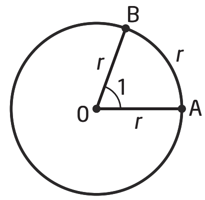
Chapter 4 Trigonometry and the Unit Circle

4.1 Angles and Angle Measure

KEY IDEAS

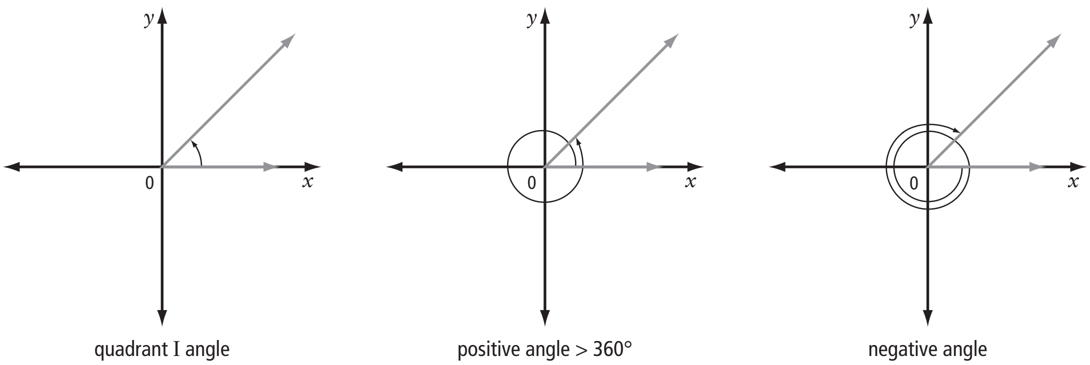


- One radian is the measure of the central angle subtended in a circle by an arc equal in length to the radius of the circle.
- Travelling one rotation around the circumference of a circle causes the terminal arm to turn $2\pi r$. Since $r = 1$ on the unit circle, $2\pi r$ can be expressed as 2π, or 2π radians.

  You can use this information to translate rotations into radian measures. For example,

1 full rotation (360°) is 2π radians	$\frac{1}{6}$ rotation (60°) is $\frac{\pi}{3}$ radians
$\frac{1}{2}$ rotation (180°) is π radians	$\frac{1}{8}$ rotation (45°) is $\frac{\pi}{4}$ radians
$\frac{1}{4}$ rotation (90°) is $\frac{\pi}{2}$ radians	$\frac{1}{12}$ rotation (30°) is $\frac{\pi}{6}$ radians

- Angles in standard position with the same terminal arms are coterminal. For an angle in standard position, an infinite number of angles coterminal with it can be determined by adding or subtracting any number of full rotations.
- Counterclockwise rotations are associated with positive angles. Clockwise rotations are associated with negative angles.



- The general form of a coterminal angle (in degrees) is $\theta \pm 360°n$, where n is a natural number (0, 1, 2, 3, …) and represents the number of revolutions. The general form (in radians) is $\theta \pm 2\pi n$, $n \in \mathrm{N}$.
- Radians are especially useful for describing circular motion. Arc length, a, means the distance travelled along the circumference of a circle of radius r. For a central angle θ, in radians, $a = \theta r$.

Working Example 1: Convert Between Degree and Radian Measure

Draw each angle in standard position. Convert each degree measure to radian measure and each radian measure to degree measure. Give answers as both exact and, if necessary, approximate measures to the nearest hundredth of a unit.

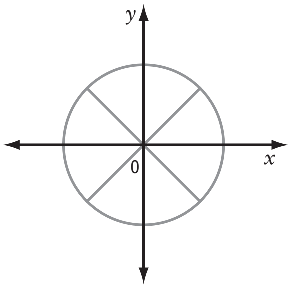
a) 135°

b) $\frac{5\pi}{6}$

c) 4

Solution

a) Draw the angle 135° in standard position. What is its reference angle? ________



The angle 135° is $\frac{3}{4}$ of a half circle, or $\frac{3}{4} \times \pi$.

Convert the degree measure to radian measure.

$360° = 2\pi$

$1° = \frac{\square}{\square}$

$= $ ________

$135° = 135$ ________

$= $ ________ (exact measure in terms of π)

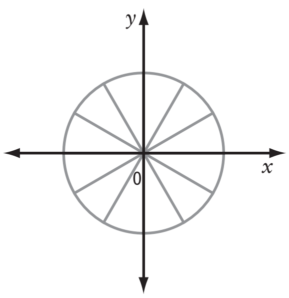
$= $ ________ (approximate measure, to two decimal places)

Is your answer reasonable? Verify using the diagram.

When expressing an angle measure in radians, no unit is necessary.

Use the π button on your calculator.

b) Draw the angle $\frac{5\pi}{6}$ in standard position.



Each half circle (π) is divided into 6 segments.

Convert the radian measure to degree measure.

$2\pi =$ ________°

$\pi =$ ________°

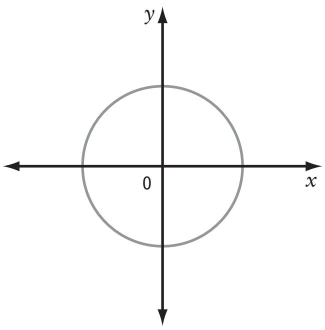
$$\frac{5\pi}{6} = \frac{5(\square^\circ)}{6}$$

$$\frac{5\pi}{6} = \text{\_\_\_\_\_\_\_\_}^\circ$$

When expressing an angle measure in degrees, the degree symbol ° is used.

Is your answer reasonable? Verify using the diagram.

c) The measure 4 has no units shown, so it represents an angle in radians.
Draw the angle 4 in standard position. Hint: $\pi \approx 3.14$ is equivalent to 180°.



Convert the radian measure to degree measure.

2π radians = 360°

$$1 \text{ radian} = \frac{\square^\circ}{\square}$$

4 radians = ________° (exact measure)

= ________° (approximate measure, to two decimal places)

Is your answer reasonable? Verify using the diagram.

Refer to pages 168–169 of *Pre-Calculus 12* for two other valid procedures for converting between degrees and radians. Use whichever method makes the most sense to you.

Working Example 2: Identify Coterminal Angles and Express Them in General Form

a) Identify the angles coterminal with 210° that satisfy the domain $-720° \leq \theta < 720°$. Express the angles coterminal with 210° in general form.

b) Identify the angles coterminal with $-\frac{3\pi}{4}$ within the domain $-4\pi \leq \theta < 4\pi$. Express angles coterminal with $-\frac{3\pi}{4}$ in general form.

Solution

a) To determine angles coterminal with 210°, add and subtract multiples of 360° (1 full rotation).

The given domain of $0° \pm 720°$ is $\pm$ 2 rotations. Determine the coterminal angles and cross out any that fall outside the given domain.

$\theta - 2(360°)$	$\theta - 360°$	$\theta + 360°$	$\theta + 2(360°)$

The values that satisfy the domain $-720° \leq \theta < 720°$ are ______________________________.

For n rotations, the general form for angles coterminal with 210° is __________________, $n \in \mathrm{N}$.

b) To determine coterminal angles, add and subtract multiples of 2π (1 full rotation).

$2\pi = \frac{\square\pi}{4}$

The given domain of $0 \pm 4\pi$ is $\pm$ _________ rotations.

Determine the coterminal angles and cross out any that fall outside the given domain.

$\theta - 2(2\pi)$	$\theta - 2\pi$	$\theta + 2\pi$	$\theta + 2(2\pi)$

The values that satisfy the domain _________ $\leq \theta <$ _________ are

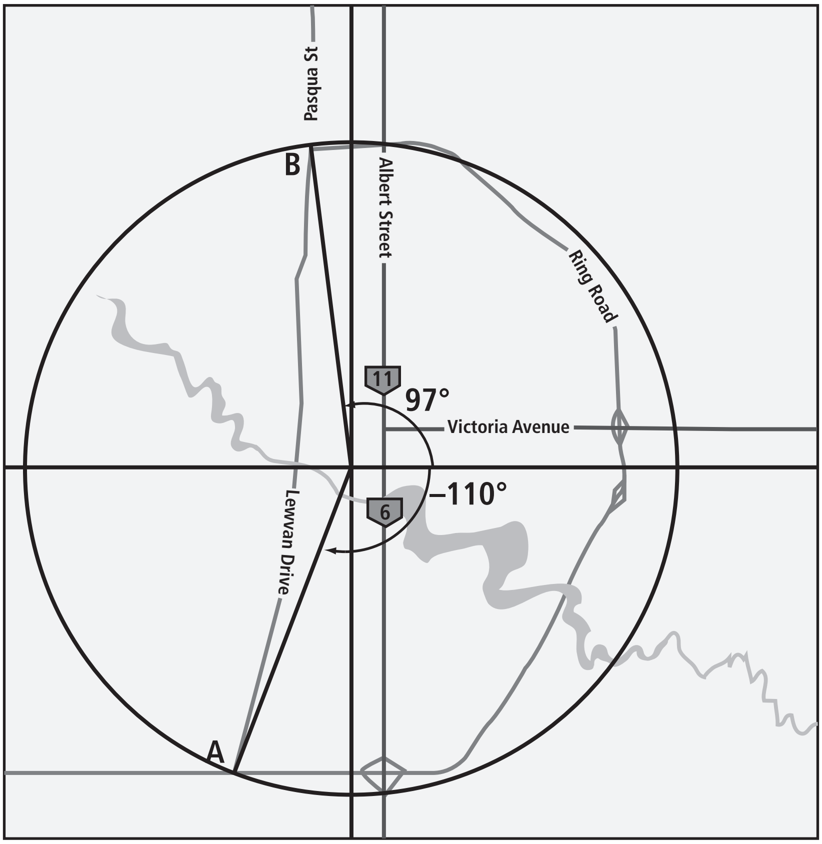
______________________________.

