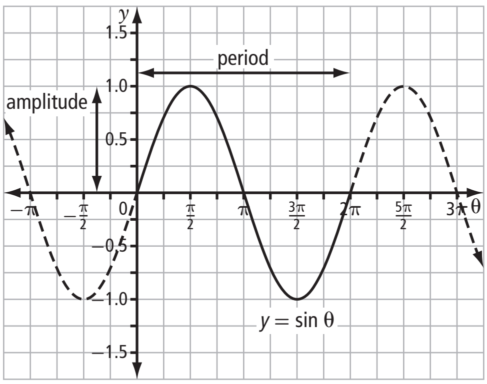
Chapter 5 Trigonometric Functions and Graphs

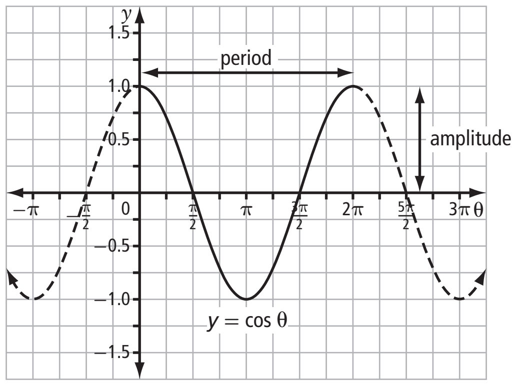
5.1 Graphing Sine and Cosine Functions

KEY IDEAS

- Sine and cosine functions are *periodic* or *sinusoidal functions*. The values of these functions repeat in a regular pattern. These functions are based on the unit circle.
- Consider the graphs of $y = \sin \theta$ and $y = \cos \theta$.



– The maximum value is +1.
– The minimum value is −1.
– The amplitude is 1.
– The period is 2π.
– The y-intercept is 0.
– The θ-intercepts on the given domain are $-\pi$, 0, π, 2π, and 3π.
– The domain of $y = \sin \theta$ is $\{\theta \mid \theta \in \mathrm{R}\}$.
– The range of $y = \sin \theta$ is $\{y \mid -1 \le y \le 1, y \in \mathrm{R}\}$.



– The maximum value is +1.
– The minimum value is −1.
– The amplitude is 1.
– The period is 2π.
– The y-intercept is 1.
– The θ-intercepts on the given domain are $-\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}$, and $\frac{5\pi}{2}$.
– The domain of $y = \cos \theta$ is $\{\theta \mid \theta \in \mathrm{R}\}$.
– The range of $y = \cos \theta$ is $\{y \mid -1 \le y \le 1, y \in \mathrm{R}\}$.

- For sinusoidal functions of the form $y = a \sin bx$ or $y = a \cos bx$, a represents a vertical stretch of factor $|a|$ and b represents a horizontal stretch of factor $\frac{1}{|b|}$. Use the following key features to sketch the graph of a sinusoidal function.
  – the maximum and minimum values
  – the amplitude, which is one half the total height of the function

  $$\text{Amplitude} = \frac{\text{maximum value} - \text{minimum value}}{2}$$

  The amplitude is given by $|a|$.
  – the period, which is the horizontal length of one cycle on the graph of a function

  $$\text{Period} = \frac{2\pi}{|b|} \text{ or } \frac{360°}{|b|}$$

  Changing the value of b changes the period of the function.
  – the coordinates of the horizontal intercepts

Working Example 1: Graph the Sine and Cosine Functions

Graph each function for the domain $0 \le \theta \le 3\pi$.

a) $y = \sin \theta$ **b)** $y = \cos \theta$

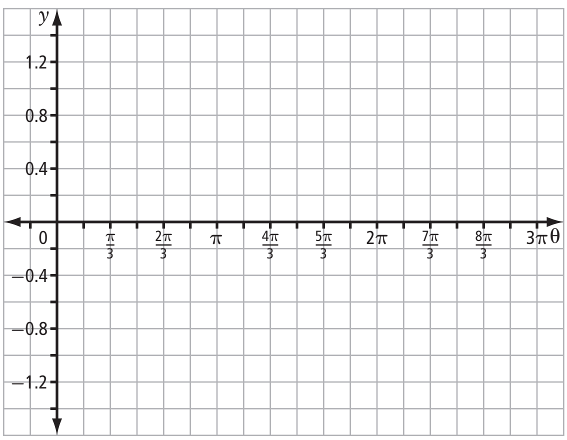
Solution

a) Complete the table of values. Round values to three decimal places.

θ	$y = \sin \theta$	θ	$y = \sin \theta$	θ	$y = \sin \theta$	θ	$y = \sin \theta$
0		$\frac{6\pi}{6} = \pi$		$\frac{12\pi}{6} = 2\pi$		$\frac{18\pi}{6} = 3\pi$	
$\frac{\pi}{6}$		$\frac{7\pi}{6}$		$\frac{13\pi}{6}$			
$\frac{2\pi}{6} = \frac{\pi}{3}$		$\frac{8\pi}{6} = \frac{4\pi}{3}$					
		$\frac{9\pi}{6} = \frac{3\pi}{2}$		$\frac{15\pi}{6} = \frac{5\pi}{2}$			
$\frac{4\pi}{6} = \frac{2\pi}{3}$				$\frac{16\pi}{6} = \frac{8\pi}{3}$			
$\frac{5\pi}{6}$							

Make sure your calculator is in radian mode.

Plot the points and join them with a smooth curve.

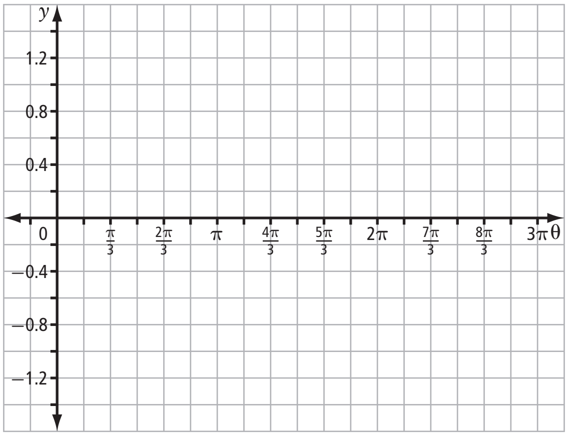


b) Complete the table of values.

θ	$y = \cos \theta$	θ	$y = \cos \theta$	θ	$y = \cos \theta$	θ	$y = \cos \theta$
0		π		2π		3π	
$\frac{\pi}{6}$		$\frac{7\pi}{6}$					
$\frac{\pi}{3}$		$\frac{4\pi}{3}$					
$\frac{\pi}{2}$		$\frac{3\pi}{2}$					
$\frac{2\pi}{3}$		$\frac{5\pi}{3}$					
$\frac{5\pi}{6}$		$\frac{11\pi}{6}$					

What will the value for 2π be? Which angle is coterminal with 2π?

Plot the points and join them with a smooth curve.

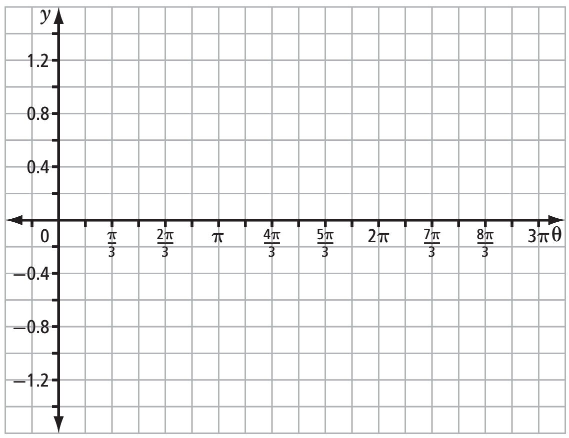


Working Example 2: Determine the Period of a Sine Function

Graph $y = \sin(3\theta)$ on the domain $0 \leq \theta \leq 3\pi$.

