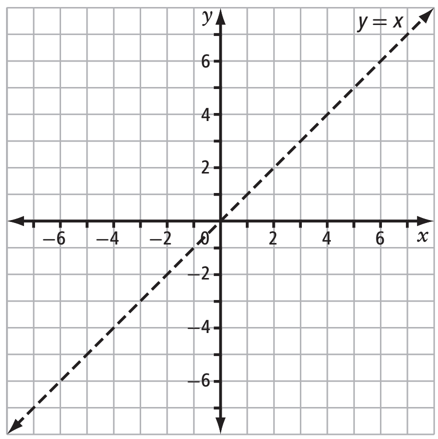
f)



The inverse of $f(x)$ ________ a function.
(*is* or *is not*)

2. Determine algebraically the inverse of each function. Verify by sketching the graph of the function and its inverse.

a) $f(x) = x - 4$



Steps:

1. Substitute y for $f(x)$.

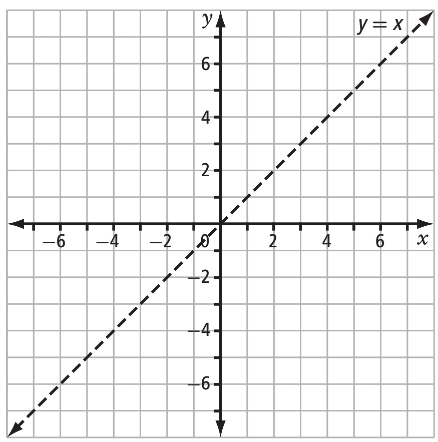
2. Interchange x and y.

3. Solve for y.

4. Restrict the domain if necessary. Then, substitute $f^{-1}(x)$ for y.

The inverse of $f(x) = x - 4$ is $f^{-1}(x) =$ ________________.

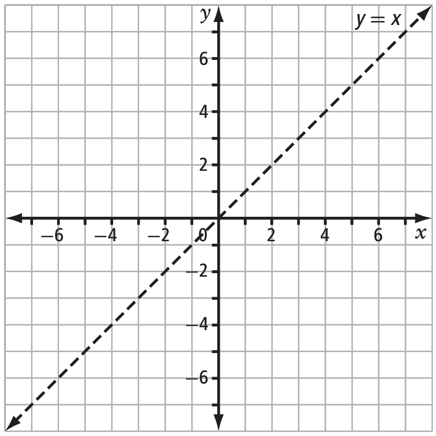
b) $f(x) = -6x - 2$



The inverse of $f(x) = -6x - 2$ is

$f^{-1}(x) =$ ________________.

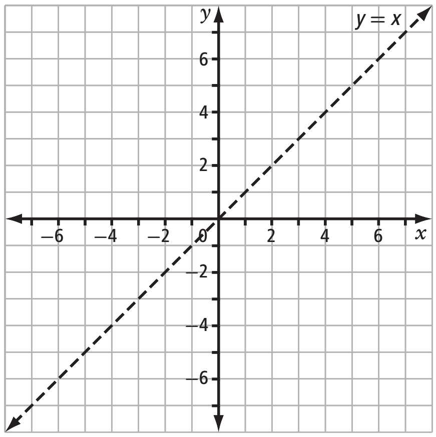
c) $f(x) = \frac{3}{5}x - 3$



The inverse of $f(x) = \frac{3}{5}x - 3$ is

$f^{-1}(x) =$ ________________.

d) $f(x) = \frac{1}{2}(x + 6)$

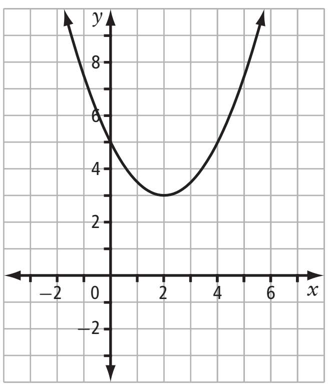


The inverse of $f(x) = \frac{1}{2}(x + 6)$ is

$f^{-1}(x) =$ ________________.