For n rotations, the general form for angles coterminal with $-\frac{3\pi}{4}$ is __________________, $n \in \mathrm{N}$.

See Examples 2 and 3 on pages 170–172 of *Pre-Calculus 12* for more examples.

Working Example 3: Determine Arc Length in a Circle

The ring road around the eastern part of the city of Regina is almost a semicircle. Estimate the length of the ring road (from A to B) if the radius of the circle is 4.9 km.



Solution

Determine the measure of the central angle in radians. Then, use the formula $a = \theta r$ to determine the arc length.

The central (obtuse) angle is ________ (in degrees).

Convert the degree measure to radian measure.

$$360° = \underline{\qquad\qquad}$$

$$1° = \underline{\qquad\qquad}$$

$$\underline{\qquad\qquad} \times 1° = \underline{\qquad\qquad} \times \frac{2\pi}{360}$$

$$\approx \underline{\qquad\qquad\qquad\qquad}$$

Write down a decimal approximation, but keep all digits in your calculator.

For central angles θ expressed in radians, the arc length of a circle of radius r is $a = \theta r$.

Therefore, the length of the ring road is approximately ________________.

What are the units?

The actual distance is 17.6 km. How accurate is your estimate?

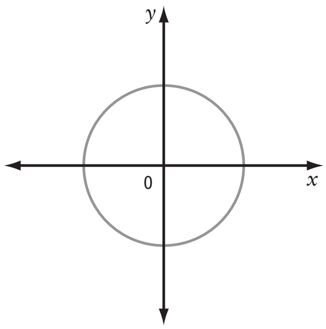
For a similar problem on a larger scale, try #20 on page 178 of *Pre-Calculus 12*.

Check Your Understanding

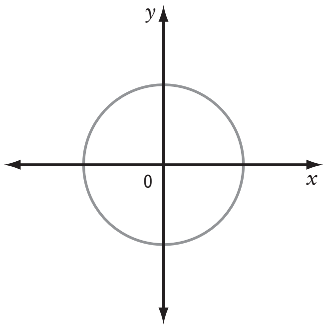
Practise

1. Sketch each angle in standard position. Change each degree measure to radian measure. Express your answer as an exact value (in terms of fractions of π).

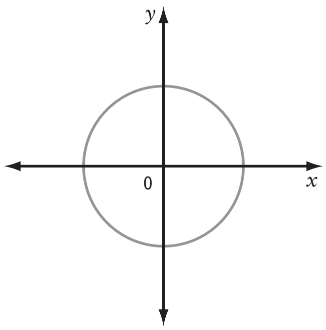
a) 60°



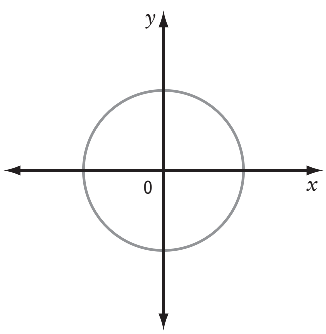
b) 315°



c) −210°

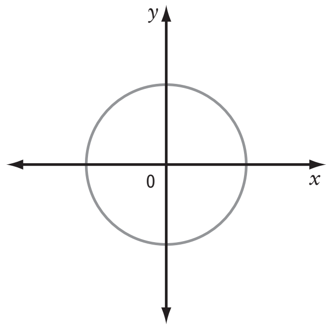


d) 600°

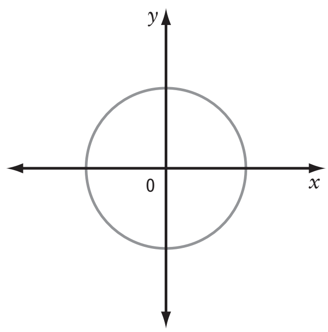


2. Draw each angle in standard position. Change each degree measure to radian measure. Express your answer as a decimal rounded to two decimal places.

a) 101°



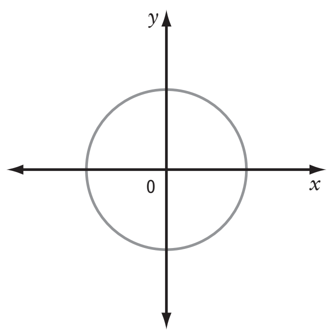
b) 57.3°



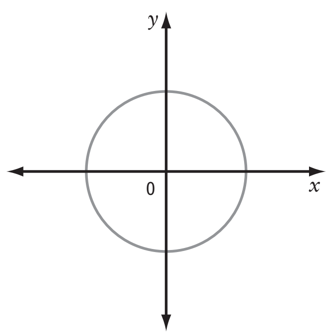
For additional practice, see #3 on page 175 of *Pre-Calculus 12*.

3. Sketch each angle in standard position. Change each radian measure to degree measure. If necessary, round your answer to two decimal places.

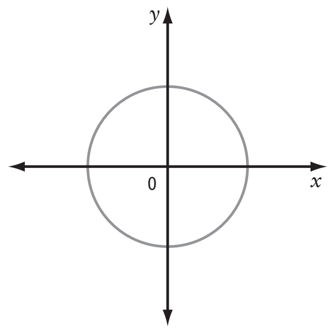
a) $\frac{\pi}{2}$



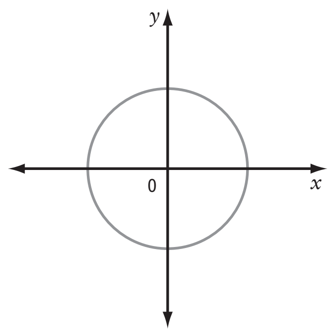
b) $\frac{4\pi}{3}$



c) $-\frac{2\pi}{9}$



d) 2



4. Determine one positive and one negative angle coterminal with each angle given.

a) 349°

b) −487°

c) $\frac{2\pi}{3}$

d) $\frac{9\pi}{4}$

5. For each angle θ, determine all coterminal angles within the given domain. Write an expression for all angles coterminal with θ in general form.

a) $\theta = 255°$ within the domain $-720° \leq \theta < 720°$

$\theta - 2(360°)$	$\theta - 360°$	$\theta + 360°$	$\theta + 2(360°)$

For n rotations, the general form for angles coterminal with 255° is

________________, $n \in N$.

b) $\theta = \pi$ within the domain $-4\pi \leq \theta < 4\pi$

$\theta - 4\pi$	$\theta - 2\pi$	$\theta + 2\pi$	$\theta + 4\pi$

For n rotations, the general form for angles coterminal with π is

________________, $n \in N$.

c) $\theta = \frac{5\pi}{6}$ within the domain $-2\pi \leq \theta < 6\pi$

For n rotations, the general form for angles coterminal with $\frac{5\pi}{6}$ is

________________, $n \in N$.

Also try #11 on page 176 of *Pre-Calculus 12*.

6. Determine the arc length subtended by each central angle. Give answers to the nearest hundredth of a unit.

a) radius 20 cm, central angle $\frac{2\pi}{3}$

b) radius 15 mm, central angle 195°

Apply

7. Angular velocity describes the rate of change in a central angle over time. For example, the change could be expressed in revolutions per minute (rpm), radians per minute, or degrees per second. To determine linear velocity from angular velocity, use the formula $v = \omega r$, where ω is the angular velocity in radians per unit of time and r is the radius of the circular motion.

a) How does the angular velocity formula compare to the formula for arc length?

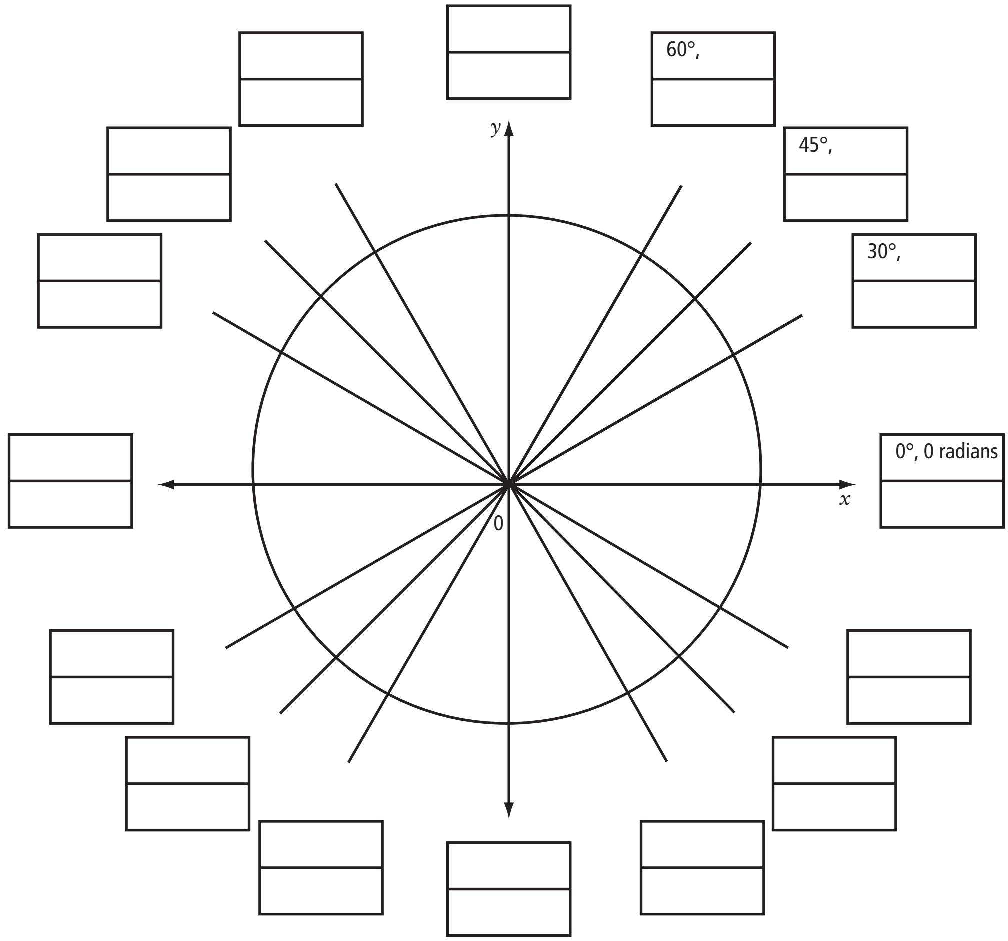
b) The Great Beijing wheel, a Ferris wheel with a diameter of 198 m, makes 1 revolution in 20 min. What is its angular velocity, in radians per minute? What is the linear velocity of a passenger, in m/s?

