Chapter 3 Polynomial Functions

3.1 Characteristics of Polynomial Functions

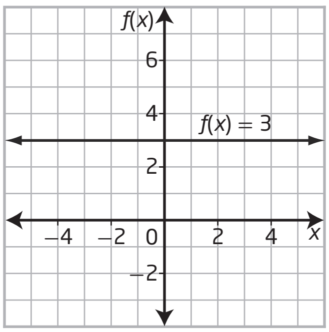
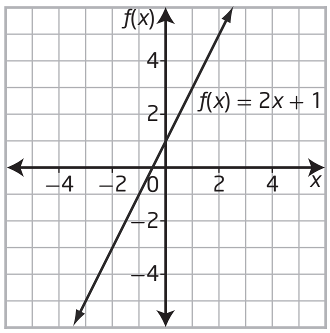
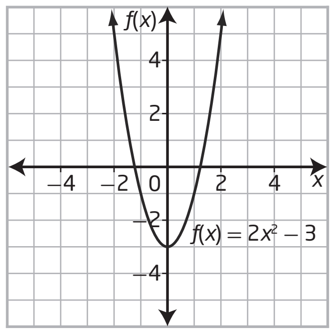
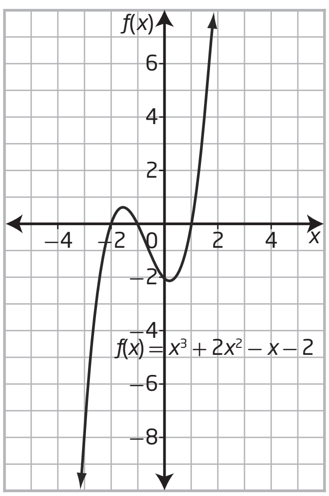
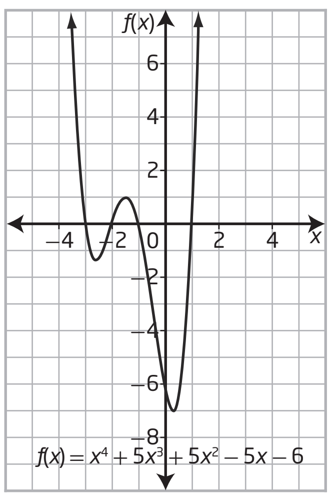
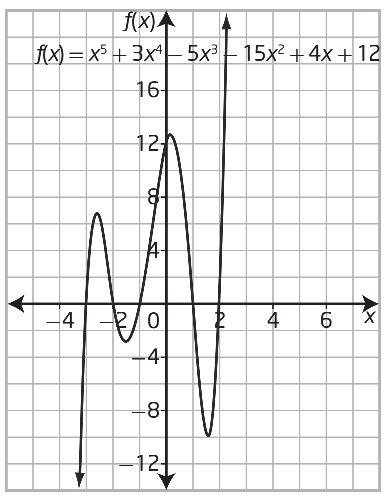
KEY IDEAS

What Is a Polynomial Function?

A polynomial function has the form $f(x) = a_nx^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_2x^2 + a_1x + a_0$ where

- n is a whole number
- x is a variable
- the coefficients a_n to a_0 are real numbers
- the degree of the polynomial function is n, the exponent of the greatest power of x
- the leading coefficient is a_n, the coefficient of the greatest power of x
- the constant term is a_0

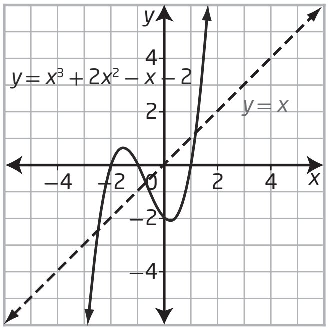
Types of Polynomial Functions

Constant Function	Linear Function	Quadratic Function
Degree 0	Degree 1	Degree 2
		
Cubic Function	**Quartic Function**	**Quintic Function**
Degree 3	Degree 4	Degree 5
		

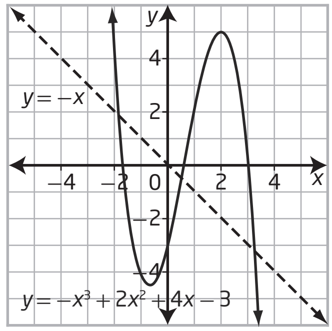
Characteristics of Polynomial Functions

Graphs of Odd-Degree Polynomial Functions

- extend from quadrant III to quadrant I when the leading coefficient is positive, similar to the graph of $y = x$



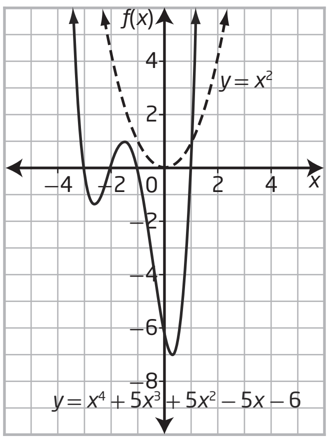
- extend from quadrant II to IV when the leading coefficient is negative, similar to the graph of $y = -x$



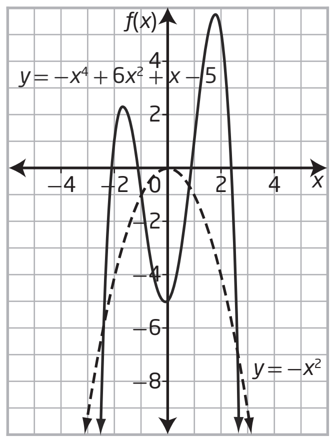
- have at least one x-intercept to a maximum of n x-intercepts, where n is the degree of the function
- have y-intercept a_0, the constant term of the function
- have domain $\{x \mid x \in \mathrm{R}\}$ and range $\{y \mid y \in \mathrm{R}\}$
- have no maximum or minimum values

Graphs of Even-Degree Polynomial Functions

- open upward and extend from quadrant II to quadrant I when the leading coefficient is positive, similar to the graph of $y = x^2$



- open downward and extend from quadrant III to IV when the leading coefficient is negative, similar to the graph of $y = -x^2$



- have from 0 to a maximum of n x-intercepts, where n is the degree of the function
- have y-intercept a_0, the constant term of the function
- have domain $\{x \mid x \in \mathrm{R}\}$; the range depends on the maximum or minimum value of the function
- have a maximum or minimum value

Working Example 1: Identify Polynomial Functions

Which of these functions are polynomials? Justify your answer. State the degree, the leading coefficient, and the constant term of each polynomial function.

a) $g(x) = 4^x$

b) $h(x) = 6 - 5x^3$

c) $y = x^{-2} + 1$

d) $y = \sqrt[3]{x} - 2$

Solution

a) $g(x) = 4^x$

The function $g(x) = 4^x$ ________ a polynomial function.
(*is* or *is not*)

It is an ________________ function.

The base of 4^x is a ________________ and the exponent is a ________________, so it is not of the form x^n.

b) $h(x) = 6 - 5x^3$

The function $h(x) = 6 - 5x^3$ ________ a polynomial function.
(*is* or *is not*)

The leading coefficient is ________, the degree is ________, and the constant term is ________.

c) $y = x^{-2} + 1$

The function $y = x^{-2} + 1$ ________ a polynomial function.
(*is* or *is not*)

x^{-2} is the same as ________, because the exponent is negative.

The function $y = x^{-2} + 1$ is a ________________ function.

d) $y = \sqrt[3]{x} - 2$

The function $y = \sqrt[3]{x} - 2$ ________ a polynomial function.
(*is* or *is not*)

$\sqrt[3]{x}$ can be rewritten as ________, which has a ________________ exponent.

The function $y = \sqrt[3]{x} - 2$ is a ________________ function.