Solution

First consider the key features of the basic sine function, $y = \sin \theta$.
Use the key features to sketch the graph of the basic sine function, $y = \sin \theta$.



Now consider the function $y = \sin(3\theta)$.

- The amplitude of $y = \sin(3\theta)$ is ________.
- The maximum value of $y = \sin(3\theta)$ is ________.
- The minimum value of $y = \sin(3\theta)$ is ________.
- The period of $y = \sin(3\theta)$ is $\frac{2\pi}{|b|}$, or ________.
- The θ-intercepts of $y = \sin(3\theta)$ on the domain $0 \leq \theta \leq 3\pi$ are ________________.
- The y-intercept of $y = \sin(3\theta)$ is ________.

How can you use the period of $y = \sin(3\theta)$ and the θ-intercepts of the basic sine function to determine the θ-intercepts of $y = \sin(3\theta)$?

Use the key features to sketch the graph of the function $y = \sin(3\theta)$ on the coordinate grid above.

Working Example 3: Determine the Amplitude of a Cosine Function

Graph $y = -4 \cos \theta$ on the domain $0 \le \theta \le 3\pi$.

Solution

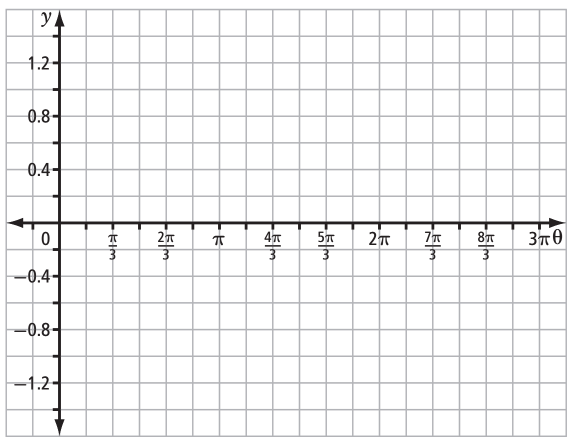
Complete the table. First, determine the key features of $y = \cos \theta$ on the specified domain.

	$y = \cos \theta$	$y = -4 \cos \theta$
Maximum value		
Minimum value		
Amplitude		
Period		
θ-intercepts		
y-intercept		

How does the amplitude of $y = -4 \cos \theta$ compare to the amplitude of $y = \cos \theta$?

How do the θ-intercepts of $y = -4 \cos \theta$ compare to the θ-intercepts of $y = \cos \theta$?

Use the key features to sketch the graph of the basic cosine function, $y = \cos \theta$. Then, sketch the graph of $y = -4 \cos \theta$ on the same coordinate grid.



Check Your Understanding

Practise

1. State the amplitude of each trigonometric function.

a) $y = 2 \cos \theta$

b) $y = \frac{1}{4} \sin \theta$

c) $y = 5 \sin (2\theta)$

d) $y = -3 \cos \left(\frac{1}{2} \theta\right)$

2. State the period of each trigonometric function in degrees and in radians.

a) $y = 3 \sin \theta$

Degrees: $\frac{360°}{|b|}$ = ________ Radians: $\frac{2\pi}{|b|}$ = ________

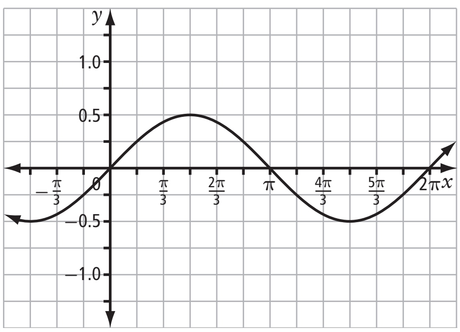
b) $y = \cos (2\theta)$

c) $y = 0.25 \sin (0.25\theta)$

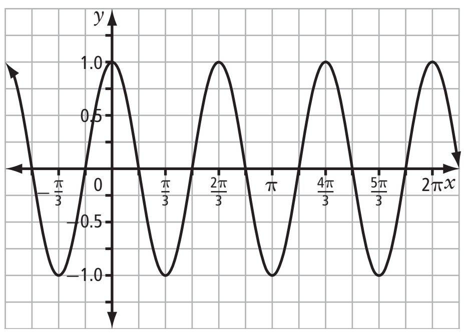
d) $y = -1.5 \cos (1.5\theta)$

3. State the period, in radians, and the amplitude of each trigonometric function.

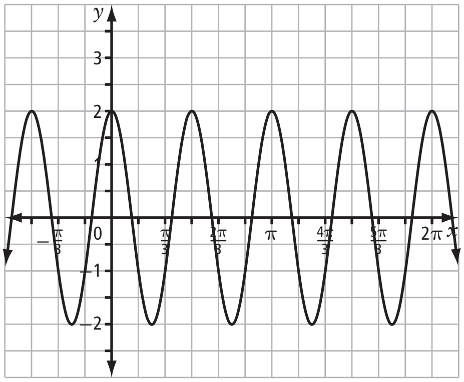
a)



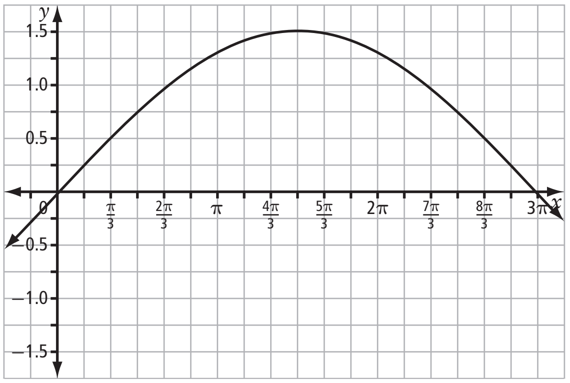
b)



c)



d)



4. Identify the key features of $y = \sin \theta$ and the transformed sine function. Then, graph at least two cycles of the transformed sine function.

a) $y = \sin\left(\frac{1}{3}\theta\right)$

Identify the key features of $y = \sin \theta$.

$a =$ ________; the amplitude is ________.

Maximum value: ________ Minimum value: ________

$b =$ ________; the period is ________.

θ-intercepts: ________________ y-intercept: ________

Identify the key features of $y = \sin\left(\frac{1}{3}\theta\right)$.

$a =$ ________; the amplitude is ________.

The graph ________ reflected in the x-axis.
(*is* or *is not*)

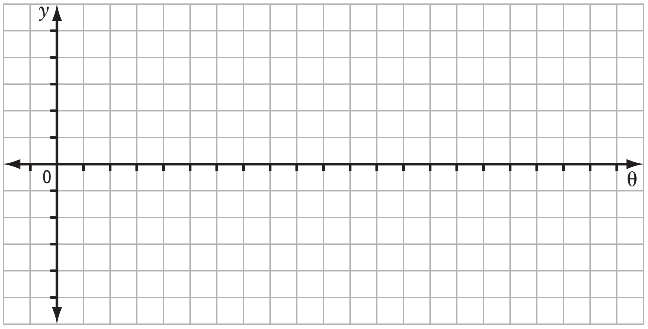
Maximum value: ________ Minimum value: ________

$b =$ ________; the period is ________.