c) A bicycle wheel turns at 60 rpm. If the wheels of the bicycle measure 650 mm across, what distance, in metres, does the bicycle travel in 1.00 min?

d) The mean distance from Earth to the moon is 385 000 km. The moon travels around Earth once every 27.2 days. Assuming a circular orbit, what is the linear velocity of the moon, in km/h?

Connect

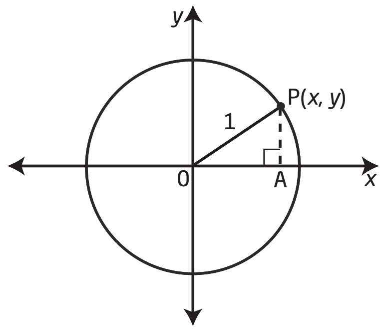
8. Label the given angles (including the axes) in both degrees and radians, $0° \leq \theta < 720°$ and $0 \leq \theta < 4\pi$.



4.2 The Unit Circle

KEY IDEAS

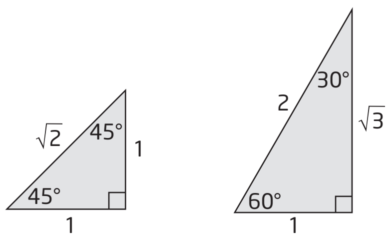
- In general, a circle of radius r centred at the origin has equation $x^2 + y^2 = r^2$.
- The unit circle has radius 1 and is centred at the origin. The equation of the unit circle is $x^2 + y^2 = 1$. All points $P(x, y)$ on the unit circle satisfy this equation.



- An arc length measured along the unit circle equals the measure of the central angle (in radians).

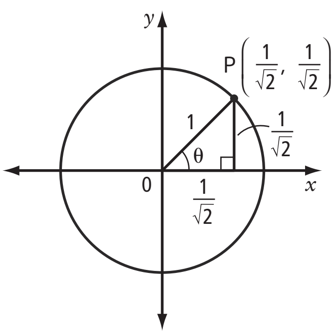
In other words, when $r = 1$, the formula $a = \theta r$ simplifies to $a = \theta$.

- Recall the special right triangles you learned about previously.

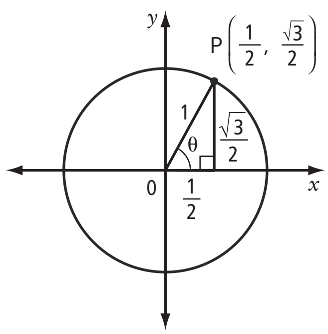


These special triangles can be scaled to fit within the unit circle ($r = 1$).

$\theta = \frac{\pi}{4}$



$\theta = \frac{\pi}{3}$



Working Example 1: Determine Coordinates for a Point on the Unit Circle

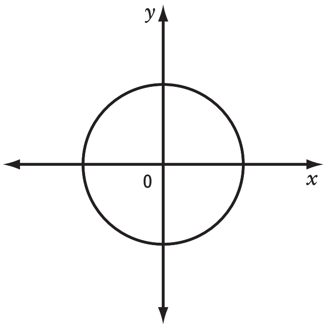
Determine the missing y-coordinate for each point on the unit circle.

a) point A $\left(-\frac{4}{5}, y\right)$ in quadrant II

b) point B $\left(\frac{3}{10}, y\right)$ in quadrant IV

Solution

a) Start by sketching the point on the unit circle. Then, solve for y in the equation of the unit circle.



The equation of the unit circle is $x^2 + y^2 = 1$.
Substitute in the known value of x.

$\boxed{}^2 + y^2 = 1$

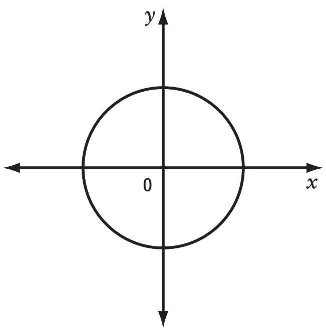
Solve for y.

Isolate y^2. Then, take the square root of both sides.

Since quadrant ________ is specified, take only the ________________ root.
(*positive* or *negative*)

Therefore, $y =$ ________.

b) Start by sketching the point on the unit circle. Then, solve for y in the equation of the unit circle.



$x^2 + y^2 = 1$

Check that your answer is reasonable using the fact that $|y| \leq 1$ on the unit circle.

Since quadrant ________ is specified, take only the ________________ root.
(*positive* or *negative*)

Therefore, $y =$ ________.

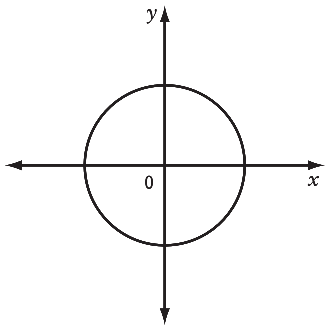
Working Example 2: Reflections of $\frac{\pi}{6}$ on the Unit Circle

Determine the coordinates of all points on the unit circle for which the reference angle is $\frac{\pi}{6}$.

Solution

Start by drawing a diagram showing $\frac{\pi}{6}$ reflected in all four quadrants.

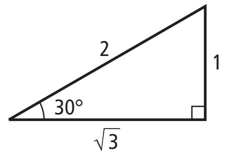
> What is $\frac{\pi}{6}$ in degrees?



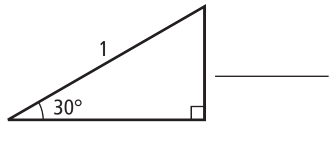
The angles (in radians) are $\frac{\pi}{6}$, $\frac{\square}{\square}$, $\frac{\square}{\square}$, $\frac{\square}{\square}$.

The points are A $\left(\frac{\pi}{6}\right)$, B $\left(\frac{\square}{\square}\right)$, C $\left(\frac{\square}{\square}\right)$, and D $\left(\frac{\square}{\square}\right)$.

The appropriate special triangle is:



Scaled down to fit within a unit circle ($r = 1$):



Therefore, point A $\left(\frac{\pi}{6}\right)$ has coordinates ________.

- Point B $\left(\frac{\square}{\square}\right)$ is a reflection in the y-axis of point A.

  The x-coordinate changes sign and the y-coordinate stays the same.

  Therefore, point B has coordinates ________.

- Point C $\left(\frac{\square}{\square}\right)$ is a reflection in the x-axis and y-axis of point A.

  The x-coordinate is ________ and the y-coordinate is ________.

  Therefore, point C has coordinates ________.

- D $\left(\frac{\square}{\square}\right)$ is a reflection in the x-axis of point A.

  The x-coordinate is ________ and the y-coordinate is ________.

  Therefore, point D has coordinates ________.

Check Your Understanding

Practise

1. Determine the equation of a circle centred at (0, 0) with each radius.

   a) 25 units

   b) 1.1 units

2. Is each point on the unit circle? Give evidence to support your answer.

   a) $(0.65, -0.76)$

   The equation of the unit circle is ______________.

Left Side	Right Side

   Conclusion: ______________________

   b) $\left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)$

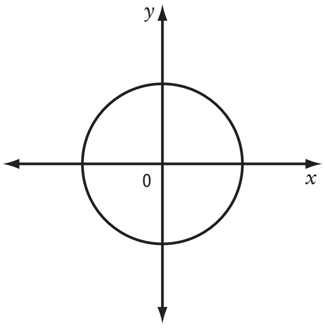
   c) $\left(\frac{\sqrt{7}}{2}, -\frac{1}{7}\right)$

What information could you use to answer part c) other than a left side/right side proof?

3. Determine the missing coordinate for each point on the unit circle. Draw a diagram to support your answer.

a) point A$\left(x, \frac{5}{13}\right)$ in quadrant II

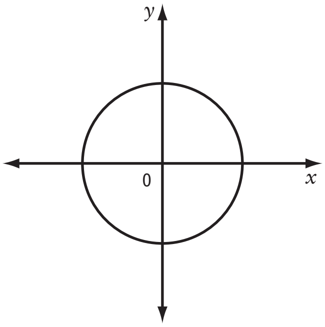
$x^2 + y^2 = 1$



Since quadrant ________ is specified, take only the ______________ root.
(*positive* or *negative*)

Therefore, x = ________.