Working Example 2: Match a Polynomial Function With Its Graph

For each polynomial function, identify the following characteristics:

- the type of function and whether it is of even or odd degree
- the end behaviour of the graph of the function
- the number of possible x-intercepts
- whether the graph has a maximum or minimum value
- the y-intercept

Then, match each function to its corresponding graph.

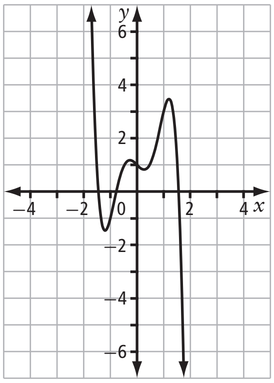
a) $f(x) = 2x^3 - 4x^2 + x + 2$

b) $g(x) = -x^4 + 10x^2 + 5x - 6$

c) $h(x) = -2x^5 + 5x^3 - x + 1$

d) $p(x) = x^4 - 5x^3 + 16$

A



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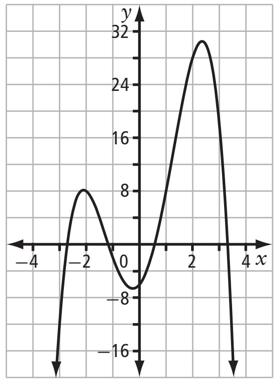

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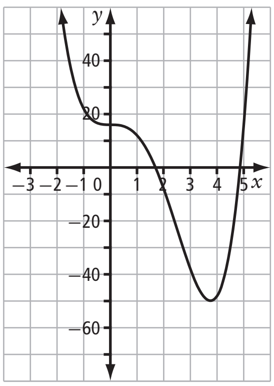
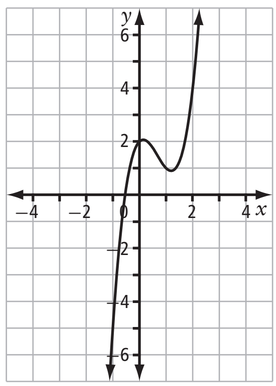


D



Solution

a) $f(x) = 2x^3 - 4x^2 + x + 2$ has degree ________, so it is an ________ function.
(*even* or *odd*)

This is a ________________ polynomial function.
(*constant* or *linear* or *quadratic* or *cubic* or *quartic* or *quintic*)

Its graph has at least ________ x-intercept(s) and at most ________ x-intercepts.

The leading coefficient is ________________, so the graph of the function extends from
(*positive* or *negative*)

quadrant ________ into quadrant ________.

Since the degree is ________, this function ______________ a maximum or minimum value.
(*even* or *odd*) (*has* or *does not have*)

The graph has a y-intercept of ________.

This function corresponds to graph ________.
(*A* or *B* or *C* or *D*)

b) $g(x) = -x^4 + 10x^2 + 5x - 6$ has degree ________, so it is an ________ function.
(*even* or *odd*)

This is a ______________ polynomial function.

Its graph has at least ________ x-intercept(s) and at most ________ x-intercepts.

Since the leading coefficient is ____________, the graph of the function opens ____________
(*positive* or *negative*) (*upward* or *downward*)

and has a ______________ value.
(*maximum* or *minimum*)

The graph has a y-intercept of ________.

This function corresponds to graph ________.

c) $h(x) = -2x^5 + 5x^3 - x + 1$ has degree ________, so it is an ________ function.
(*even* or *odd*)

This is a ______________ polynomial function.

Its graph has at least ________ x-intercept(s) and at most ________ x-intercepts.

The leading coefficient is ______________, so the graph of the function extends from
(*positive* or *negative*)

quadrant ________ into quadrant ________.

Since the degree is ________, this function ______________ a maximum or minimum value.
(*even* or *odd*) (*has* or *does not have*)

The graph has a y-intercept of ________.

This function corresponds to graph ________.

d) $p(x) = x^4 - 5x^3 + 16$ has degree ________, so it is an ________ function.
(*even* or *odd*)

This is a ______________ polynomial function.

Its graph has at least ________ x-intercept(s) and at most ________ x-intercepts.

The leading coefficient is ______________, so the graph of the function opens ______________
(*positive* or *negative*) (*upward* or *downward*)

and has a ______________ value.
(*maximum* or *minimum*)

The graph has a y-intercept of ________.

This function corresponds to graph ________.

Working Example 3: Application of a Polynomial Function

An antibacterial spray is tested on a bacterial culture. The population, P, of bacteria t minutes after the spray is applied is modelled by the function $P(t) = -2t^3 - 2t^2 + 3t + 800$.

a) What is the population of the bacteria 3 min after the spray is applied?

b) How many bacteria were in the culture before the spray was applied?

c) What is the population of the bacteria 8 min after the spray is applied? Why is this not realistic for this situation? Explain.

Solution

a) Substitute $t =$ ________ into the function and evaluate the result.

$P($________$) = -2($________$)^3 - 2($________$)^2 + 3($________$) + 800$

$=$ ________

After 3 min there are ________ bacteria in the culture.

See page 112 of *Pre-Calculus 12* for a different method of solving this question.

b) The number of bacteria before the spray was applied occurs when $t =$ ________.

This is the ________-intercept of the graph.

It is the ________________ term of the function.
(*constant* or *variable*)

There were ________ bacteria before the spray was applied.

c) Substitute $t =$ ________ into the function and evaluate the result.

$P($________$) = -2($________$)^3 - 2($________$)^2 + 3($________$) + 800$

$=$ ________

After 8 min there are ________ bacteria in the culture.

It is not realistic for the number of bacteria to be ________________.
(*positive* or *negative*)

This means that the spray has worked and there are ________ bacteria remaining.

Check Your Understanding

Practise

1. Determine whether each function is a polynomial function. Justify your answers.

a) $f(x) = 2x^4 - 3x + 2$

The degree is ________, which is an ________________ number.

$f(x)$ ________ a polynomial function.
(*is* or *is not*)

b) $y = 3^x + 5$

The term 3^x means this is an ________________ function.

This function ________ a polynomial function.
(*is* or *is not*)

c) $g(x) = 9$

$g(x)$ has degree ________.

This function ________ a polynomial function.
(*is* or *is not*)

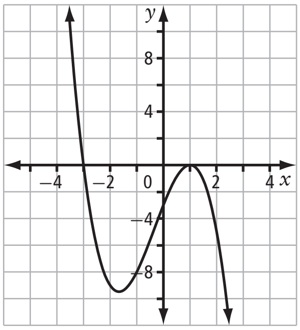
d) $y = x^{-2} + 7x^3 + 1$

2. Complete the table for each polynomial function.

Polynomial Function	Degree	Type	Leading Coefficient	Constant Term
a) $f(x) = 6x^3 - 5x^2 + 2x - 8$	3			
b) $y = -2x^5 + 5x^3 + x^2 + 1$		Quintic		
c) $g(x) = x^3 - 7x^4$				0
d) $p(x) = 10x - 9$				
e) $y = -0.5x^2 + 4x + 3$				
f) $h(x) = 3x^4 - 8x^3 + x^2 + 2$			3	
g) $y = -5$		Constant		

3. For each graph of a polynomial function,
- determine whether the function has odd or even degree
- determine whether the leading coefficient is positive or negative
- state the number of x-intercepts
- state the domain and range

a) $f(x) = -x^3 - x^2 + 5x - 3$



The graph extends from quadrant ________ to quadrant ________.

The function has ________ degree.

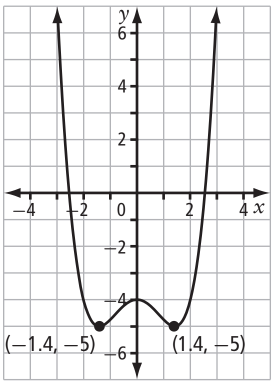
The leading coefficient is ________.