The graph is stretched ________________ by a factor of ________.
(*horizontally* or *vertically*)

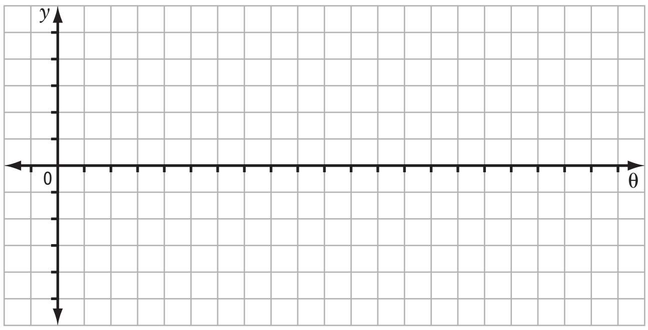
θ-intercepts: ________________ y-intercept: ________

Use the key features to sketch the graph of the function.

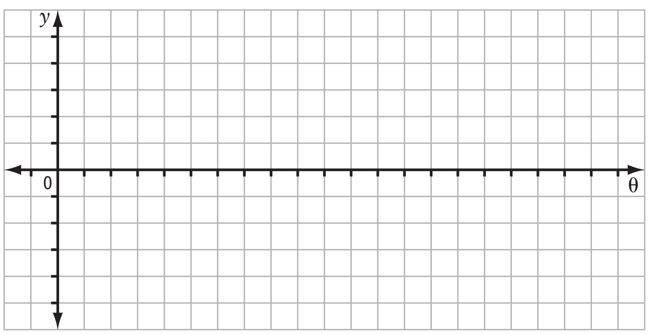


Consider the key features of the function when choosing the scales.

b) $y = 1.5 \sin (2\theta)$

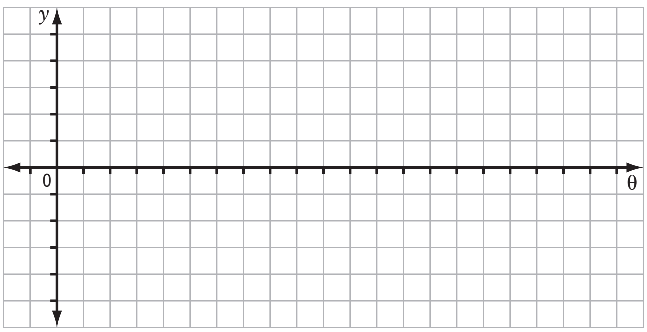


c) $y = -2 \sin (4\theta)$

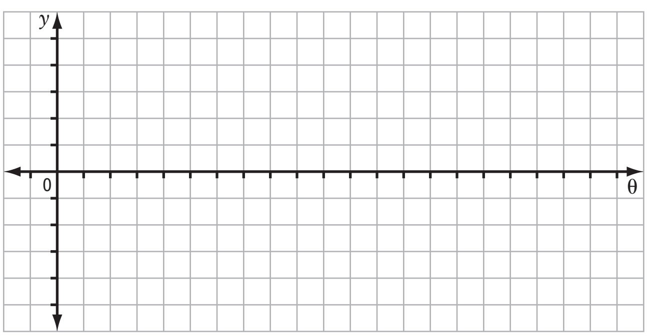


5. Identify the key features of each cosine function. Then, graph at least two cycles of each cosine function.

a) $y = 2\cos\left(\frac{1}{2}\theta\right)$



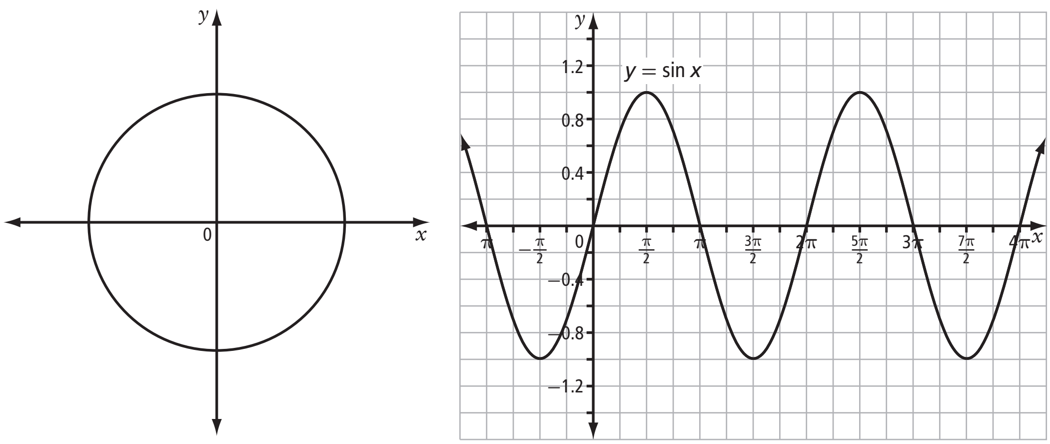
b) $y = -\cos(2\theta)$



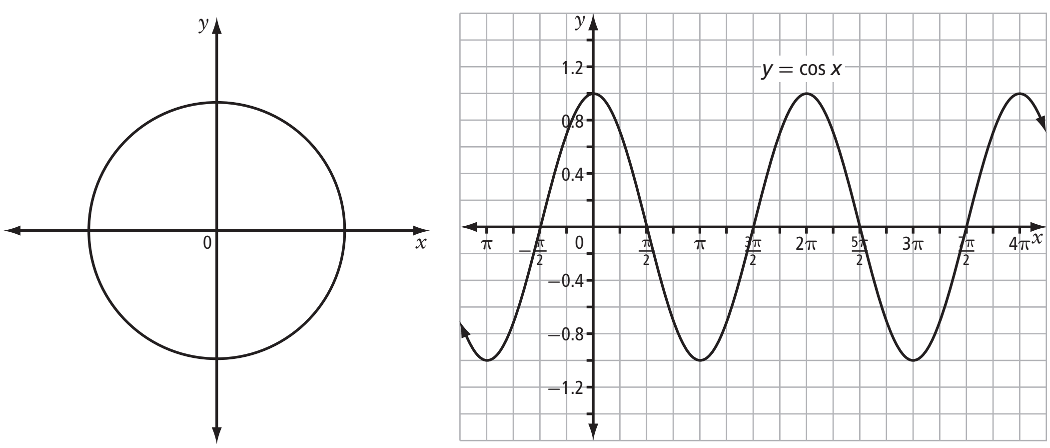
Connect

6. Consider the graphs of $y = \sin x$ and $y = \cos x$ below.
   a) Divide the graph of each trigonometric function into quadrants. Label the quadrants on the graphs and on the unit circles.
   b) Shade in the regions where $\sin x$ or $\cos x$ is positive on each graph and on each unit circle. Identify the maximum values.
   c) Using a different colour, shade in the regions where $\sin x$ or $\cos x$ is negative on each graph and on each unit circle. Identify the minimum values.
   d) Identify the x-intercepts and the corresponding angles on the unit circle.

Sine Function



Cosine Function



5.2 Transformations of Sinusoidal Functions

KEY IDEAS

- You can apply the same transformation rules to sinusoidal functions of the form $y = a \sin b(\theta - c) + d$ or $y = a \cos b(\theta - c) + d$.
  - A vertical stretch by a factor of $|a|$ changes the amplitude to $|a|$.