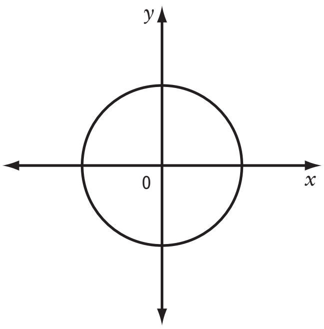
b) point B$\left(\frac{1}{6}, y\right)$ in quadrant IV



Since quadrant ________ is specified, take only the ______________ root.
(*positive* or *negative*)

Therefore, y = ________.

c) point C$\left(x, -\frac{1}{2}\right)$ in quadrant III

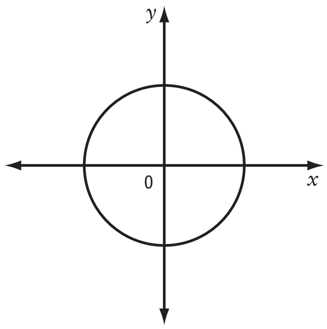


Do you recognize this pair of values? What angle is associated with it?

4. If P(θ) is the point at which the terminal arm of angle θ in standard position intersects the unit circle, determine the exact coordinates of each of the following.

The word *exact* in the question is a clue to use special triangles.

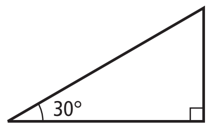
a) $P\left(\frac{\pi}{2}\right)$



b) $P(2\pi)$

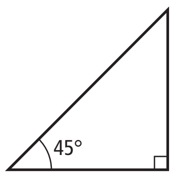
c) $P\left(\frac{2\pi}{3}\right)$

d) $P\left(\frac{5\pi}{4}\right)$



e) $P\left(-\frac{\pi}{4}\right)$

f) $P\left(\frac{23\pi}{6}\right)$



5. Determine the value of angle θ in standard position, $0 \leq \theta < 2\pi$, given the coordinates of $P(\theta)$, the point at which the terminal arm intersects the unit circle.

> The domain is given in radians, so your answers should also be in radians.

a) $P(\theta) = (-1, 0)$ $\qquad \theta =$ ________

b) $P(\theta) = \left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ $\qquad \theta =$ ________

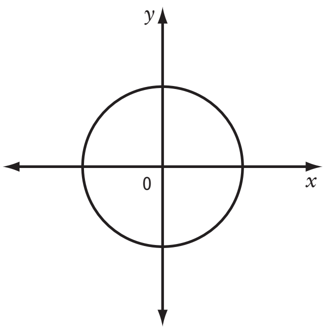
c) $P(\theta) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ $\qquad \theta =$ ________

d) $P(\theta) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$ $\qquad \theta =$ ________

This question should help you complete #5 and #6 on page 187 of *Pre-Calculus 12*.

6. Determine the arc length on the unit circle from (1, 0) to each point.

a) $P\left(\frac{\pi}{2}\right)$

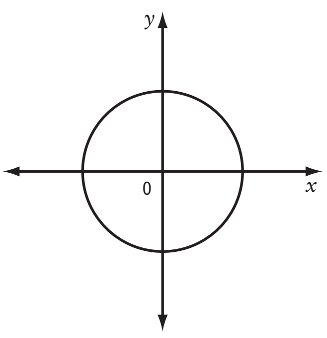


$\theta =$ ________

On the unit circle, $r = 1$.

So, $a =$ ________.

b) $P(\theta)\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$



$\theta =$ ________

On the unit circle, $r = 1$.

So, $a =$ ________.

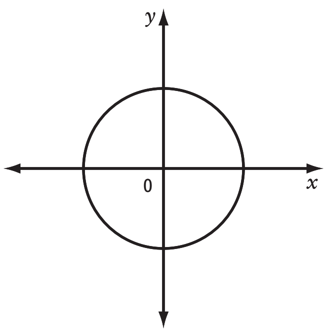
Apply

7. Earth moves in a roughly circular orbit around the sun at a distance of approximately 150 000 000 km. To estimate longer distances in space, scientists measure in multiples of the Earth–sun radius, or astronomical units (1 AU = 149 597 870.691 km). NASA defines 1 AU as the radius of an unperturbed circular orbit of a massless theoretical body revolving about the sun in $\frac{2\pi}{k}$ days, where k is a constant exactly equal to 0.017 202 098 95.

a) Placing the sun at the origin, write an equation representing Earth's orbit, in kilometres, assuming a circular orbit.

b) Placing the sun at the origin, write an equation representing Earth's orbit (in AU), assuming a circular orbit.

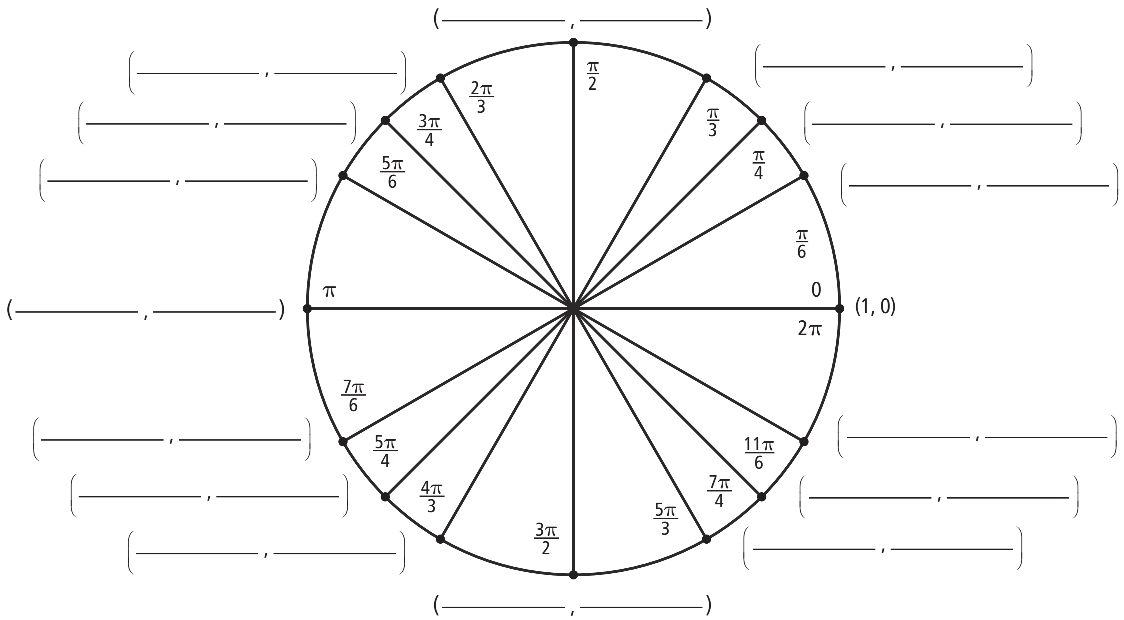
c) Mars is 1.38 AU from the sun. If the unit circle shown at right describes Earth's orbit, sketch the orbit of Mars.

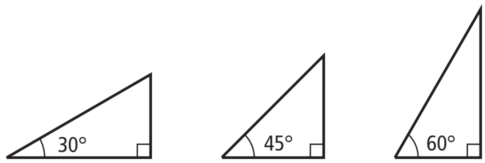


d) One year on Mars represents one complete rotation. How long (in radians) is 1 Mars-year on Earth? How long is this in Earth-days?

Connect

8. Give the exact coordinates of each P(θ) listed on the diagram below.





Check your work by referring to the Key Ideas on page 186 of *Pre-Calculus 12*.

4.3 Trigonometric Ratios

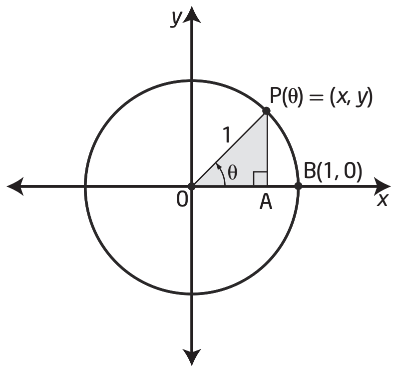
KEY IDEAS

- These are the primary trigonometric ratios:

sine	**cosine**	**tangent**
$\sin\theta = \frac{y}{r}$	$\cos\theta = \frac{x}{r}$	$\tan\theta = \frac{y}{x}$

- For points on the unit circle, $r = 1$. Therefore, the primary trigonometric ratios can be expressed as:

  $\sin\theta = \frac{y}{1} = y \qquad \cos\theta = \frac{x}{1} = x \qquad \tan\theta = \frac{y}{x}$

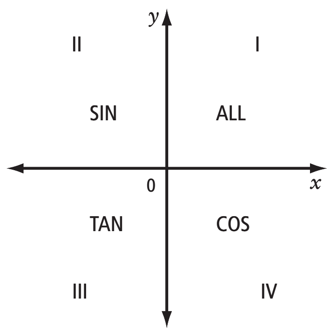


- Since $\cos\theta$ simplifies to x and $\sin\theta$ simplifies to y, you can write the coordinates of $P(\theta)$ as $P(\theta) = (\cos\theta, \sin\theta)$ for any point $P(\theta)$ at the intersection of the terminal arm of θ and the unit circle.