There are ________ x-intercepts.

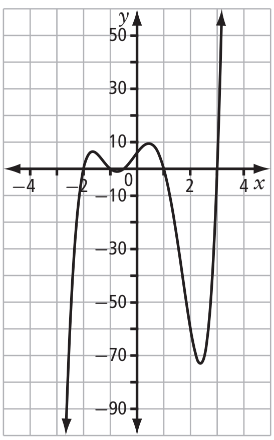
Domain: ________________

Range: ________________

b)



c)



4. For each function, use the degree and the sign of the leading coefficient to describe the end behaviour of its graph. State the possible number of x-intercepts and the value of the y-intercept.

a) $g(x) = 4x^5 - x^3 + 3x^2 - 6x + 2$

The degree is ________ with a ________________ leading coefficient.

The graph extends from quadrant ________ to quadrant ________.

There are a maximum of ________ x-intercepts. The y-intercept is ________.

b) $y = -x^4 - 2x^5 + x^3 - 3x^2 + x$

c) $h(x) = x - 7x^3 - 6$

d) $y = 3x^5 - 2x^4 + 5x^3 - x^2 + x + 3$

e) $p(x) = 5x^4 - 6x - 1$

Apply

5. Sonja claims that all graphs of polynomial functions of the form $y = ax^n + x + b$, where a, n, and b are odd integers, extend from quadrant II to quadrant IV. Do you agree? Use examples to explain your answer.

6. A skateboard manufacturer determines that its profit, P, in dollars, can be modelled by the function $P(x) = 1000x + 1.25x^4 - 3200$, where x represents the number, in hundreds, of skateboards sold.

a) What is the degree of the function $P(x)$?

b) What are the leading coefficient and the constant of this function? What does the constant represent in this context?

c) Describe the end behaviour of the graph of this function.

d) What are the restrictions on the domain of this function? Explain how you determined those restrictions.

e) What do the x-intercept(s) of the graph represent in this context?

f) What is the profit from the sale of 1200 skateboards?

7. Ali moves forward and backward along a straight path. Ali's distance, D, in metres, from a tree is modelled by the function $D(t) = t^3 - 12t^2 + 36t + 5$, where t represents the time, in seconds.

a) What is the degree of function $D(t)$?

b) What are the leading coefficient and the constant of this function? What does the constant represent in this situation?

c) Describe the end behaviour of the graph of this function.

d) What are the restrictions on the domain of this function? Explain how you determined the restrictions.

e) How far is Ali from the tree after 7 s?

f) Sketch the function. Then, graph the function using technology. How does the graph compare to your sketch?

8. By analysing the effect of growing economic conditions, the predicted population, P, of a town in t years from now can be modelled by the function $P(t) = 6t^4 - 6t^3 + 200t + 12\,000$. Assume this model can be used for the next 15 years.

a) What are the key features of the graph of this function?

b) What is the current population of this town?

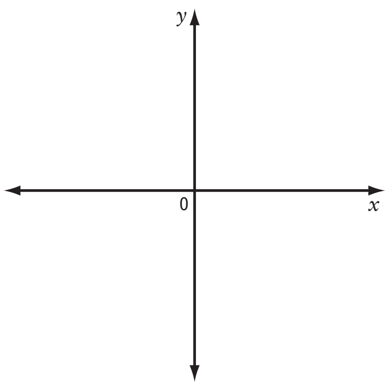
c) What will the population be 10 years from now?

d) When will the population of the town be approximately 175 000?

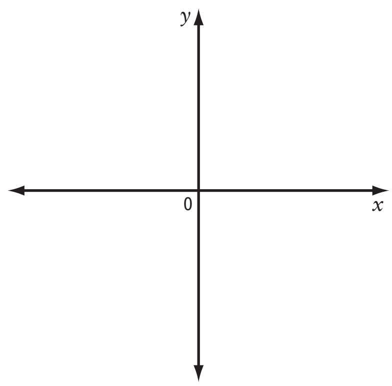
Connect

9. On each set of axes, sketch a polynomial function with the given characteristics.

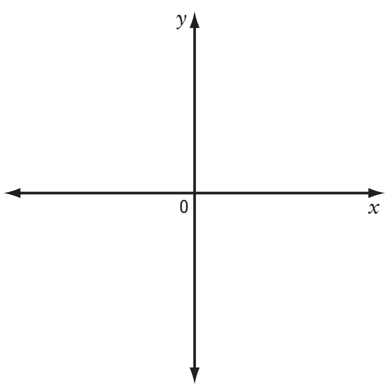
a) A polynomial function with degree 3, a positive leading coefficient, and 2 x-intercepts.



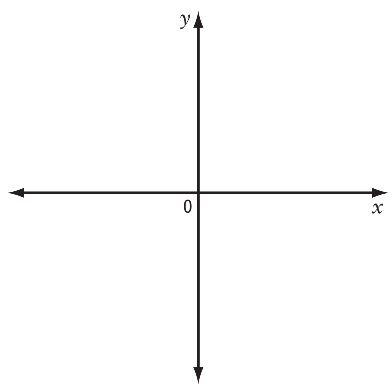
b) A polynomial function with degree 4, a negative leading coefficient, and 4 x-intercepts.



c) A polynomial function with degree 5, a negative leading coefficient, and 3 x-intercepts.



d) A polynomial function with degree 4, a positive leading coefficient, and 2 x-intercepts.



3.2 The Remainder Theorem

KEY IDEAS

Long Division
You can use long division to divide a polynomial by a binomial: $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$

The components of long division are
- the dividend, $P(x)$, which is the polynomial that is being divided
- the divisor, $x - a$, which is the binomial that the polynomial is divided by
- the quotient, $Q(x)$, which is the expression that results from the division
- the remainder, R, which is the value or expression that is left over after dividing

To check the division of a polynomial, verify the statement $P(x) = (x - a)Q(x) + R$.

Synthetic Division
- a short form of division that uses only the coefficients of the terms
- it involves fewer calculations

Remainder Theorem
- When a polynomial $P(x)$ is divided by a binomial $x - a$, the remainder is $P(a)$.
- If the remainder is 0, then the binomial $x - a$ is a factor of $P(x)$.
- If the remainder is *not* 0, then the binomial $x - a$ is *not* a factor of $P(x)$.

Working Example 1: Divide a Polynomial by a Binomial of the Form *x* − *a*

a) Divide $P(x) = 9x + 4x^3 - 12$ by $x + 2$. Express the result in the form $\frac{P(x)}{x-a} = Q(x) + \frac{R}{x-a}$.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division.

Solution

a) $x + 2\overline{)4x^3 + 0x^2 + 9x - 12}$

Why is the order of the terms different?
Why is it necessary to include the term $0x^2$?

See Example 1 on page 120 of *Pre-Calculus 12* for help with long division.

$\frac{4x^3 + 9x - 12}{x + 2} =$ ______________________

b) Since division by ________ is not defined, the divisor cannot be ________:

$x + 2 \neq$ ________ or $x \neq$ ________.

c) The corresponding statement that can be used to check the division is

____________________________ = ____________________________.

Working Example 2: Apply Polynomial Division to Solve a Problem

The volume, V, in cubic centimetres, of gift boxes is given by $V(x) = 2x^3 + x^2 - 27x - 36$. The height, h, in centimetres, is $x + 3$. What are the possible dimensions of the boxes in terms of x?

Solution

Divide the volume of the box by the height to obtain an expression for the area of the base of the box.