    $y = a \sin \theta$ $\qquad\qquad$ $y = a \cos \theta$

    If $a < 0$, the function is reflected through the horizontal mid-line of the function.
  - A horizontal stretch by a factor of $\frac{1}{|b|}$ changes the period to $\frac{360°}{|b|}$ or $\frac{2\pi}{|b|}$ radians.

    $y = \sin (b\theta)$ $\qquad\qquad$ $y = \cos (b\theta)$

    If $b < 0$, the function is reflected in the y-axis.
  - For sinusoidal functions, a horizontal translation is called the *phase shift*.

    $y = \sin (\theta - c)$ $\qquad\qquad$ $y = \cos (\theta - c)$

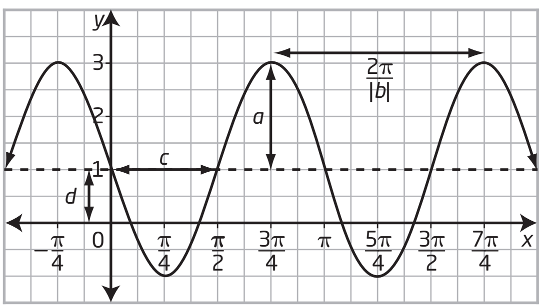
    If $c > 0$, the function shifts c units to the right.
    If $c < 0$, the function shifts c units to the left.
  - The *vertical displacement* is a vertical translation.

    $y = \sin \theta + d$ $\qquad\qquad$ $y = \cos \theta + d$
  - If $d > 0$, the function shifts d units up.
  - If $d < 0$, the function shifts d units down.

    $$d = \frac{\text{maximum value} + \text{minimum value}}{2}$$
  - The *sinusoidal axis* is defined by the line $y = d$. It represents the mid-line of the function.
- Apply transformations of sinusoidal functions in the same order as for any other functions:
  i) horizontal stretches and reflections, $\frac{1}{|b|}$
  ii) vertical stretches and reflections, $|a|$
  iii) translations, c and d
- The domain of a sinusoidal function is not affected by transformations.

  The range of a sinusoidal function, normally $\{y \mid -1 \leq y \leq 1, y \in \mathrm{R}\}$, is affected by changes to the amplitude and vertical displacement.

  Consider the graph of $y = 2 \sin 2\left(x - \frac{\pi}{2}\right) + 1$.



$a = 2$, so the amplitude is 2

$b = 2$, so the period is $\frac{2\pi}{2}$, or π

$c = \frac{\pi}{2}$, so the graph is shifted $\frac{\pi}{2}$ units right

$d = 1$, so the graph is shifted 1 unit up

domain: $\{x \mid x \in \mathrm{R}\}$

range: $\{y \mid -1 \leq y \leq 3, y \in \mathrm{R}\}$

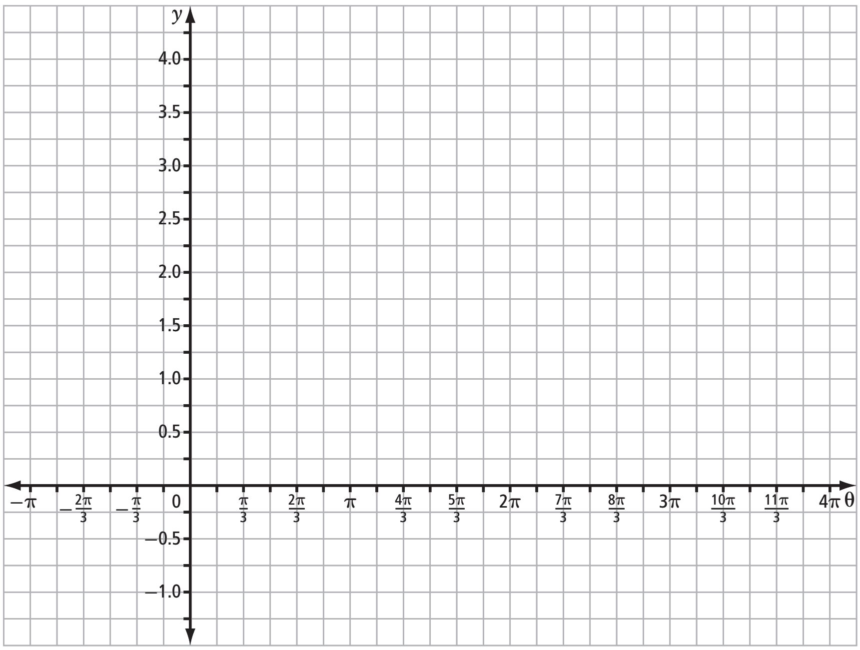
Working Example 1: Graph $y = \sin(\theta - c) + d$

a) Sketch the graph of the function $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$.

b) State the domain and range.

Solution

a) Sketch two cycles of the graph of the base function, $y = \sin\theta$.



Next, consider the transformed function $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$.

The amplitude is ________ and the period is ________.

$c =$ ________. This represents a phase shift of ________ units to the ________________.
(*left* or *right*)

$d =$ ________. This represents a vertical displacement of ________ units ________________.
(*up* or *down*)

On the grid above, sketch the sinusoidal axis at $y = 3$.

Use the sinusoidal axis, amplitude, period, and phase shift to sketch the graph of $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$ on the grid above.

b) The domain of $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$ is ____.

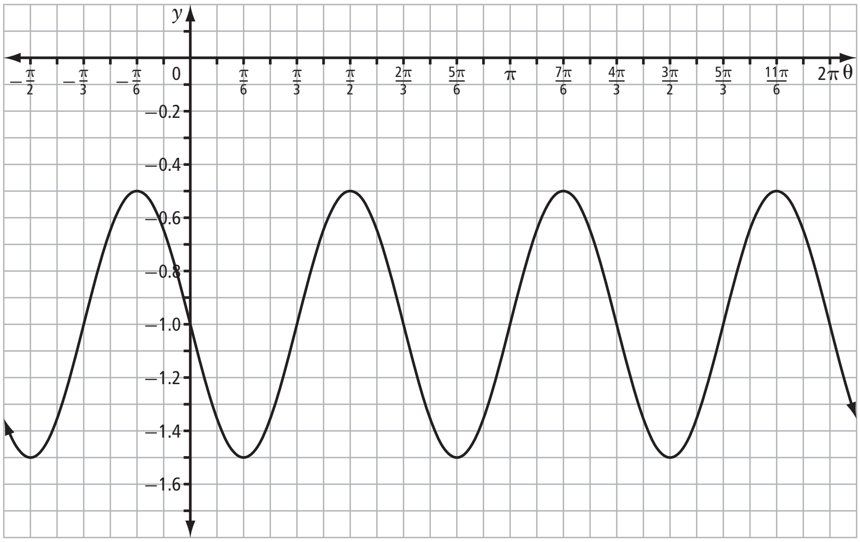
The range of $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$ is

____.

> Compare the graph of $y = \sin\left(\theta + \frac{\pi}{2}\right) + 3$ to the graph of $y = \cos\theta$. What do you notice?

Working Example 2: Determine an Equation From a Graph

Write two sinusoidal equations of the form $y = a \sin b(\theta - c) + d$ and $y = a \cos b(\theta - c) + d$ to represent the function shown in the graph below.



Solution

Determine the equation of a sine function.
In $y = a \sin b(\theta - c) + d$, there are four parameters to determine: a, b, c, d.

- Determine the amplitude, a.

$$|a| = \frac{\text{maximum value} - \boxed{}}{2}$$

$$= \frac{\boxed{} - \boxed{}}{2}$$

$$= \_\_\_\_\_\_\_\_$$

- Determine the displacement, d.