- These are the reciprocal trigonometric ratios:

cosecant	**secant**	**cotangent**
$\csc\theta = \frac{1}{\sin\theta}$	$\sec\theta = \frac{1}{\cos\theta}$	$\cot\theta = \frac{1}{\tan\theta}$
$\csc\theta = \frac{r}{y}$	$\sec\theta = \frac{r}{x}$	$\cot\theta = \frac{x}{y}$

- Recall from the CAST rule that
  – $\sin\theta$ and $\csc\theta$ are positive in quadrants I and II
  – $\cos\theta$ and $\sec\theta$ are positive in quadrants I and IV
  – $\tan\theta$ and $\cot\theta$ are positive in quadrants I and III



Working Example 1: Determine the Trigonometric Ratios for Angles in the Unit Circle

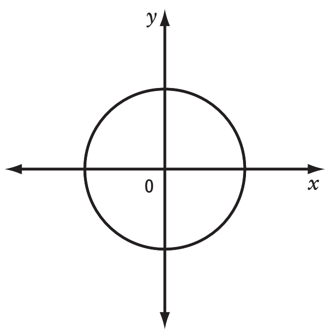
Point A $\left(-\frac{12}{13}, \frac{5}{13}\right)$ is on the unit circle and on the terminal arm of an angle θ in standard position.

Determine the values of the six trigonometric ratios for angle θ.

Solution

Method 1: Use the Unit Circle

Start by sketching point A on the unit circle.



Since point A is on the unit circle,

- the ________-coordinate is defined as cos θ
- the y-coordinate is defined as ________

The primary trigonometric ratios are

sin θ = ________ cos θ = ________ tan θ = $\frac{y}{x}$

= ________

= ________

Now, take the reciprocal of each ratio to determine the reciprocal trigonometric ratios.

csc θ = $\frac{1}{\sin\theta}$ sec θ = $\frac{1}{\boxed{}}$ ________ θ = $\frac{1}{\tan\theta}$

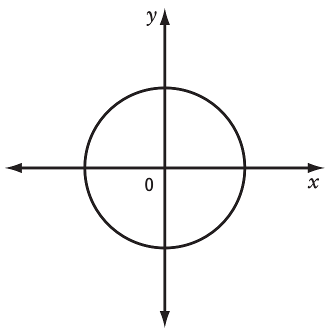
= ________ = ________ = ________

= ________ = ________ = ________

> Leave your answer as a fraction in lowest terms, unless "approximate" (meaning decimal) value is specified.

Method 2: Use a Right Triangle

Start by sketching point A on the unit circle. Draw a vertical line from point A to the x-axis to form right △ABO.



The angle is in quadrant ________.

x = ________ y = ________ $r = 1$

> How could you determine r for a point not on the unit circle?

Substitute these values into the definitions of the primary trigonometric ratios.

sin θ = $\frac{y}{r}$ cos θ = $\frac{x}{r}$ tan θ = $\frac{y}{x}$

= ________ = ________ = ________

Take the reciprocal of each ratio to determine the reciprocal trigonometric ratios.

csc θ = $\frac{1}{\sin\theta}$ sec θ = $\frac{1}{\cos\theta}$ cot θ = $\frac{1}{\tan\theta}$

= ________ = ________ = ________

Note that this process can be followed for points that are not on the unit circle.

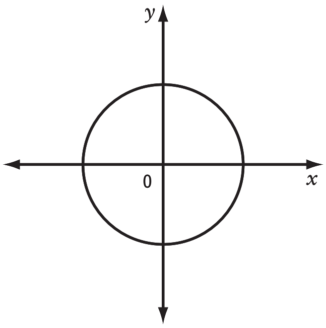
Working Example 2: Exact and Approximate Values for Trigonometric Ratios

a) Determine the exact value of csc $\frac{5\pi}{4}$.

b) Determine the value of cot (−0.5) to four decimal places.

Solution

a) Sketch the angle $\frac{5\pi}{4}$ in standard position on the unit circle.



Point $P\left(\frac{5\pi}{4}\right)$ is in quadrant ________. The reference angle for $\frac{5\pi}{4}$ is θ_R = ________.

Sketch the special triangle for the reference angle. Determine the values of x (cos θ_R) and y (sin θ_R).

For the reference angle θ_R (in quadrant I), $P\left(\frac{\pi}{4}\right)$ = (________, ________).

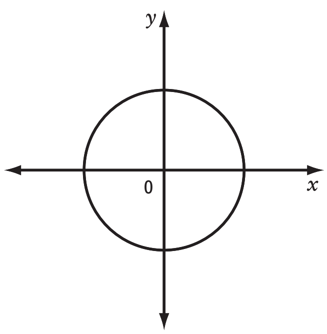
Therefore, for the angle $\frac{5\pi}{4}$, $P\left(\frac{5\pi}{4}\right)$ = (________, ________).

csc θ is the reciprocal of ________.

csc θ = ________.

> The expression *exact value* is a clue that you should be thinking in terms of special triangles and fractions of π.

b) Draw the angle −0.5 in standard position on the unit circle.



–0.5 is in quadrant ________. The reference angle

θ_R is ________.

cot θ is the reciprocal of ________.

> Is the angle in radians or degrees? Make sure your calculator is on the correct setting.

This is not one of the special angles for which you know exact values of x and y using special triangles. Therefore, calculate the primary trigonometric ratio of the reference angle using your calculator.

Then, take the reciprocal.

Determine the sign (+ or −).

Final answer: cot (–0.5) ≈ ________________

> Your scientific or graphing calculator has a reciprocal button, usually labelled $\frac{1}{x}$ or x^{-1}. Do not round until the final answer.

Working Example 3: Determine Angles Given Their Trigonometric Ratios

Determine the measure of all angles that satisfy the following conditions. Give exact answers where possible. Otherwise, round to two decimal places.

a) $\sec \theta = \sqrt{2}$ in the domain $0 \leq \theta < 4\pi$

b) $\csc \theta = -1.5557$ in the domain $-360° \leq \theta < 360°$

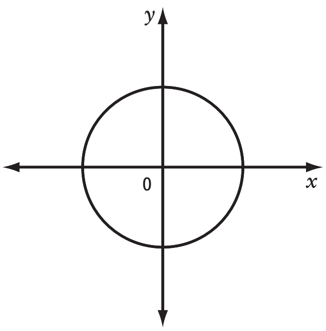
Solution

Degrees or radians?

a) $\sec \theta = \sqrt{2}$. Therefore, $\cos \theta =$ ________.

Draw the special triangle.

Then, draw the angle on the unit circle.



The reference angle θ_R is ________.

Cosine and secant are positive in quadrants ________ and ________.

Add the associated angles to your diagram of the unit circle.

quadrant I angle: ________

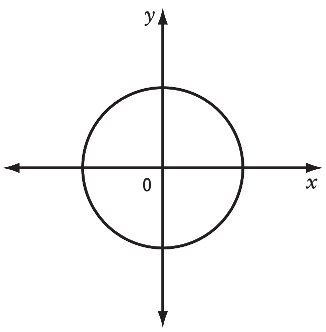
quadrant IV angle: ________

Now, check the domain and determine all additional relevant coterminal angles.

Therefore, $\sec \theta = \sqrt{2}$ when $\theta =$ ________, ________, ________, and ________, $0 \leq \theta < 4\pi$.

b) $\csc \theta = -1.5557$. Therefore, $\sin \theta =$ ________.

Draw the angle on the unit circle and calculate θ_R.



The reference angle θ_R is ________.

Sine and cosecant are negative in quadrants ________ and ________.

Add the associated angles to your diagram of the unit circle.

quadrant III angle: ________

quadrant IV angle: ________

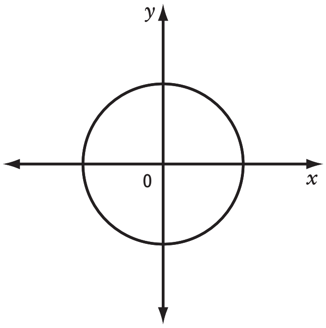
Now, check the domain and determine all additional relevant coterminal angles.

Therefore, $\csc \theta = -1.5557$ when $\theta =$ ________, ________, ________, and ________, $-360° \leq \theta < 360°$.

Check Your Understanding

Practise

1. Point P$\left(\frac{7}{25}, -\frac{24}{25}\right)$ is on the unit circle and on the terminal arm of an angle θ in standard position. Determine the values of the six trigonometric ratios for angle θ.