$x + 3 \overline{)2x^3 + x^2 - 27x - 36}$

$\frac{V(x)}{h} = lw$, where lw is the area of the base

Since the remainder is ________, express the volume $2x^3 + x^2 - 27x - 36$ as (________)

(________________).

The quotient ________________ represents the area of the base.

This expression can be factored as ____________________________.

The factors represent the possible ________________ and ________________ of the base.

Expressions for the dimensions, in centimetres, are ________, ________, and ________.

Working Example 3: Divide a Polynomial Using Synthetic Division

a) Use synthetic division to divide $5x^2 - x + 2x^3 - 6$ by $x + 2$.

b) Check your results using long division.

Solution

a) Write the terms of the dividend in order of ________________ powers.
(*ascending* or *descending*)

____

Fill in the missing values and perform the division.

	____	____	____	____
	↓	↓	↓	↓
+2				
−				
×				

See Example 3 on page 122 of *Pre-Calculus 12* for help with synthetic division.

$(2x^3 + 5x^2 - x - 6) \div (x + 2) =$ ______________________; Restriction: ______________

b) $x + 2\overline{)2x^3 + 5x^2 - x - 6}$

The result obtained from long division is ______________ that using
(*the same as* or *different from*)
______________ division.

Working Example 4: Apply the Remainder Theorem

a) Use the remainder theorem to determine the remainder when $P(x) = 3x^4 - x^3 - 5$ is divided by $x - 3$.

b) Verify your answer using synthetic division.

Solution

a) Since the binomial is $x - 3$, determine the remainder by evaluating $P(x)$ at $x =$ ________,

or $P($________$)$.

$P($________$) = 3($________$)^4 - ($________$)^3 - 5$

$=$ ________________________

Why is it necessary to write the polynomial this way?

The remainder when $3x^4 - x^3 - 5$ is divided by $x - 3$ is ________.

b) To use synthetic division, first rewrite $P(x)$ as $P(x) =$ ______________.

	____	____	____	____	____
	↓	↓	↓	↓	↓
________					
−					
×					

The remainder when using synthetic division is ________.

Check Your Understanding

Practise

1. **a)** Use long division to divide $x^3 + 3x^2 - 2x + 5$ by $x + 1$. Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.

   b) Identify any restrictions on the variable.

   c) Write the corresponding statement that can be used to check the division. Then, verify your answer.

2. Divide using long division. Then, verify your answer using synthetic division.

   a) $(2x^2 - x + 5) \div (x + 3)$

   b) $(x^3 - x - 10) \div (x + 4)$

c) $(3x^4 + 2x^3 - 6x + 1) \div x$

d) $(-4x^4 + 11x - 7) \div (x - 3)$

3. Express each result in #2 above in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.

Identify any restrictions on the variable.

a) $\frac{2x^2 - x + 5}{x + 3}$

b)

c)

d)

4. Determine the remainder when each polynomial function is divided by $x - 2$. Use the remainder theorem.

a) $P(x) = 2x^3 + 3x^2 - 17x - 30$

b) $P(x) = x^3 + x^2 - 4x + 4$

5. Determine each remainder.

a) $(6x^2 - x + 15) \div (x + 1)$

b) $(x^3 - x^2 - 2x - 1) \div (x + 2)$

c) $(2x^3 - 5x^2 - 13x + 2) \div (x - 4)$

d) $(x^4 - 3x^2 - 5x + 2) \div (x - 2)$

Apply

6. For each dividend, determine the value of k if the remainder is −2.

a) $(2x^3 - 5x^2 - 4x + k) \div (x + 1)$

b) $(x^3 - 4x^2 + kx + 10) \div (x - 3)$

c) $(3x^3 + kx^2 - 13x + 4) \div (x + 2)$

d) $(kx^3 - 4x^2 - 5x + 8) \div (x - 2)$

7. For what value of m will the polynomial $P(x) = x^3 + 6x^2 + mx - 4$ have the same remainder when it is divided by $x - 1$ and $x + 2$?

Since the remainder is the same, determine the value of m by solving $P(1) = P(\_\_\_\_\_\_\_\_)$.

8. You can model the volume, in cubic centimetres, of a rectangular box by the polynomial function $V(x) = 3x^3 + x^2 - 12x - 4$. Determine expressions for the other dimensions of the box if the height is $x + 2$.

Connect

9. When the polynomial $bx^3 + cx^2 + dx + e$ is divided by $x - a$, the remainder is zero.

a) What can you conclude from this result?

b) Write an expression for the remainder in terms of a, b, c, d, and e.

3.3 The Factor Theorem

KEY IDEAS

Factor Theorem

The factor theorem states that $x - a$ is a factor of a polynomial $P(x)$ if and only if $P(a) = 0$.
If and only if means that the result works both ways. That is,

- if $x - a$ is a factor then, $P(a) = 0$
- if $P(a) = 0$, then $x - a$ is a factor of a polynomial $P(x)$

Integral Zero Theorem

- The integral zero theorem describes the relationship between the factors and the constant term of a polynomial. The theorem states that if $x - a$ is a factor of a polynomial $P(x)$ with integral coefficients, then a is a factor of the constant term of $P(x)$ and $x = a$ is an integral zero of $P(x)$.

Factor by Grouping

- If a polynomial $P(x)$ has an even number of terms, it may be possible to group two terms at a time and remove a common factor. If the binomial that results from common factoring is the same for each pair of terms, then $P(x)$ may be factored by grouping.

Steps for Factoring Polynomial Functions

To factor polynomial functions using the factor theorem and the integral zero theorem,

- use the integral zero theorem to list possible integer values for the zeros
- next, apply the factor theorem to determine one factor
- then, use division to determine the remaining factor
- repeat the above steps until all factors are found

Working Example 1: Use the Factor Theorem to Test for Factors of a Polynomial

Which binomials are factors of the polynomial $P(x) = x^3 + 4x^2 + x - 6$? Justify your answers.

a) $x - 1$ **b)** $x - 2$ **c)** $x + 2$ **d)** $x + 3$

Solution

Use the factor theorem to evaluate $P(a)$ given $x - a$.

a) For $x - 1$, substitute $x =$ ________ into the polynomial expression.

$P($________$) =$

Since the remainder is ________, $x - 1$ ______________ a factor of $P(x)$.
(*is* or *is not*)

b) For $x - 2$, substitute $x =$ ________ into the polynomial expression.

$P($________$) =$

Since the remainder is ________, $x - 2$ ________________ a factor of $P(x)$.
(*is* or *is not*)

c) For $x + 2$, substitute $x =$ ________ into the polynomial expression.

$P($________$) =$

Since the remainder is ________, $x + 2$ ________________ a factor of $P(x)$.
(*is* or *is not*)

d) For $x + 3$, substitute $x =$ ________ into the polynomial expression.

$P($________$) =$

Since the remainder is ________, $x + 3$ ________________ a factor of $P(x)$.
(*is* or *is not*)

Working Example 2: Factor Using the Integral Zero Theorem

a) Factor $2x^3 + 3x^2 - 3x - 2$ fully.

b) Describe how to use the factors of the polynomial expression to determine the zeros of the corresponding polynomial function.

Solution

a) Let $P(x) =$ ________________. Find a factor by evaluating $P(x)$ for values of x that are factors of ________________: ________________ and ________________.

Test the values until you find one that gives a remainder of zero.

$P($________$) =$

Since $P($________$) = 0$, ________ is a factor of $P(x)$.

Use synthetic or long division to find the other factors.

Which method of division do you prefer? Why?

Therefore, $2x^3 + 3x^2 - 3x - 2 = ($________________$)($________________$)($________________$)$.

b) Since the factors of $2x^3 + 3x^2 - 3x - 2$ are ______________, ______________, and ______________, the corresponding zeros of the function are ________, ________, and ________.