  The equation of the sinusoidal axis is $y =$ ________. Therefore, $d =$ ________.

- Determine the value of b.

  The period of the graphed function is ________ radians.

$$\text{Period} = \frac{2\pi}{|b|}$$

$$\_\_\_\_\_\_\_\_ = \frac{2\pi}{|b|}$$

$$b = \_\_\_\_\_\_\_\_$$

Choose b to be positive.

• Determine the phase shift, c.

The graph is shifted ________ units to the right, so $c =$ ________.

A sine equation that represents the graphed function is $y =$ ____________________________.

To determine an equation of a cosine function, $y = a \cos b(\theta - c) + d$, the values for a, b, and d are the same, but the phase shift is different.

The graph is shifted ________ units to the right, so $c =$ ________.

A cosine equation that represents the graphed function is $y =$ ____________________________.

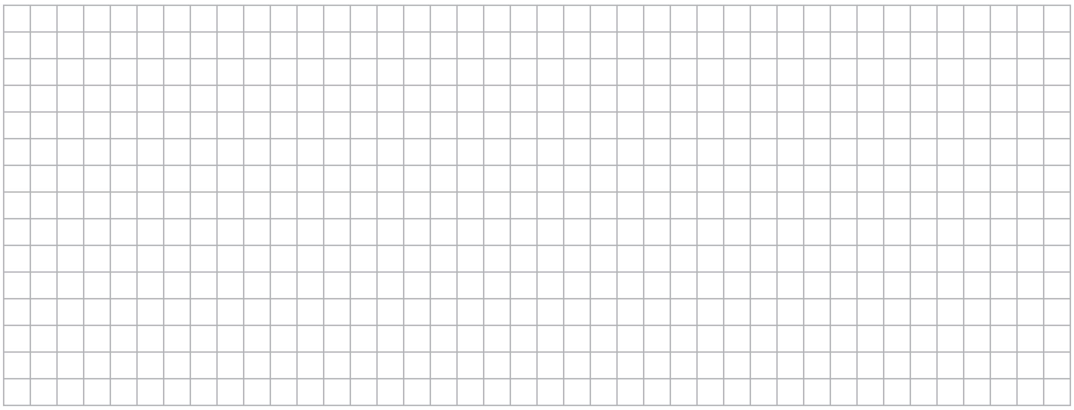
Check Your Understanding

Practise

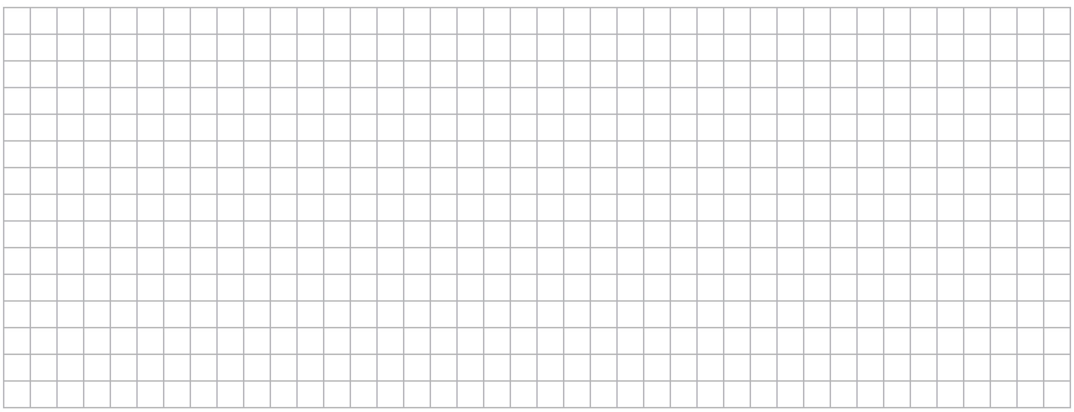
1. Determine the phase shift and the vertical displacement. Then, graph the function. Choose appropriate scales for the axes.

a) $y = \cos\left(\theta - \frac{\pi}{3}\right) - 1$

Phase shift: ________________ Vertical displacement: ________________



b) $y = \sin\left(\theta + \frac{\pi}{4}\right) + 2$



For more practice, see #1 and #2 on page 250 of *Pre-Calculus 12*.

2. Determine the key features of each sine function.

a) $y = -5 \sin\left(\frac{1}{2}(\theta - 90°)\right) + 15$

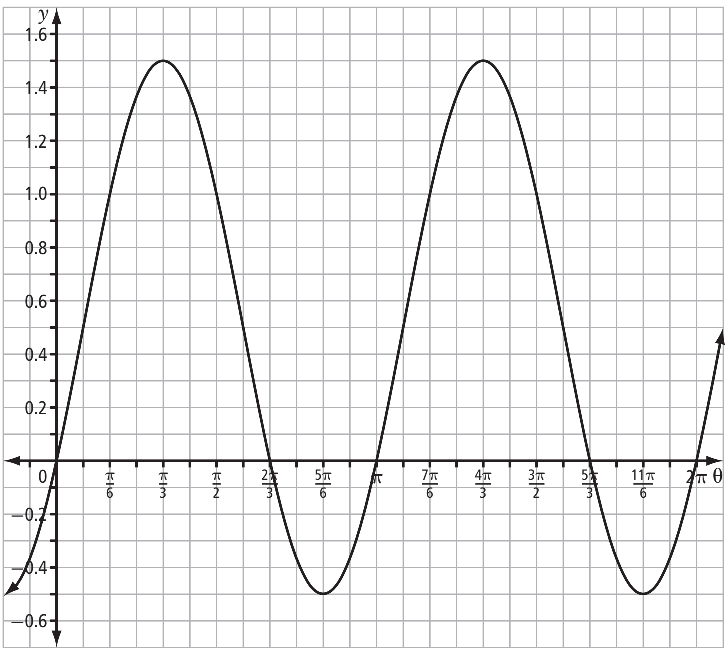
Amplitude: ______________ Period: ______________

Phase shift: ______________ Vertical displacement: ______________

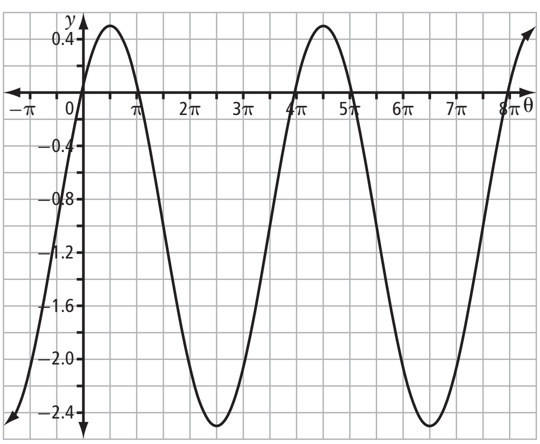
Domain: ______________ Range: ______________

b) $y = 0.1 \sin(2\theta + 90°) - 1$

c)



d)



3. Write the equation of each sine function in the form $y = a \sin b(x - c) + d$ given its characteristics.

a) amplitude 2, period π, phase shift $\frac{\pi}{3}$ to the left, vertical displacement 1 unit down

b) amplitude $\frac{1}{4}$, period 6π, phase shift π to the left, vertical displacement 2 units up

c) amplitude 4, period 540°, phase shift 60° to the right, no vertical displacement