3. For each graph, identify a restricted domain for which the function has an inverse that is also a function.

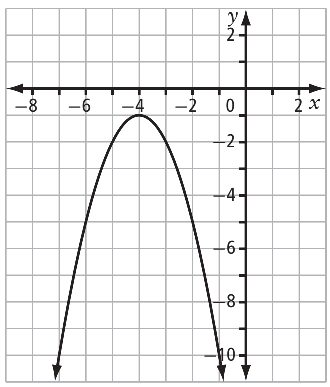
a)



Axis of symmetry: ________________

Domain: ____________________________

b)



Axis of symmetry: ________________

Domain: ____________________________

4. Determine algebraically the inverse of each function. Restrict the domain of the base function so that the inverse is a function. Verify by sketching the graph of the function and its inverse.

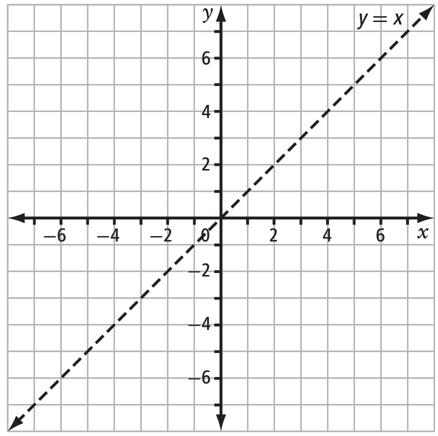
a) $f(x) = -x^2 + 6$

Steps:

1. Substitute y for $f(x)$.

2. Interchange x and y.

3. Solve for y.



4. Restrict the domain if necessary. Then, substitute $f^{-1}(x)$ for y.

The inverse of $f(x) = -x^2 + 6$, ________________, is $f^{-1}(x) =$ ________________________.

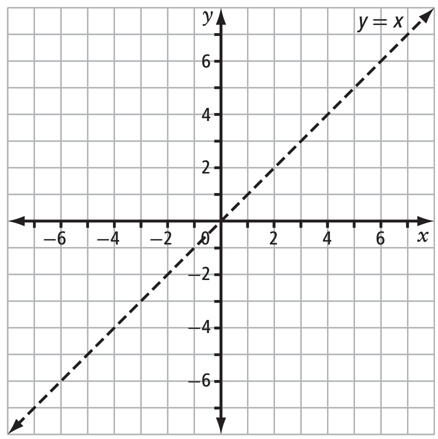
b) $f(x) = \frac{1}{2}x^2 + 4$

Steps:

1. Substitute y for $f(x)$.

2. Interchange x and y.

3. Solve for y.



4. Restrict the domain if necessary. Then, substitute $f^{-1}(x)$ for y.

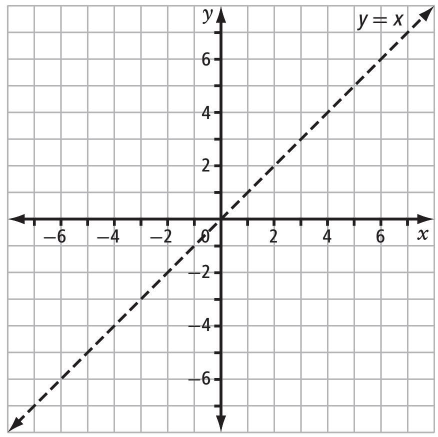
The inverse of $f(x) = \frac{1}{2}x^2 + 4$, ________________, is $f^{-1}(x) =$ ________________________.

For more practice with quadratics, try #12 on page 54 of *Pre-Calculus 12*.

Apply

5. Determine the equation of the inverse of $f(x) = x^2 + 6x + 7$. Verify by sketching the graph of the function and its inverse.

Hint: Complete the square.



See also #10 on page 53 of *Pre-Calculus 12*.

6. One of the factors that doctors use to determine the age of a fetus is the crown-to-rump length (CRL) measured during an ultrasound. A recent study determined that the average CRL, in millimetres, of a fetus with gestational age x days could be represented by the function $f(x) = 0.016\,34(x - 26.643)^2$. This formula applies between 6 and 15 weeks of gestation.