For an explanation of how the zeros can be confirmed, refer to Example 2b) on page 130 of *Pre-Calculus 12*.

Working Example 3: Factor Higher–Degree Polynomials

Fully factor $x^4 + 3x^3 - 7x^2 - 27x - 18$.

Solution

Let $P(x) =$ ______________.

Find a factor by testing factors of ______________: ________________________.
Test the values until you find one that gives a remainder of zero.

$P($________$) =$

Since $P($________$) = 0$, ______________ is a factor of $P(x)$.
Use division to find the other factors.

The remainder factor is ______________.

Let $f(x) =$ ________________________.
Use the factor theorem again.

Since $f($________$) = 0$, ______________ is a second factor.
Use division to find the other factors.

Combine all the factors to write the fully factored form.

Therefore, $x^4 + 3x^3 - 7x^2 - 27x - 18 =$ ________________________.

Compare this method with Method 2 of Example 3 on page 131 of *Pre-Calculus 12*. Is it possible to use factor by grouping in this situation? Explain.

Working Example 4: Solve Problems Involving Polynomial Expressions

An artist creates a carving from a block of soapstone. The soapstone is in the shape of a rectangular prism whose volume, in cubic feet, is represented by $V(x) = 6x^3 + 25x^2 + 2x - 8$, where x is a positive real number. What are the factors that represent possible dimensions, in terms of x, of the block of soapstone?

Solution

The possible integral factors correspond to the factors of the ______________ term of the polynomial, ______________: ________________________.

Use the factor theorem to determine which of these values correspond to the factors of the polynomial.

The values of x that result in a remainder of _______ are ______________, and the corresponding factors are ________________________.

The possible dimensions of the block of soapstone are ________________________.

See Example 4 on page 132 of *Pre-Calculus 12* for another method of solving this problem.

Check Your Understanding

Practise

1. What is the corresponding binomial factor of a polynomial, $P(x)$, given the value of the zero?

a) $P(2) = 0$

b) $P(-4) = 0$

c) $P(b) = 0$

d) $P(-d) = 0$

2. Determine whether $x + 1$ is a factor of each polynomial.

a) $x^3 + x^2 - x - 1$

b) $x^4 - 3x^3 - 4x^2 + x + 1$

c) $2x^3 - x^2 - 3x - 1$

d) $4x^4 + 7x + 3$

3. State whether each polynomial has $x + 3$ as a factor.

a) $x^3 + x^2 - x + 6$

b) $2x^3 + 9x^2 + 10x + 3$

c) $x^3 + 27$

d) $x^4 - 9x^2 + 2x + 6$

4. What are the possible integral zeros of each polynomial?

a) $x^3 - 3x^2 + 4x - 16$

b) $x^3 + 2x^2 + 8x + 12$

c) $x^3 - 3x^2 + 10x - 32$

d) $x^4 + 8x^3 - 9x^2 + 2x + 18$

5. Factor fully.

a) $x^3 - x^2 - 4x + 4$

b) $x^3 - 2x^2 - 4x + 8$

c) $x^3 + 3x^2 + 3x + 1$

d) $x^4 + 2x^3 - x - 2$

6. Factor fully.

a) $x^3 + 2x^2 - 9x - 18$

b) $4x^3 - 8x^2 + x + 3$

c) $6x^3 + x^2 - 31x + 10$

d) $x^4 + x^3 - 13x^2 - 25x - 12$

Apply

7. Determine the value(s) of k so that the binomial is a factor of the polynomial.

a) $P(x) = x^3 + 5x^2 + kx + 6$ $x + 2$

If $x + 2$ is a factor, then $P(\_\_\_\_\_\_\_\_) = \_\_\_\_\_\_\_\_$.

b) $P(x) = kx^3 - 10x^2 + 2x + 3$ $x - 3$

8. The product of four integers is $x^4 + 7x^3 + 7x^2 - 15x$, where x is one of the integers. What are the possible expressions for the other three integers?

9. A sculptor creates a carving from a block of marble. The marble is in the shape of a rectangular prism whose volume, in cubic feet, is represented by $V(x) = 3x^3 + 2x^2 - 7x + 2$, where x is a positive real number. What are the factors that represent possible dimensions, in terms of x, of the block of marble?

Connect

10. Describe the steps required to factor the polynomial $x^4 - 10x^3 + 24x^2 + 10x - 25$.

3.4 Equations and Graphs of Polynomial Functions

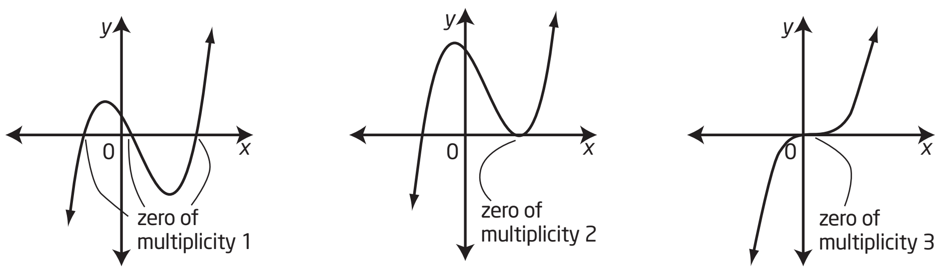
KEY IDEAS

Sketching Graphs of Polynomial Functions

- To sketch the graph of a polynomial function, use the x-intercepts, the y-intercept, the degree of the function, and the sign of the leading coefficient.
- The x-intercepts of the graph of a polynomial function are the roots of the corresponding polynomial equation.
- Determine the zeros of a polynomial function from the factors.
- Use the factor theorem to express a polynomial function in factored form.

Multiplicity of a Zero

- If a polynomial has a factor $x - a$ that is repeated n times, then $x = a$ is a zero of multiplicity n.
- The multiplicity of a zero or root can also be referred to as the *order* of the zero or root.
- The shape of a graph of a polynomial function close to a zero of $x = a$ (multiplicity n) is similar to the shape of the graph of a function with degree equal to n of the form $y = (x - a)^n$.
- Polynomial functions change sign at x-intercepts that correspond to *odd* multiplicity. The graph crosses over the x-axis at these intercepts.
- Polynomial functions do not change sign at x-intercepts of *even* multiplicity. The graph touches, but does not cross, the x-axis at these intercepts.



Transformation of Polynomial Functions

To sketch the graph of a polynomial function of the form $y = a[b(x - h)]^n + k$ or $y - k = a[b(x - h)]^n$, where $n \in \mathrm{N}$, apply the following transformations to the graph of $y = x^n$.

Note: You may apply the transformations represented by a and b in any order before the transformations represented by h and k.

Parameter	Transformation
k	• Vertical translation up or down • $(x, y) \rightarrow (x, y + k)$
h	• Horizontal translation left or right • $(x, y) \rightarrow (x + h, y)$
a	• Vertical stretch about the x-axis by a factor of $\|a\|$ • For $a < 0$, the graph is also reflected in the x-axis • $(x, y) \rightarrow (x, ay)$

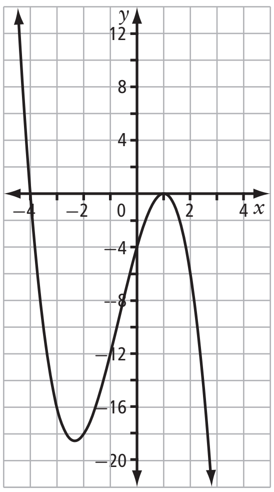
b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$ • For $b < 0$, the graph is also reflected in the y-axis • $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$

Working Example 1: Analyse Graphs of Polynomial Functions

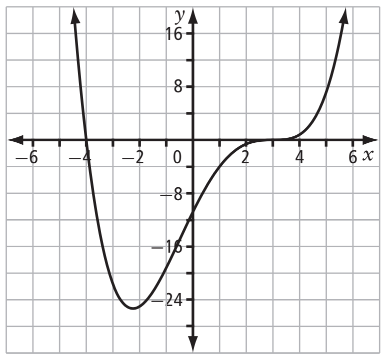
For each graph of a polynomial function, determine

- the least possible degree
- the sign of the leading coefficient
- the x-intercepts and the factors of the function with least possible degree
- the intervals where the function is positive and the intervals where it is negative

a)



b)



Solution

a) There are ________ x-intercepts; they are ____________________________.