4. Graph each function in the space provided. Show at least two cycles.

a) $y = 5 \sin 0.5(\theta + \pi) + 3$

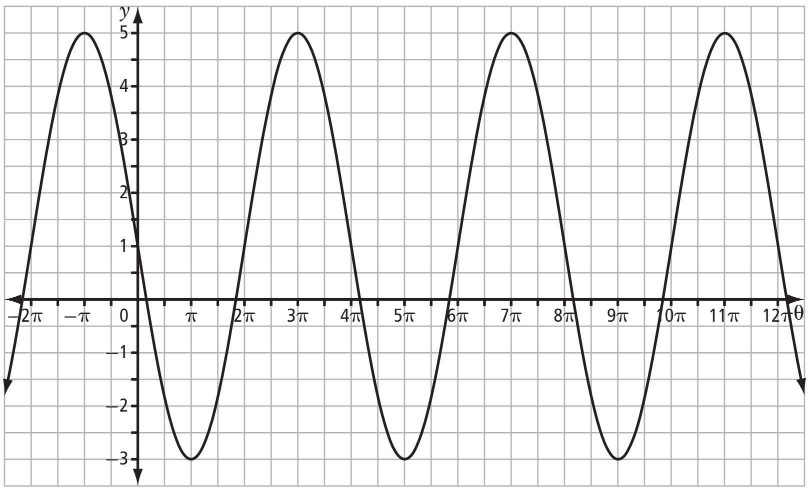
b) $y = -2 \sin 2\left(\theta - \frac{\pi}{3}\right) + 4$

c) $y = 1.5 \cos 3\left(\theta + \frac{\pi}{2}\right) - 1$

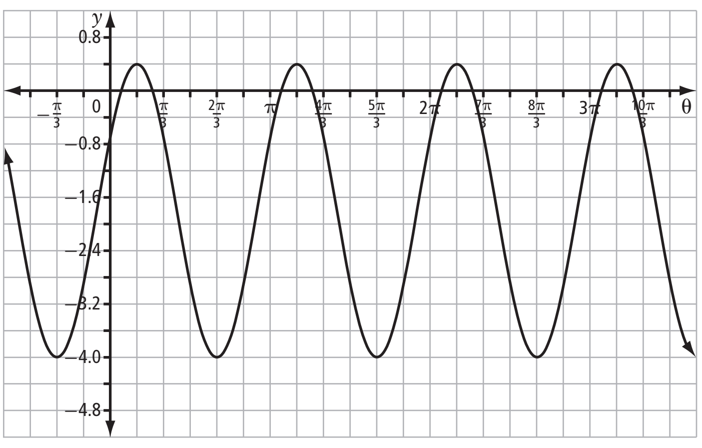
d) $y = -\cos \frac{1}{3}(\theta - \pi) + 3$

Apply

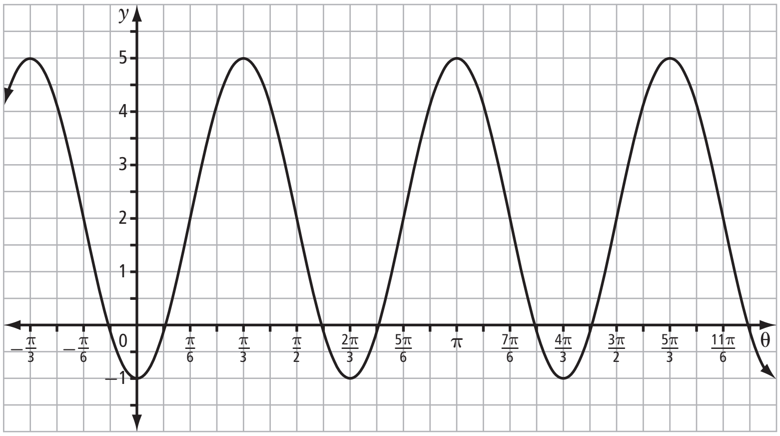
5. Write two different equations of the form $y = a \sin b(\theta - c) + d$ for the function graphed below. Use technology to check that your equations are correct.



6. Write two different equations of the form $y = a \cos b(\theta - c) + d$ to represent the function graphed below. Use technology to verify that your equations are correct.



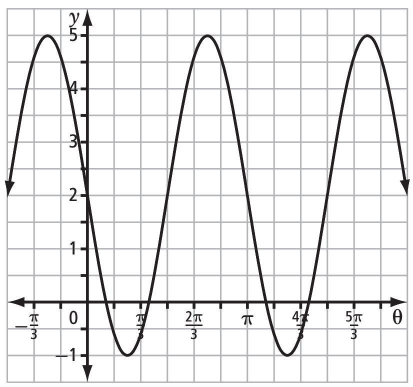
7. Write an equation of the form $y = a \sin b(\theta - c) + d$ and an equation of the form $y = a \cos b(\theta - c) + d$ to represent the function graphed below.



Connect

8. The graphed function is represented by an equation of each of the following forms. Determine the values of a, b, c, and d.

a) $y = a \sin b(\theta - c) + d; a > 0$



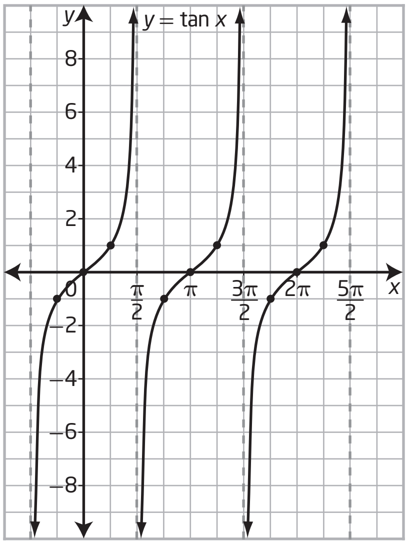
b) $y = a \sin b(\theta - c) + d; a < 0$

c) $y = a \cos b(\theta - c) + d; a > 0$

5.3 The Tangent Function

KEY IDEAS

- The graph of the tangent function, $y = \tan x$, is periodic, but it is *not* sinusoidal.



- These are the characteristics of the tangent function graph, $y = \tan x$:
  - It has period π or 180°.
  - It is discontinuous where tan x is undefined, that is, when $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{\pi}{2} + n\pi$, $n \in I$. The discontinuity is represented on the graph of $y = \tan x$ as *vertical asymptotes*.
  - The domain is $\left(x \mid x \neq \frac{\pi}{2} + n\pi, x \in R, n \in I\right)$.
  - It has no maximum or minimum values.
  - The range is $\{y \mid y \in R\}$.
  - It has x-intercepts at every multiple of π: 0, π, 2π, … , $n\pi$, $n \in I$. Each of the x-intercepts is a turning point, where the slope changes from decreasing to increasing.
- On the unit circle, you can express the coordinates of the point P on the terminal arm of angle θ as (x, y) or (sin θ, cos θ). The slope of the terminal arm is represented by the tangent function:

$$\begin{aligned} \text{slope} &= \frac{\Delta y}{\Delta x} \\ &= \frac{y-0}{x-0} \\ &= \frac{y}{x} \\ &= \tan\theta \end{aligned}$$

OR

$$\begin{aligned} \text{slope} &= \frac{\sin\theta}{\cos\theta} \\ &= \tan\theta \end{aligned}$$

Therefore, you can use the tangent function to model the slope of a line from a fixed point to a moving object as the object moves through a range of angles.

Working Example 1: Graph $y = \tan \theta$ Using Key Points

Graph $y = \tan \theta$ over the domain $-\pi \leq \theta \leq 4\pi$.