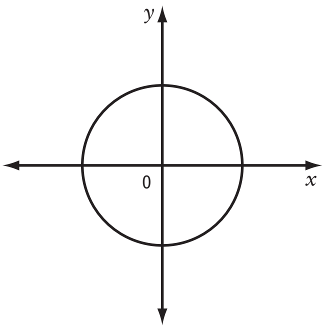


$\sin\theta =$ ________ $\cos\theta =$ ________ $\tan\theta =$ ________

$\csc\theta =$ ________ $\sec\theta =$ ________ $\cot\theta =$ ________

2. Without using a calculator, determine the sign (+ or −) of each of the following.

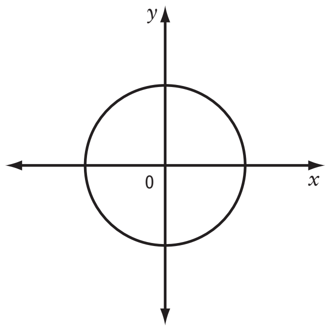
a) sin 580°



quadrant ________

sign is ________

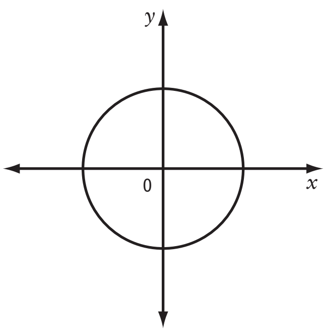
b) tan 1



quadrant ________

sign is ________

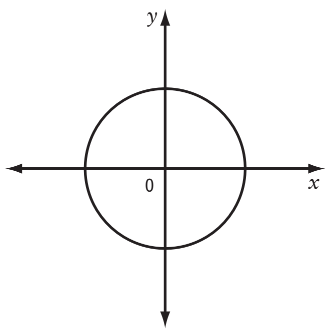
c) $\csc\theta = \frac{2\pi}{3}$



quadrant ________

sign is ________

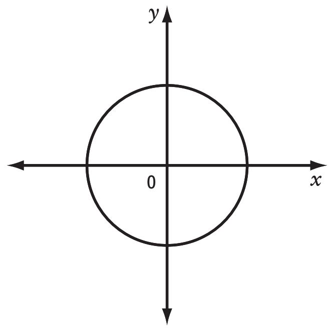
d) $\sec\theta = \frac{5\pi}{4}$



quadrant ________

sign is ________

3. In which quadrant(s) is (are) the terminal arm(s) of angle θ given the following conditions?



a) cot θ is positive ________

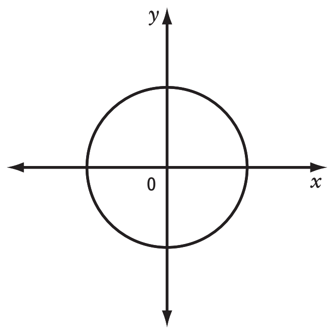
b) cot θ is positive and sin θ is negative ________

c) csc θ = 1.2 ________

d) csc θ = 1.2 and cos θ = −0.574 ________

4. What is the exact value for each trigonometric ratio?

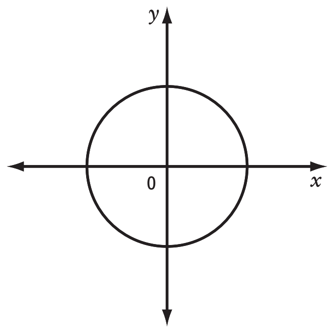
a) $\cos \frac{\pi}{3}$



$P\left(\frac{\pi}{3}\right)$ is in quadrant ________.

$\theta_R =$ ________

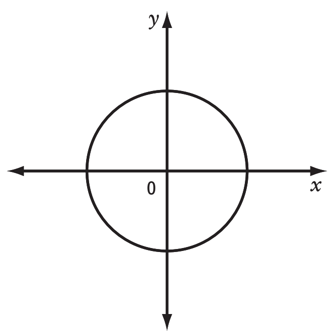
b) $\sin \frac{\pi}{4}$



$P\left(\frac{\pi}{4}\right)$ is in quadrant ________.

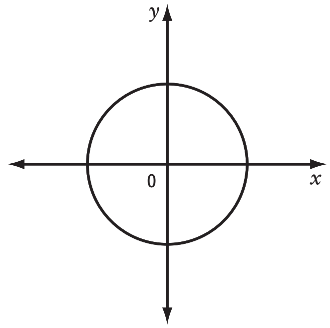
$\theta_R =$ ________

c) $\tan 3\pi$



$P(3\pi)$ is a quadrantal angle.

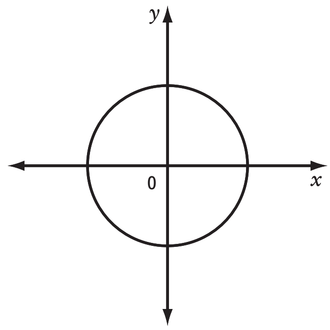
d) $\cot\left(-\frac{2\pi}{3}\right)$



$P\left(-\frac{2\pi}{3}\right)$ is in quadrant ________.

$\theta_R =$ ________

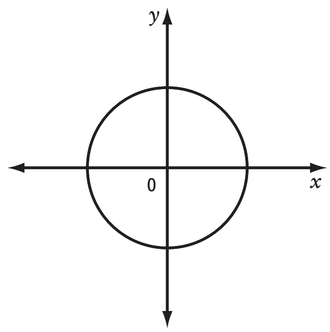
e) $\sec \frac{5\pi}{6}$



$P\left(\frac{5\pi}{6}\right)$ is in quadrant ________.

$\theta_R =$ ________

f) $\csc\left(-\frac{9\pi}{4}\right)$



$P\left(-\frac{9\pi}{4}\right)$ is in quadrant ________.

$\theta_R =$ ________

5. Determine the approximate value for each trigonometric ratio, to three decimal places.

a) sec 74°

sec θ is the reciprocal of ________

quadrant: ________

sign (+ or −): ________

b) cot 104°

c) csc 2.8

d) $\sec\left(-\frac{7\pi}{10}\right)$

These questions are similar to #1 and #2 on page 201 of *Pre-Calculus 12*.

Apply

6. Determine the measure of all angles that satisfy the following conditions. Round your answers to the nearest degree.

a) $\tan \theta = -3.078$ in the domain $0° \le \theta < 720°$

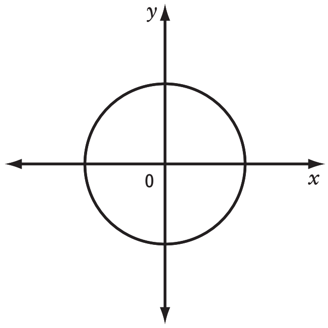
$\theta_R = \tan^{-1}(+3.078)$

$\approx$ ________

Tangent is negative in quadrants ________ and ________.

Therefore, $\tan \theta = -3.078$ when $\theta \approx$ ________,

________, ________, and ________, $0° \le \theta < 720°$.



Which other coterminal angles fall within the domain?

b) $\sec \theta = -1.046$ in the domain $-360° \le \theta < 360°$

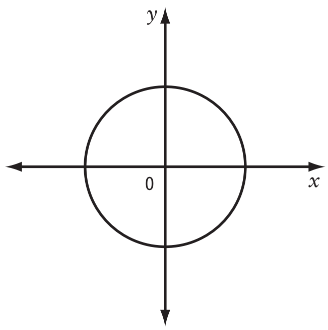
$\sec \theta$ is the reciprocal of ________.

$\theta_R \approx$ ________

Secant is negative in quadrants ________ and ________.

Therefore, $\sec \theta = -1.046$ when $\theta \approx$ ________,

________, ________, and ________, $-360° \le \theta < 360°$.



7. Determine the measure of all angles that satisfy the following conditions. Give exact answers.

Drawing the special triangle may help.

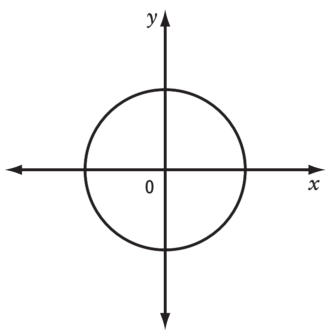
a) $\sin \theta = -\frac{\sqrt{3}}{2}$ in the domain $0 \le \theta < 4\pi$

$\theta_R =$ ________

Sine is negative in quadrants ________ and ________.

Therefore, $\sin \theta = -\frac{\sqrt{3}}{2}$ when $\theta =$ ________, ________,

________, and ________, $0 \le \theta < 4\pi$.



b) $\csc \theta = 2$ in the domain $-2\pi \le \theta < 2\pi$

What is the reciprocal of 2?

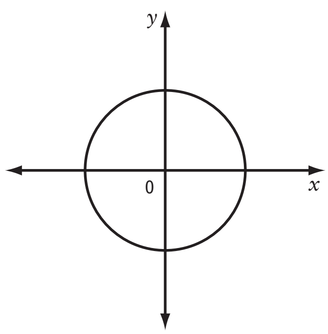
$\csc \theta$ is the reciprocal of ________.

$\theta_R =$ ________

Cosecant is positive in quadrants ________ and ________.

Therefore, $\csc \theta = 2$ when $\theta =$ ________, ________,

________, and ________, $-2\pi \le \theta < 2\pi$.



8. Determine the value of the five other trigonometric ratios if $\csc \theta = \frac{5}{3}$, $90° \leq \theta < 180°$.

The angle is in quadrant ________.

$x =$ ________ $y =$ ________ $r =$ ________

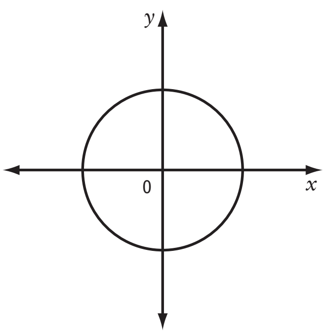
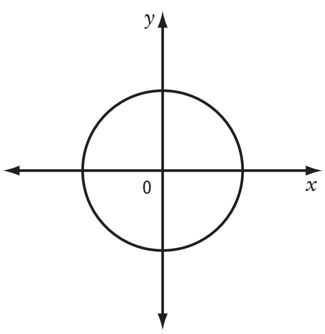
$\sin \theta =$ ________ $\cos \theta =$ ________ $\tan \theta =$ ________

$\sec \theta =$ ________ $\cot \theta =$ ________

This question will help you with #12 on page 202 of *Pre-Calculus 12*.