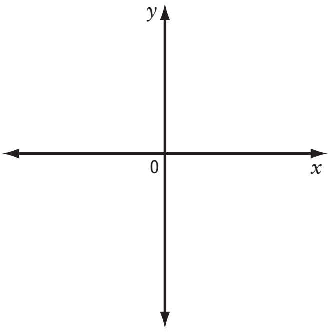
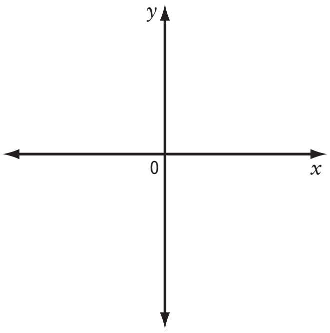
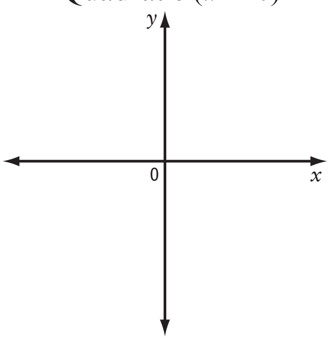
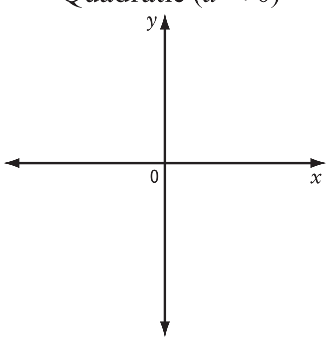
a) What are the restrictions on the domain of this function?

b) Determine an equation that would allow a doctor to determine gestational age, in days, if the CRL, in millimetres, is known.

c) If the CRL of a fetus is 7.4 cm, predict the gestational age in weeks.

Connect

7. Complete the table using words, equations, and diagrams. A few prompts are included to help you get started.

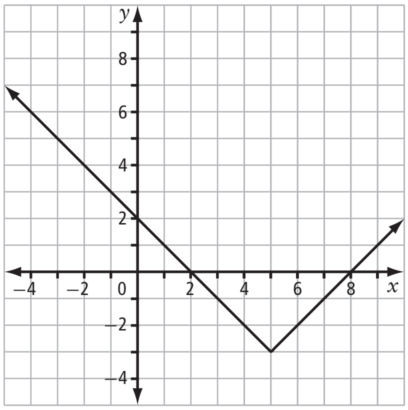
$f(x)$ is …	Key Features of $f(x)$	$f^{-1}(x)$ is …	Key Features of $f^{-1}(x)$
Linear (slope > 0) 	slope: y-intercept: x-intercept:		slope: y-intercept: x-intercept:
Linear (slope < 0) 			
Quadratic ($a > 0$) 	vertex:		vertex:
Quadratic ($a < 0$) 			

Chapter 1 Review

1.1 Horizontal and Vertical Translations, pages 1–8

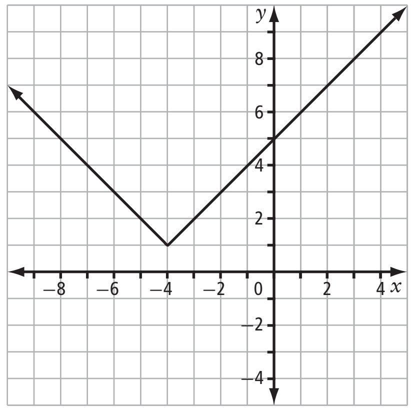
1. Write an equation to represent each translation of the function $y = |x|$.

a)