Solution

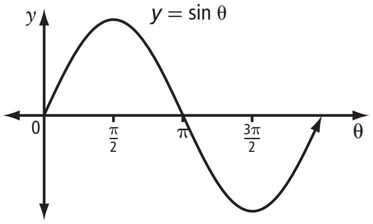
The key features needed to sketch the graph of the tangent function are the zeros, ones, and asymptotes.

The *ones* of a tangent function are the values of θ when $y = \pm 1$.

Determine the Zeros

Given $\tan \theta = \frac{\sin \theta}{\cos \theta}$, when $\sin \theta = 0$, $\tan \theta =$ ________.

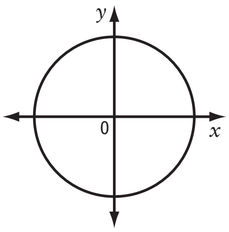
Therefore, $y = \tan \theta$ has the same zeros (x-intercepts) as $y = \sin \theta$.



The zeros of $y = \tan \theta$ over the domain $-\pi \leq \theta \leq 4\pi$ are ________, ________, ________, ________, ________, and ________.

Determine the Ones

The tangent function represents the slope of the terminal arm of an angle in standard position.



Given slope $= \frac{\sin \theta}{\cos \theta}$, the slope is 1 when $\sin \theta =$________.

Over the domain $-\pi \leq \theta \leq 4\pi$, the slope of the terminal arm is 1 when $\theta =$ ________, ________, ________, ________, and ________.

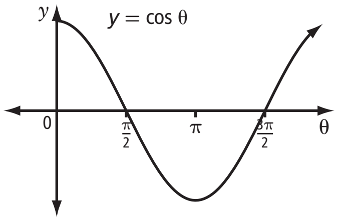
Given slope $= \frac{\sin \theta}{\cos \theta}$, the slope is -1 when $\sin \theta =$ ________.

Over the domain $-\pi \leq \theta \leq 4\pi$, the slope of the terminal arm is -1 when $\theta =$ ________, ________, ________, ________, and ________.

Determine the Asymptotes

Given $\tan \theta = \frac{\sin \theta}{\cos \theta}$, tan θ is undefined when ________ = 0.

Therefore, $y = \tan \theta$ has non-permissible values wherever $y =$ ________ has zeros (x-intercepts).

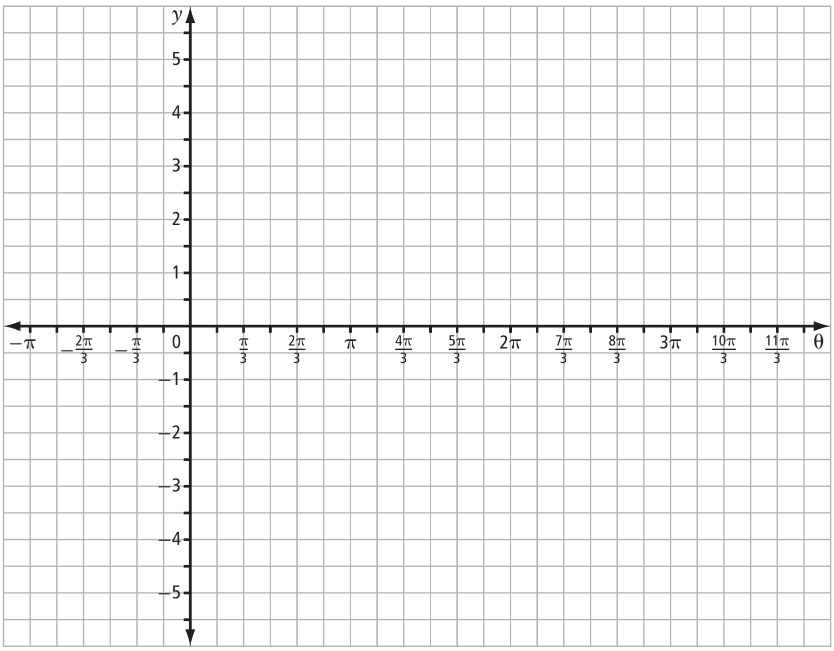


The non-permissible values of $y = \tan \theta$ over the domain $-\pi \le \theta \le 4\pi$ are ________, ________, ________, ________, and ________.

Use a broken line to draw a vertical asymptote on the graph at each non-permissible value of $y = \tan \theta$.

Plot the zeros and the ones.

Draw the tangent function starting at the lower edge of your graph near an asymptote, passing through the points plotted, and continuing to the upper edge of your graph near an asymptote. Be sure not to cross the asymptotes. In the given domain, how many cycles are shown?

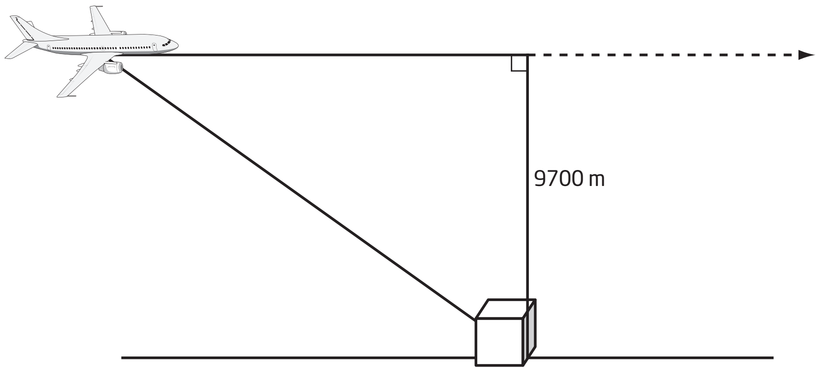


Working Example 2: Model a Problem Using the Tangent Function

An airplane is flying at a constant altitude of 9700 m in a straight line directly above a radar station. How does the horizontal distance between the plane and the radar station change as the plane crosses overhead?

Solution

Draw a diagram to illustrate the situation.



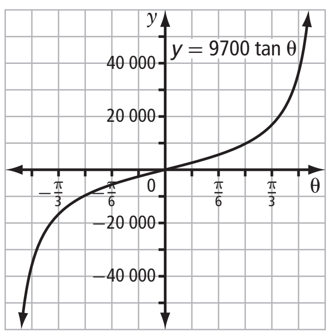
Label the complementary angle to the angle of elevation as θ, and label the horizontal distance as *h*.

Write an equation expressing *h* in terms of θ.

$\tan \theta =$

$h =$

As θ approaches 0, what happens to the horizontal distance?
Indicate this point on the graph of $y = 9700 \tan \theta$ below, and on the diagram above.



As the plane continues its flight away from the radar station,

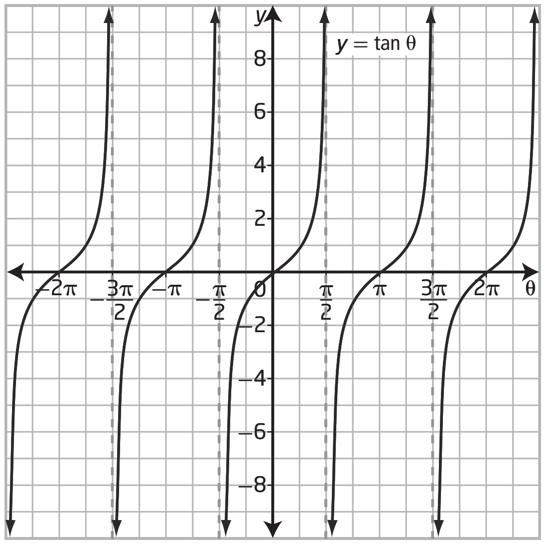
- what happens to θ?