The x-intercept of multiplicity 1 is ________.

The x-intercept of multiplicity 2 is ________.

The least possible degree of the graph is ________.

The graph extends from quadrant ________ to quadrant ________.

The leading coefficient is ________________.
(*positive* or *negative*)

The factors are ____________________________.

The function is positive for values of x in the interval(s) ____________________________.

The function is negative for values of x in the interval(s) ____________________________.

b) The x-intercepts are ______________________.

The x-intercept of multiplicity ________ is ________.

The x-intercept of multiplicity ________ is ________.

The least possible degree of the graph is ________.

The graph extends from quadrant ________ to quadrant ________.

The leading coefficient is ________________.
(*positive* or *negative*)

The factors are ______________________.

The function is positive for values of x in the interval(s) ______________________.

The function is negative for values of x in the interval(s) ______________________.

To see additional graphs, refer to Example 1 on page 138 of *Pre-Calculus 12*.

Working Example 2: Analyse Equations to Sketch Graphs of Polynomial Functions

Sketch the graph of each polynomial function.

a) $y = -(x + 1)^3(x - 3)$

b) $y = 2x^5 + x^4 - 18x^3 - 9x^2$

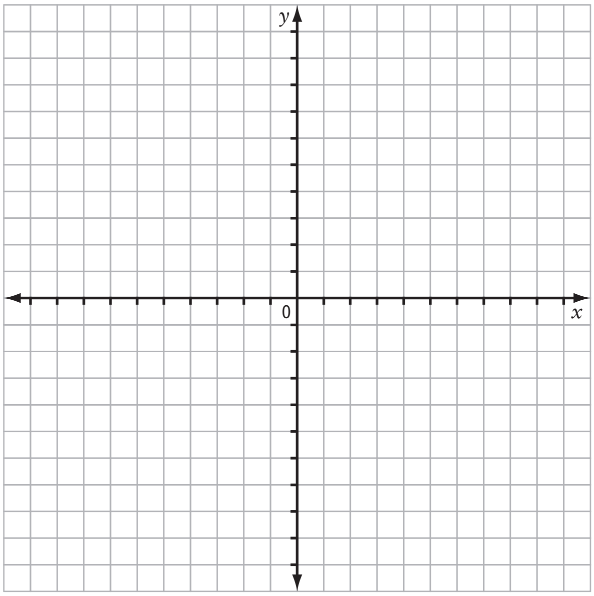
Solution

a) The function $y = -(x + 1)^3(x - 3)$ is in ________________ form.

Degree	
Leading coefficient	
End behaviour	
Zeros/x-intercepts	
Multiplicity of zeros	
y-intercept	
Interval(s) where the function is positive	
Interval(s) where the function is negative	

Use the information from the table to sketch the graph.

How can you check whether the function is positive or negative?



b) The function $y = 2x^5 + x^4 - 18x^3 - 9x^2$ is not in

________________ form.

First, factor out the common factor.

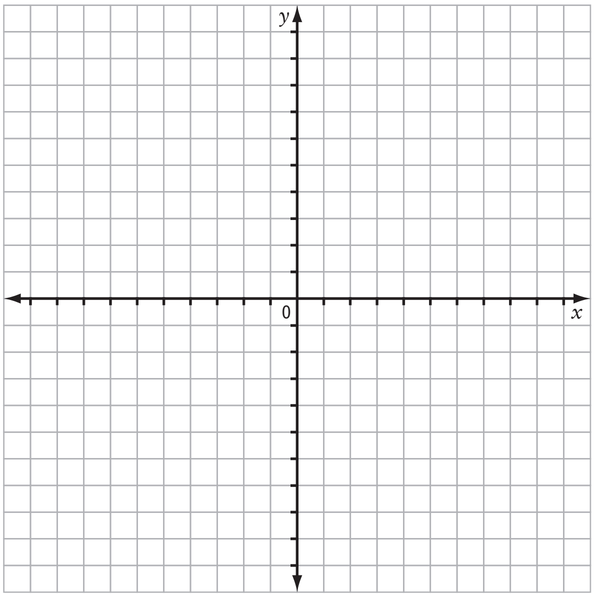
What zero corresponds to the common factor? What multiplicity does it have?

Next, use the integral zero theorem and the factor theorem to factor the polynomial.

The factored form of $y = 2x^5 + x^4 - 18x^3 - 9x^2$ is ______________________________.

Degree	
Leading coefficient	
End behaviour	
Zeros/x-intercepts	
Multiplicity of zeros	
y-intercept	
Interval(s) where the function is positive	
Interval(s) where the function is negative	

Use the information from the table to sketch the graph.



Working Example 3: Apply Transformations to Sketch a Graph

The graph of $y = x^4$ is transformed to obtain the graph of $y = 3\left[-\frac{1}{2}(x + 1)\right]^4 - 4$.

a) State the parameters and describe the corresponding transformations.

b) Complete a table to show what happens to the given points under each transformation.

c) Sketch the graph of $y = 3\left[-\frac{1}{2}(x + 1)\right]^4 - 4$.

Solution

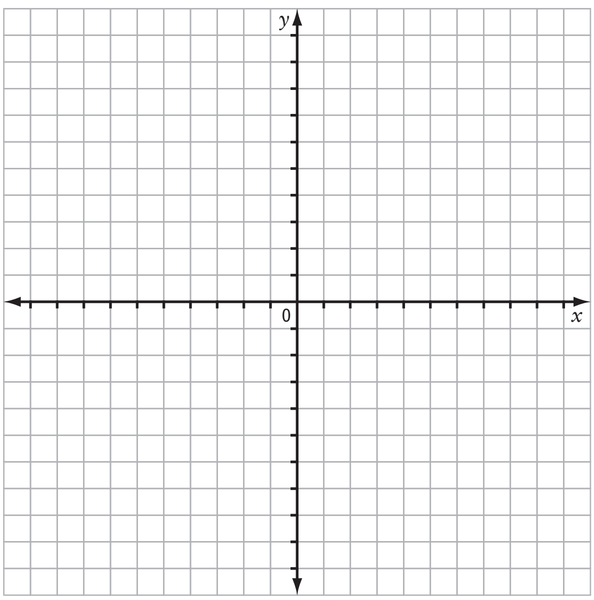
a)

Parameter	Description of Transformation
$b =$	
$a =$	
$h =$	
$k =$	

b) Complete the table to show what happens to the given points under each transformation.

$y = x^4$	$y = \left(-\frac{1}{2}x\right)^4$	$y = 3\left(-\frac{1}{2}x\right)^4$	$y = 3\left[-\frac{1}{2}(x + 1)\right]^4 - 4$
(−2, 16)			
(−1, 1)			
(0, 0)			
(1, 1)			
(2, 16)			

c) To sketch the graph, plot the points from column 4 and draw a smooth curve through them.