Equation: ______________________

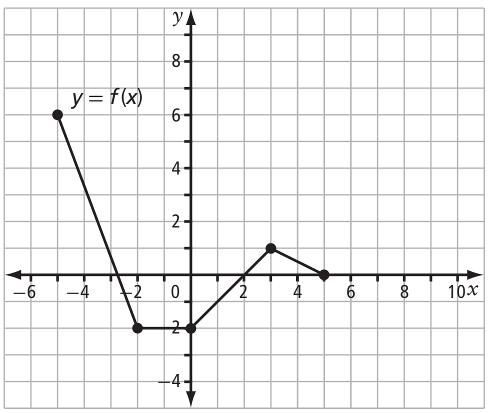
b)



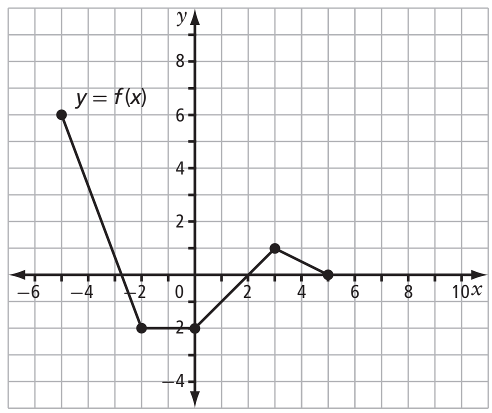
Equation: ______________________

2. For $y = f(x)$ as shown, graph the following.

a) $y - 2 = f(x - 3)$



b) $y + 2 = f(x + 1)$



1.2 Reflections and Stretches, pages 9–17

3. The key point $(12, -5)$ is on the graph of $y = f(x)$. Determine the coordinates of its image point under each transformation.

a) $y = -f(x)$

$(x, y) \rightarrow$

$(12, -5) \rightarrow$

b) $y = f(-4x)$

$(x, y) \rightarrow$

$(12, -5) \rightarrow$

c) $y = 2f\left(\frac{1}{3}x\right)$

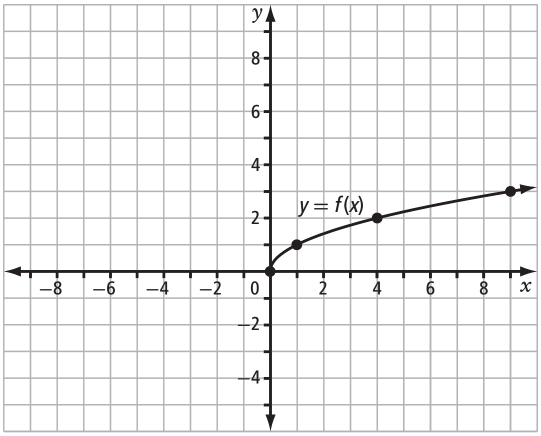
$(x, y) \rightarrow$

$(12, -5) \rightarrow$

4. Describe the following transformations of $y = f(x)$ and sketch a graph of each transformation.

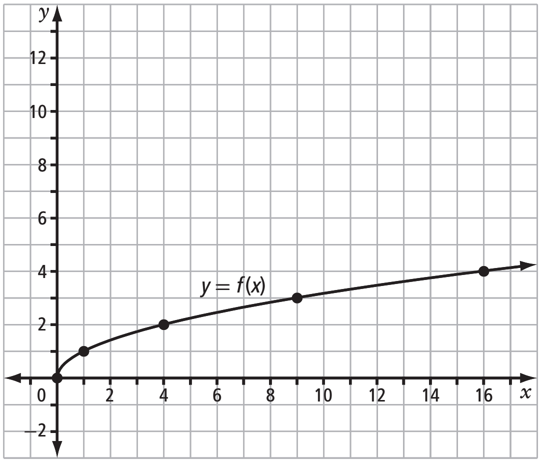
a) $y = -f(-x)$

Description:



b) $y = 3f(2x)$

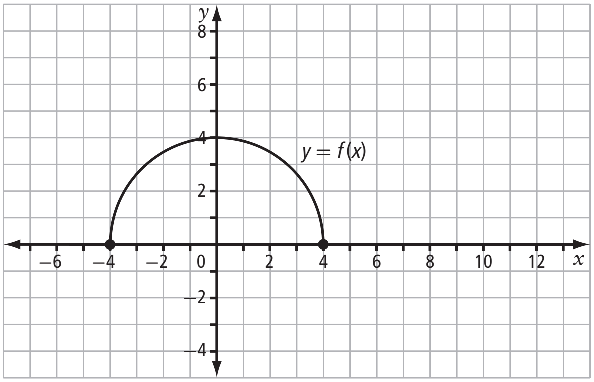
Description:



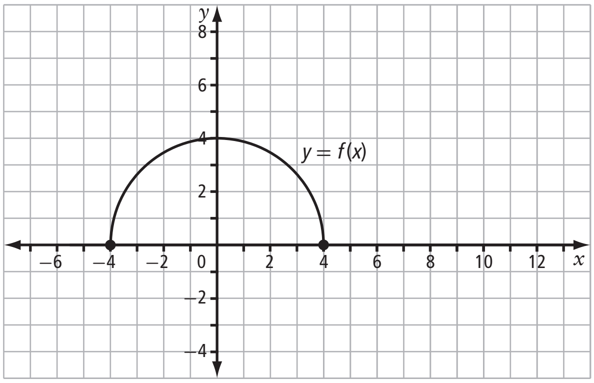
1.3 Combining Transformations, pages 18–25

5. The graph of the function $y = f(x)$ is given. Graph each of the following transformations of the function. Show each stage of the transformation in a different colour.

a) $y - 5 = \frac{1}{2}f\left(\frac{2}{3}(x - 6)\right)$



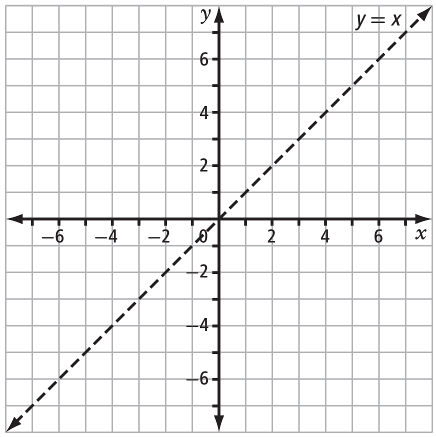
b) $y = -f(4x + 12)$



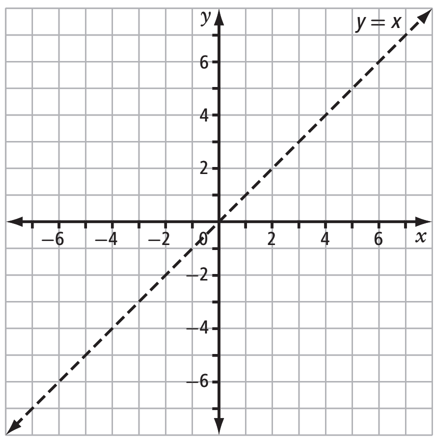
1.4 Inverse of a Relation, pages 26–34

6. Determine algebraically the inverse of each function. If necessary, restrict the domain so that the inverse of $f(x)$ is also a function. Verify by sketching the graph of the function and its inverse.

a) $f(x) = -\frac{1}{2}x + 5$



b) $f(x) = 2(x - 1)^2$



Chapter 1 Skills Organizer

Complete the tables to review the key concepts you have learned in this chapter about transformations and functions.

Transformation of Functions

Transformation	Parameter	Effect on Graph of $y = f(x)$		
		Location	Shape	Orientation
Vertical translation	________ > 0			
	________ < 0			
Horizontal translation	________ > 0			
	________ < 0			
Reflection: $y = -f(x)$	________ < 0			
Reflection: $y = f(-x)$	________ < 0			
Vertical stretch	factor of $\lvert\_\_\_\_\_\_\rvert$			
Horizontal stretch	factor of $\frac{1}{\lvert\_\_\_\_\_\_\rvert}$			
Order of transformations: ________________________ ________________________ ________________________				

Inverse of a Relation

Function	Inverse
$y = f(x)$	________________
Domain: A	Domain: ________
Range: B	Range: ________
The inverse, f^{-1}, of the function f maps y to x if and only if ________________________.	

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