Connect

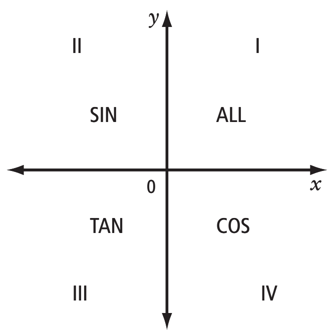
9. Choose any two of the special angles $\frac{\pi}{6}$, $\frac{\pi}{4}$, and $\frac{\pi}{3}$. Complete the table below. You may also choose the quadrantal angles (on the axes), but then you will have to change the headings on the table.

$\theta_R =$	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
	sin csc			
	cos sec			
	tan cot			
$\theta_R =$	Quadrant I	Quadrant II	Quadrant III	Quadrant IV
	sin csc			
	cos sec			
	tan cot			

4.4 Introduction to Trigonometric Equations

KEY IDEAS

- Solving an equation means to determine the value (or values) of a variable that make an equation true (Left Side = Right Side).
  For example, $\sin \theta = \frac{1}{2}$ is true when $\theta = 30°$ or $\theta = 150°$, and for every angle coterminal with 30° or 150°. These angles are solutions to a very simple trigonometric equation.
- The variable θ is often used to represent the unknown angle, but any other variable is allowed.
- In general, solve for the trigonometric ratio, and then determine
  – all solutions within a given domain, such as $0 \leq \theta < 2\pi$
  or
  – all possible solutions, expressed in general form, $\theta + 2\pi n, n \in \mathrm{I}$
- Unless the angle is a multiple of 90° or $\frac{\pi}{2}$, there will be two angles per solution of the equation within each full rotation of 360° or 2π. As well, there will be two expressions in general form per solution, one for each angle. It is sometimes possible to write a combined expression representing both angles in general form.
- If the angle is a multiple of 90° or $\frac{\pi}{2}$ (that is, the terminal arm coincides with an axis), then there will be at least one angle within each full rotation that is a correct solution to the equation.
- Note that $\sin^2 \theta = (\sin \theta)^2$. Also, recall that
  – $\sin \theta$ and $\csc \theta$ are positive in quadrants I and II
  – $\cos \theta$ and $\sec \theta$ are positive in quadrants I and IV
  – $\tan \theta$ and $\cot \theta$ are positive in quadrants I and III



Working Example 1: Solve a First-Degree Trigonometric Equation

Solve for the angle (in degrees). Round answers to the nearest tenth of a degree.

a) $7 \cos \theta + 5 = 2 - 3 \cos \theta$, $0° \le \theta < 360°$

b) $17 + 3 \cot \theta = 29$, in general form

Solution

a) Start by solving for the trigonometric ratio.

$7 \cos \theta + 5 = 2 - 3 \cos \theta$

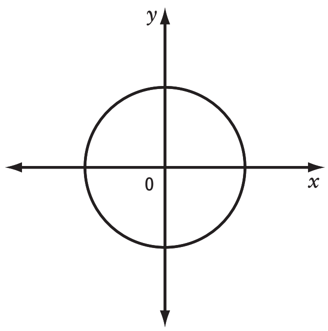
Treat $\cos \theta$ as a variable and isolate it on one side of the equation.

$\cos \theta =$ ________

Determine the reference angle.

$\theta_R = \cos^{-1}(+0.3)$

$\approx$ ________



Cosine is negative in quadrants ________ and ________.

Add the associated angles to your diagram of the unit circle.

quadrant II angle: ________

quadrant III angle: ________

Therefore, the solutions are $\theta =$ ________ and ________, $0° \le \theta < 360°$.

b) Start by solving for the trigonometric ratio.

$17 + 3 \cot \theta = 29$

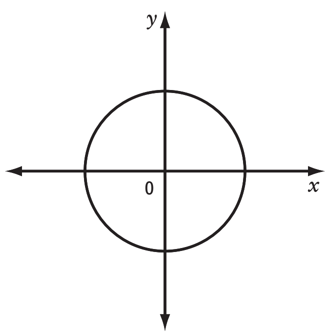
$3 \cot \theta =$ ________

$\cot \theta =$ ________

$\cot \theta$ is the reciprocal of ________. Therefore, $\tan \theta =$ ________.

Determine the reference angle.

$\theta_R =$ ________



Tangent and cotangent are positive in quadrants ________

and ________.

Add the associated angles to your diagram of the unit circle.

$\theta_1 =$ ________ $\theta_2 =$ ________

Can θ_1 and θ_2 be generalized in a single expression?

Notice on your diagram that $\theta_2 = \theta_1 + 180°$. There is a solution every 180°.

The general form is ________________ $+ 180°n$, $n \in I$.

Working Example 2: Solve Second-Degree Equations

Solve for the unknown value. If necessary, round your answer to two decimal places.

a) $2 \sin^2 \theta = 1, 0 \leq \theta < 2\pi$

b) $\tan^2 \theta - 4 \tan \theta + 3 = 0, 0 \leq \theta < 2\pi$

Solution

a) Isolate the trigonometric ratio $\sin^2 \theta$. Then, take the square root of both sides.

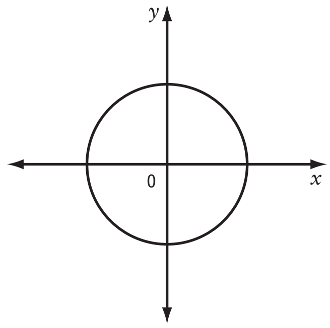
What is the degree of the equation? How many solutions are there?

$2 \sin^2 \theta = 1$

$\sin^2 \theta =$ ________________

$\sin \theta =$ ________ or $\sin \theta =$ ________

Determine the reference angles for both solutions.



$\theta_R =$ ________ $\theta_R =$ ________

sin > 0 (positive) in Q ________ and Q ________

$\theta_1 =$ ________ $\theta_2 =$ ________

or

sin < 0 (negative) in Q ________ and Q ________

$\theta_3 =$ ________ $\theta_4 =$ ________

Therefore, the solutions are ______________________________, $0 \leq \theta < 2\pi$.

b) Factor and solve for $\tan \theta$.

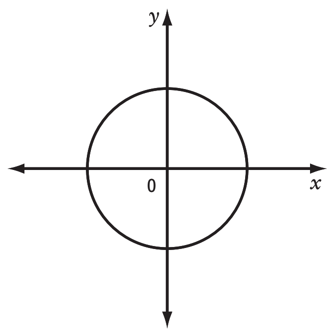
Treat $\tan \theta$ as a variable.

$$\tan^2 \theta - 4 \tan \theta + 3 = 0$$

$(\tan \theta -$ ________$)(\tan \theta -$ ________$) = 0$

Determine the reference angles for both solutions.

Is your calculator in radians?



$\theta_R =$ ________ $\theta_R =$ ________

tan > 0 (positive) in Q ________ and Q ________

$\theta_1 =$ ________ $\theta_2 =$ ________

or

tan < 0 (negative) in Q ________ and Q ________

$\theta_3 =$ ________ $\theta_4 =$ ________

Therefore, the solutions are ______________________________, $0 \leq \theta < 2\pi$.

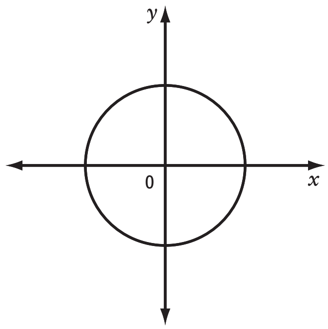
Also see Example 2 on page 208–209 of *Pre-Calculus 12*.

Check Your Understanding

Practise

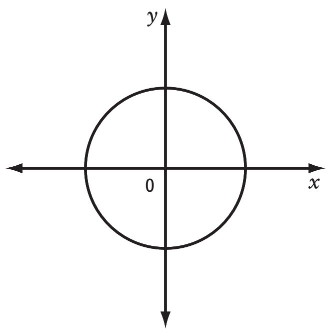
1. Determine the exact solutions for each trigonometric equation in the specified domain.

   a) $4 \sin \theta - 5 = 3$, $0° \leq \theta < 360°$



   b) $7 \cot \theta - 4 = 6 \cot \theta - 5$, $0 \leq \theta < 4\pi$

   cot θ is the reciprocal of ________.

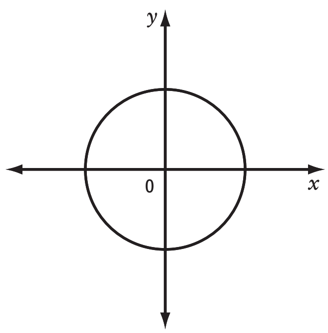


   Therefore, the solutions are

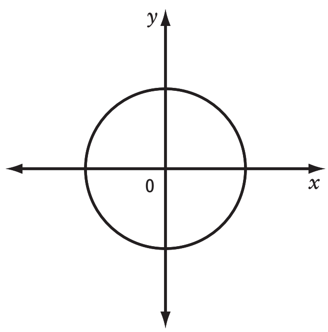
   $\theta =$ ______________________________, $0 \leq \theta < 4\pi$.