Check Your Understanding

Practise

1. Solve.

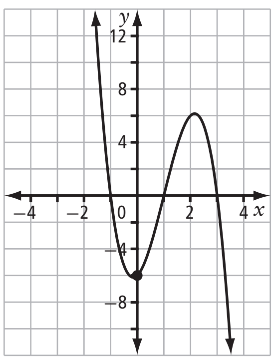
a) $4x^3(x + 2)(2x - 1) = 0$

b) $(x + 1)^2(x - 3)(x - 5) = 0$

c) $x^3 - 8 = 0$

2. Use the graph of the given function to write the corresponding polynomial equation. State the roots of the equation. The roots are all integral values.

a)



The graph of the function has ________ x-intercepts.

It crosses the x-axis at each of the x-intercepts. All the x-intercepts are of ______________ multiplicity.
(*even* or *odd*)

The least possible multiplicity of each x-intercept is ________, so the least possible degree is ________.

The graph extends up into quadrant ________ and down into quadrant ________, so the leading coefficient is ______________.
(*positive* or *negative*)

The y-intercept is ________; this is the ______________ term in the equation of the function.

The zeros, or x-intercepts, are ________, ________, and ________. The product of the roots is ______________.

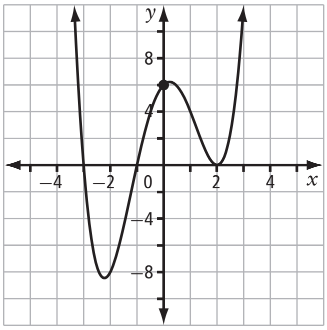
Compare the product of the roots to the y-intercept to determine the vertical stretch, a.

$a =$ ________

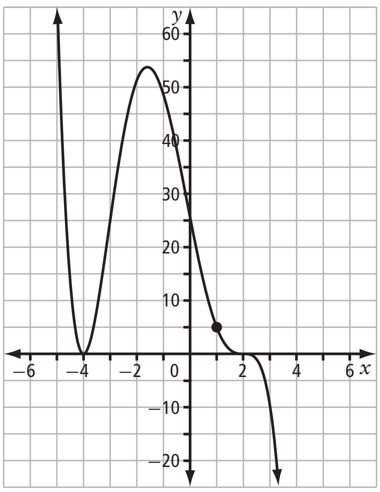
The equation of the polynomial function is

$f(x) =$ ________(________)(________)(________).

b)



c)

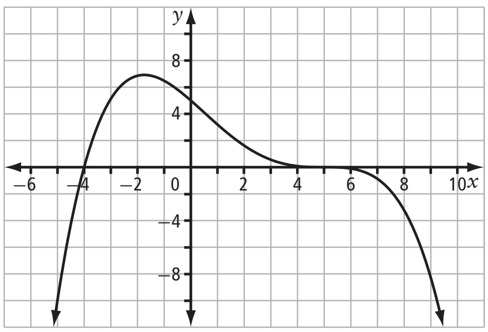


In this case, use the coordinates of the point (1, 5) to solve for *a*.

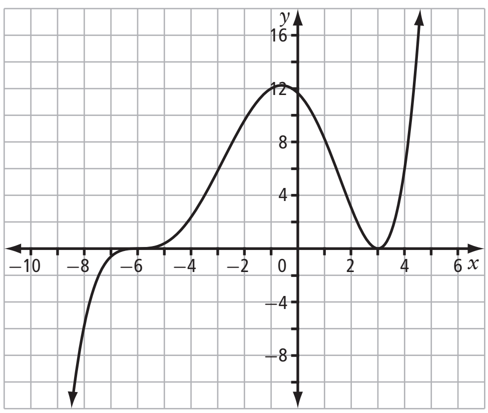
3. For each graph,

- state the x-intercepts
- state the intervals where the function is positive and the intervals where the function is negative
- explain whether the graph might represent a polynomial that has zero(s) of multiplicity 1, 2, or 3.

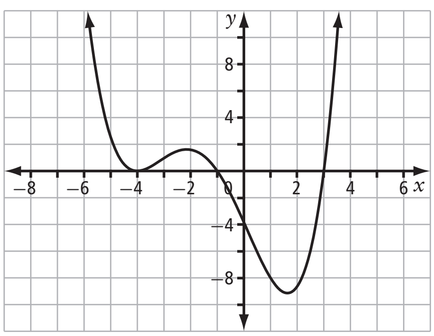
a)



b)

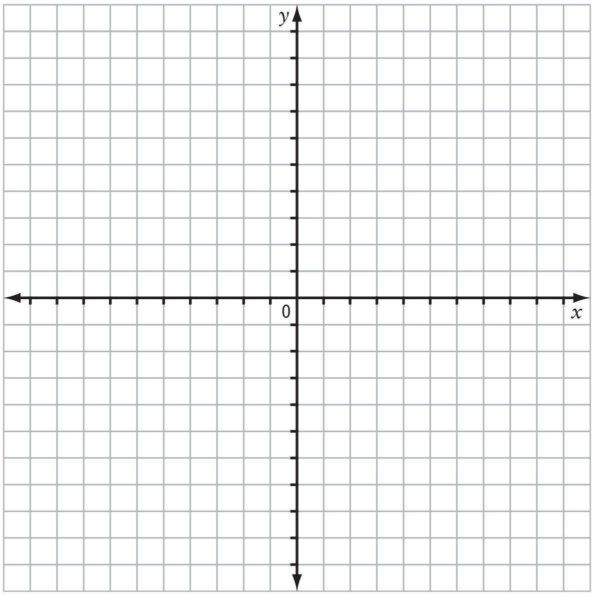


c)

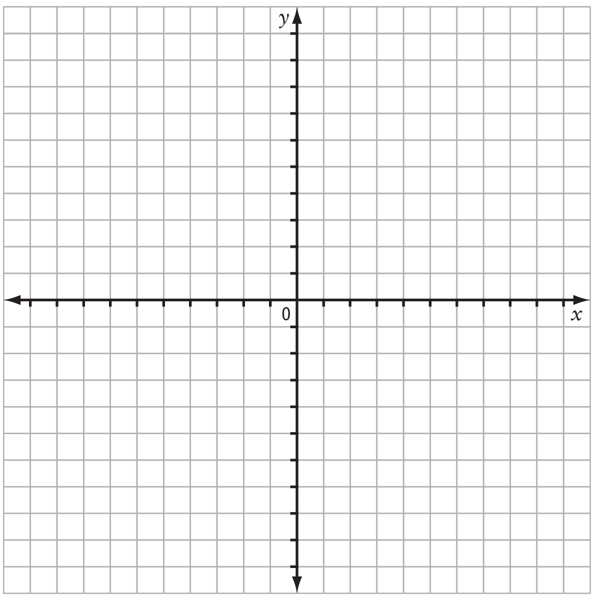


4. Without using technology, sketch the graph of each function. Label all intercepts. (Hint: Factor.)

a) $f(x) = -2x^3 + 3x^2 + 11x - 6$



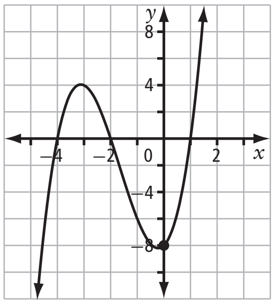
b) $g(x) = x^4 + 5x^3 + 6x^2 - 4x - 8$



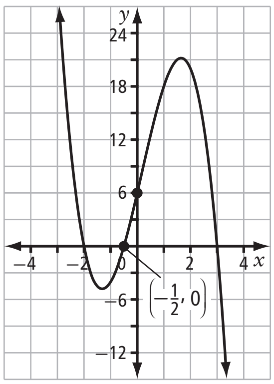
Apply

5. Determine the equation for each graph of a polynomial function shown.

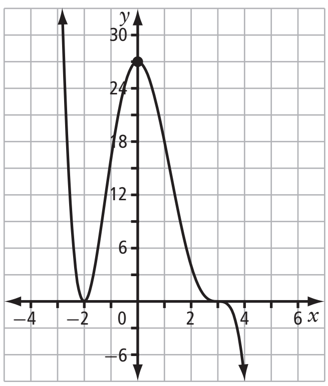
a)



b)



c)



6. **a)** Given the function $y = x^3$, list the parameters of the transformed polynomial function $y = \frac{1}{2}\left(3(x + 4)\right)^3 - 5$ and describe how each parameter transforms the graph of the function $y = x^3$.

b) Determine the domain and range for the transformed function.

7. Determine the equation with least degree for each polynomial function.
a) quartic function with zeros 2 (multiplicity 3) and -5, and y-intercept 30

b) quintic function with zeros -1 (multiplicity 2), 3 (multiplicity 1), and -2 (multiplicity 2), and constant term -12

8. An interlocking stone path that is x feet wide is built around a rectangular garden. The garden is 20 ft wide and 40 ft long. The combined surface area of the garden and the walking path is 1196 ft^2. What are the dimensions of the stone path?

For help with #8, see Example 4 on page 145 of *Pre-Calculus 12* for an example of how to solve a problem involving polynomial functions.

Connect

9. Given a polynomial function of the form $y = a[b(x - h)]^n + k$, which parameters do not change the shape of the graph of the function? Explain.

Chapter 3 Review

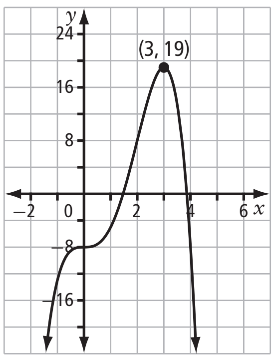
3.1 Characteristics of Polynomial Functions, pages 66–77

1. Complete the chart for each polynomial function.

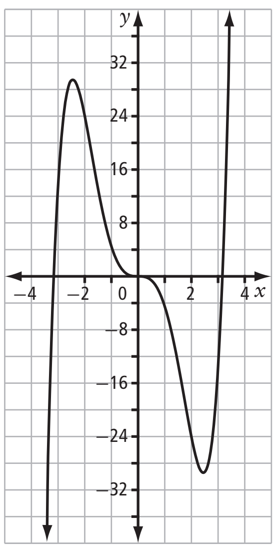
Polynomial Function	Degree	Type	Leading Coefficient	Constant Term
a) $f(x) = -2x^4 - x^3 + 3x - 7$				
b) $y = 3x^5 + 2x^4 - x^3 + 3$				
c) $g(x) = 0.5x^3 - 8x^2$				
d) $p(x) = 10$				

2. For each of the following,
- determine whether the graph represents on odd-degree or an even-degree polynomial function
- determine whether the leading coefficient of the corresponding function is positive or negative
- state the number of x-intercepts
- state the domain and range

a)



b)



3. The distance, d, in metres, travelled by a boat from the moment it leaves shore can be modelled by the function $d(t) = 0.002t^3 + 0.05t^2 + 0.3t$, where t is the time, in seconds.

a) What is the degree of the function $d(t)$?

b) What are the leading coefficient and constant of this function? What does the constant represent?

c) Describe the end behaviour of the graph of this function.

d) What are the restrictions on the domain of this function? Explain why you selected those restrictions.

e) What distance has the boat travelled after 15 s?

f) Make a sketch of what you think the function will look like. Then, graph the function using technology. How does it compare to your sketch?

3.2 The Remainder Theorem, pages 78–83

4. a) Use long division to divide $5x^3 - 7x^2 - x + 6$ by $x - 1$.

Express the result in the form $\frac{P(x)}{x - a} = Q(x) + \frac{R}{x - a}$.

b) Identify any restrictions on the variable.

c) Write the corresponding statement that can be used to check the division. Then, verify your answer.

5. Determine the remainder resulting from each division.

a) $(x^3 + 2x^2 - 3x + 9) \div (x + 3)$

b) $(2x^3 + 7x^2 - x + 1) \div (x + 2)$

c) $(x^3 + 2x^2 - 3x + 5) \div (x - 3)$

d) $(2x^4 + 7x^2 - 8x + 3) \div (x - 4)$

6. a) Determine the value of m such that when $f(x) = x^4 - mx^3 + 7x - 6$ is divided by $x - 2$, the remainder is -8.

b) Use the value of m from part a) to determine the remainder when $f(x)$ is divided by $x + 2$.

7. When a polynomial $P(x)$ is divided by $x - 2$, the quotient is $x^2 + 4x - 7$ and the remainder is -4. What is the polynomial?

3.3 The Factor Theorem, pages 84–90

8. What is the corresponding binomial factor of a polynomial, $P(x)$, given the value of the zero?

a) $P(7) = 0$

b) $P(-6) = 0$

c) $P(c) = 0$

9. Determine whether $x + 2$ is a factor of each polynomial.

a) $x^3 + 2x^2 - x - 2$

b) $x^4 + 2x^3 - 4x^2 + x + 10$

10. What are the possible integral zeros of each polynomial?

a) $x^3 - 5x^2 + 3x - 27$

b) $x^3 + 6x^2 + 2x + 36$

11. Factor fully.

a) $x^3 - 4x^2 + x + 6$

b) $3x^3 - 5x^2 - 26x - 8$

c) $5x^4 + 12x^3 - 101x^2 + 48x + 36$

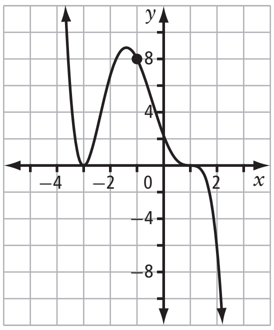
d) $2x^4 + 5x^3 - 8x^2 - 20x$

12. Rectangular blocks of ice are cut up and used to build the front entrance of an ice castle. The volume, in cubic feet, of each block is represented by $V(x) = 5x^3 + 7x^2 - 8x - 4$, where x is a positive real number. What are the factors that represent possible dimensions, in terms of x, of the blocks?

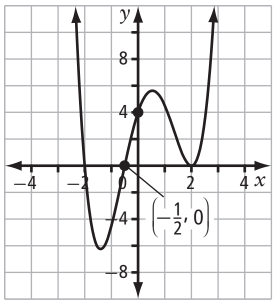
3.4 Equations and Graphs of Polynomial Functions, pages 91–102

13. For each graph of a polynomial function, determine
- the least possible degree
- the sign of the leading coefficient
- the x-intercepts and their multiplicity
- the intervals where the function is positive and the intervals where it is negative
- the equation for the polynomial function

a)



b)



14. a) Given the function $y = x^5$, list the parameters of the transformed polynomial function $y = -2\left(\frac{1}{3}(x - 1)\right)^5 + 4$ and describe how each parameter transforms the graph of the function $y = x^5$.

b) Determine the domain and range for the transformed function.

15. Determine the equation with least degree for a cubic function with zeros −2 (multiplicity 2) and 3 (multiplicity 1), and y-intercept 36.

Chapter 3 Skills Organizer

Make note of some of the key details and things to remember about the processes you have learned in this unit. Use your class notes, textbook, or questions from this workbook to help you choose examples (or create your own). Some information is provided below to help you get started.

Process	Example	Things to Remember
Analysing graphs of odd-degree polynomial functions		
Analysing graphs of even-degree polynomial functions		
Using synthetic division		
Applying the remainder theorem		
Using the factor theorem		
Factoring using the integral zero theorem		
Determining the multiplicity of zeros		
Applying transformations of functions $y = a[b(x - h)]^n + k$		a: vertical stretch/reflection b: horizontal stretch/reflection h: horizontal translation k: vertical translation