2. Solve for θ within the domain $0° \leq \theta < 360°$. Round answers to one decimal place.

   a) $-3(5 - 4 \sec \theta) = \sec \theta$



   b) $\csc \theta + \frac{3}{4} = -\frac{2}{3}$

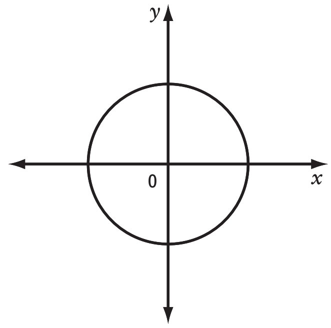


3. Solve for θ within the given domain. Give exact answers where possible. Otherwise, round your answer to two decimal places.

a) $4\cos^2\theta = 3,\ 0 \le \theta < 2\pi$

Treat cos θ as a variable.

Determine the reference angles for both solutions.

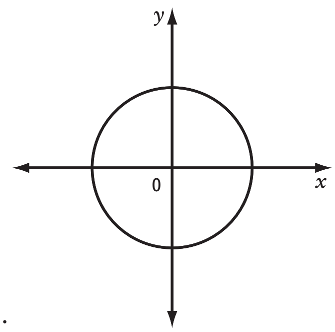


The solutions are __________________________, $0 \le \theta < 2\pi$.

b) $\csc^2\theta - 3\csc\theta - 10 = 0,\ 0 \le \theta < 360°$

Degrees or radians?

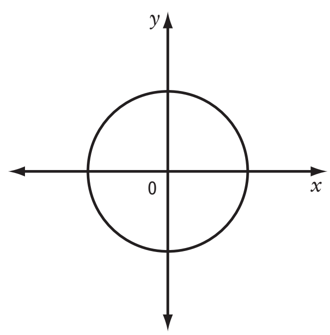
Determine the reference angles for both solutions.



The solutions are __________________________, $0 \le \theta < 360°$.

4. The equation $\cos\theta = \frac{\sqrt{3}}{2},\ 0 \le \theta < 2\pi$, has solutions $\frac{\pi}{6}$ and $\frac{11\pi}{6}$. Suppose the domain is not restricted.

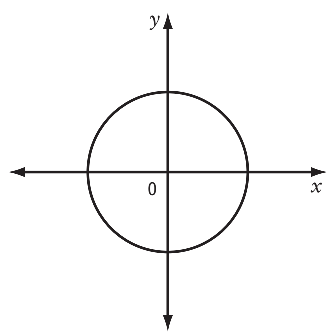
a) Write the general solution corresponding to $\frac{\pi}{6}$.



b) Write the general solution corresponding to $\frac{11\pi}{6}$.

5. The equation $\tan\theta = 1,\ 0 \le \theta < 2\pi$, has solutions $\frac{\pi}{4}$ and $\frac{5\pi}{4}$.

a) Write the solutions for $\tan\theta = 1$ if the domain is $0 \le \theta < 4\pi$.



b) Suppose the domain is unrestricted. Write the general solution.

Apply

6. The following solution contains one or more errors. Identify the errors and correct them.

Solve $\tan^2 \theta - 3 \tan \theta = 0$ in the domain $0 \leq \theta < 2\pi$.
Round answers to two decimal places.

$\tan^2 \theta - 3 \tan \theta = 0$

$\tan^2 \theta = 3 \tan \theta \quad (\div \tan \theta)$

$\tan \theta = 3$

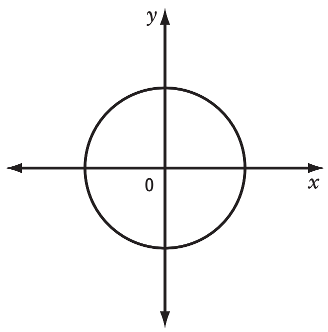
$\theta = \tan^{-1} (3)$

$= 1.25$

$\therefore$ The solution is 1.25 radians.

Correct solution:

7. a) Solve $\cos^2 \theta = \cos \theta$, $0 \leq \theta < 2\pi$. Give exact answers.



b) Suppose the domain of solutions to $\cos^2 \theta = \cos \theta$ is unrestricted. Write a general expression representing each solution. Can the expressions be combined to a single general expression representing all solutions? Why or why not?

Also try #13 on page 212 of *Pre-Calculus 12*.

8. Verify whether the expression $\dfrac{(1 + 2n)\pi}{3}$, $n \in I$ represents the general form for solutions to $\sec \theta = 2$.

Try various cases, such as $n = 0$.

Connect

9. Fill in the table with information related to solving trigonometric equations.

				Diagram
If	sin θ csc θ	< 0	the solutions will be in quadrants ________ and ________.	
If	sin θ csc θ	> 0	the solutions will be in quadrants ________ and ________.	
If	cos θ ________ θ	< 0	the solutions will be in quadrants ________ and ________.	
If	cos θ ________ θ	> 0	the solutions will be in quadrants ________ and ________.	
If	tan θ ________ θ	> 0	the solutions will be in quadrants ________ and ________.	
If	tan θ ________ θ	< 0	the solutions will be in quadrants ________ and ________.	

Chapter 4 Review

4.1 Angles and Angle Measure, pages 109–119

1. Convert each degree measure to radian measure and each radian measure to degree measure. Give exact values.

a) 270°

b) $\frac{5\pi}{3}$

c) 300°

d) -4

e) 495°

f) $\frac{13\pi}{4}$

2. Identify one positive and one negative angle measure that is coterminal with each angle. Then, write a general expression for all the coterminal angles in each case.

a) $\frac{11\pi}{6}$

b) −375°

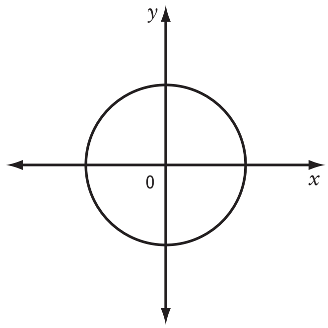
3. Determine the measure of the central angle subtended by each arc to one decimal place.

a) arc length 31.4 cm, radius 5.0 cm, in radians

b) arc length 11.3 m, radius 22.6 m, in degrees

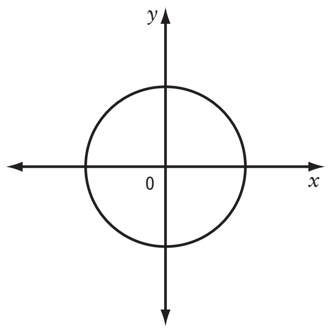
4.2 The Unit Circle, pages 120–128

4. Determine the missing coordinate for point $P\left(x, -\frac{2}{3}\right)$ in quadrant III on the unit circle.

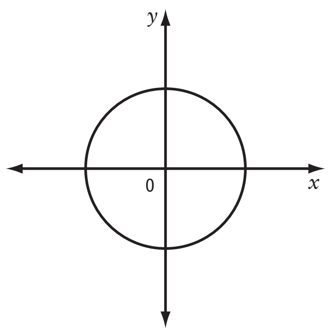


5. Determine the value of angle θ in standard position, $0 \leq \theta < 2\pi$, given the coordinates of $P(\theta)$, the point at which the terminal arm of θ intersects the unit circle.

a) $P\left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)$ $\theta =$

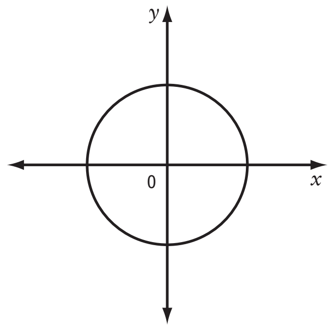


b) $P\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ $\theta =$



4.3 Trigonometric Ratios, pages 129–137

6. Determine the measure of all angles that satisfy $\sec \theta = 1.788$, $0° \leq \theta < 720°$. Round your answers to the nearest degree.



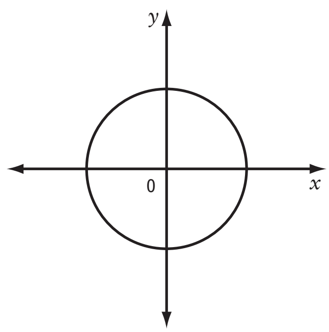
7. Determine the exact value of

a) $\cot\left(\frac{5\pi}{6}\right)$

b) $\csc\left(\frac{5\pi}{3}\right)$

4.4 Introduction to Trigonometric Equations, pages 138–144

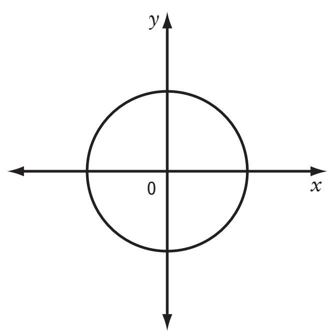
8. Write the general form of the solutions to $\sec \theta + 10 = 2 - 4 \sec \theta$ (in degrees).



$\theta_1 \approx$ ________________, $n \in I$

$\theta_2 \approx$ ________________, $n \in I$

9. Solve $2 \sin^2 \theta + \sin \theta = 1$, $0 \leq \theta < 2\pi$. Give exact solutions.



Chapter 4 Skills Organizer

Make note of some of the key things you remember about the processes you have learned in this chapter. Use your class notes, textbook, or questions from this workbook to help you choose examples (or create your own).

Process	Example	Things to Remember
Converting angle measures • from degrees to radians • from radian to degrees		
Determining coterminal angles		
Determining the six trigonometric ratios for angles in the unit circle		
Solving trigonometric equations • for a restricted domain • a general solution		