- what happens to *h*? Indicate this region on the graph and on the diagram.

Check Your Understanding

Practise

1. Use the graph of $y = \tan \theta$ to determine each value.



a) $\tan 2\pi$

b) $\tan \frac{3\pi}{2}$

c) $\tan \frac{\pi}{4}$

d) $\tan \left(-\frac{\pi}{4}\right)$

2. Use the graph of $y = \tan \theta$ from #1 and your knowledge of the properties of the tangent function to determine each value.

a) $\tan (-\pi)$

b) $\tan (-3\pi)$

c) $\tan (-100\pi)$

3. Use the graph of $y = \tan \theta$ from #1 and your knowledge of the properties of the tangent function to determine each value.

a) $\tan \left(\frac{9\pi}{4}\right)$

b) $\tan \left(\frac{13\pi}{4}\right)$

c) $\tan \left(\frac{17\pi}{4}\right)$

4. Refer to your answers in #2 and #3. Write a general expression for all solutions in each case.

a) $\tan \theta = 0$

b) $\tan \theta = 1$

5. Use graphing technology to graph $y = \tan x$, where x is measured in degrees. Trace along the graph to locate the approximate value of the function when $x = 35°$. Predict the following values. Verify your predictions by tracing along the graph.

$\tan 35° \approx$ ________

a) $\tan (-325°)$

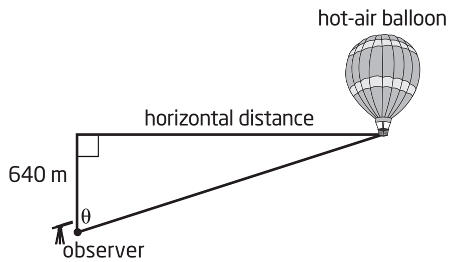
b) $\tan 395°$

c) $\tan (-35°)$

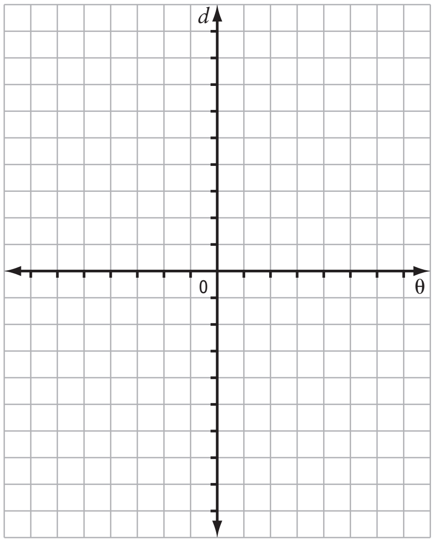
Apply

6. An observer watches a hot-air balloon pass directly overhead at a constant altitude of 640 m.

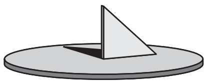
a) Determine the relation between the horizontal distance, d, in metres, from the observer to the hot-air balloon and the angle, in degrees, formed from the vertical to the balloon.



b) Graph the function. What are reasonable limits on the domain and range?



7. A simple sundial has a gnomon 10 cm high. Suppose the angle of the sun with respect to the gnomon changes from $-75°$ at 7:00 a.m. to $+75°$ at 5:00 p.m.



a) Write an expression for the length of the shadow in terms of the angle of the sun.

b) Describe any assumptions you made in creating this model.

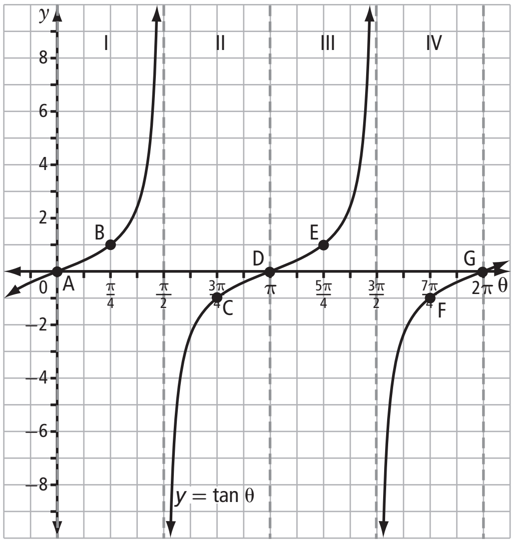
c) Determine the length of the shadow at 9:00 a.m.

d) Determine the length of the shadow at 4:00 p.m.

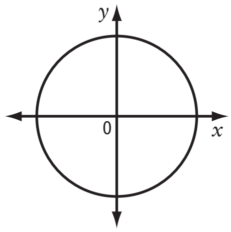
For more practice modelling using the tangent function, try #9 to #12 on pages 264–265 of *Pre-Calculus 12*.

Connect

8. a) For each labelled point on the graph of $y = \tan \theta$, sketch the terminal arm of angle θ on a unit circle.



A



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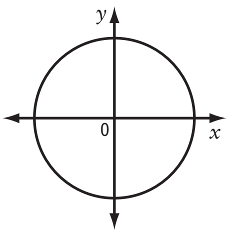

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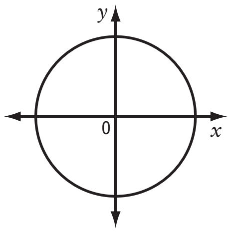

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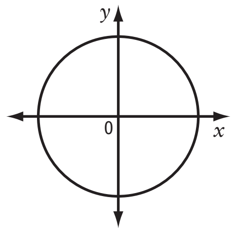

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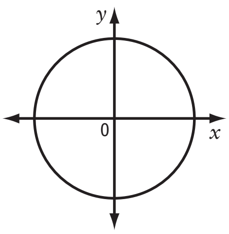

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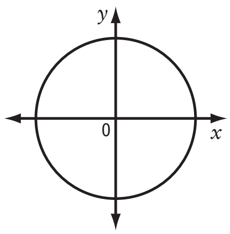

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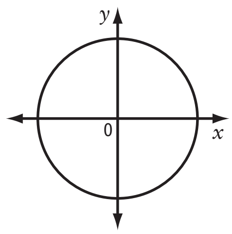
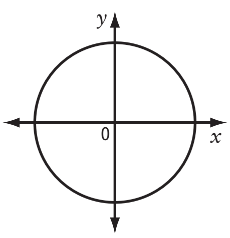
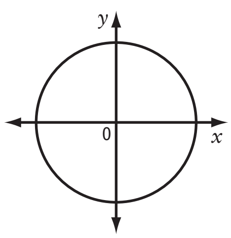


b) For each asymptote of $y = \tan \theta$, sketch the terminal arm of angle θ on a unit circle.

i) $\theta = \frac{\pi}{2}$



ii) $\theta = \frac{3\pi}{2}$



5.4 Equations and Graphs of Trigonometric Functions

KEY IDEAS

- You can use sinusoidal functions to model periodic phenomena that do not involve angles as the independent variable. Examples of such phenomena include
  – wave shapes, such as a heartbeat or ocean waves
  – pistons in a machine or the swing of a pendulum
  – circular motion, such as a Ferris wheel
- You can adjust the parameters a, b, c, and d in sinusoidal equations of the form $y = a \sin b(\theta - c) + d$ or $y = a \cos b(\theta - c) + d$ to fit the characteristics of the phenomenon being modelled.
- Graphing technology allows you examine how well the model represents the data. It also allows you to extrapolate or interpolate solutions from the model.
- You can find approximate solutions to trigonometric equations using the graphs of the trigonometric functions. Express solutions over a specific interval or give a general solution.

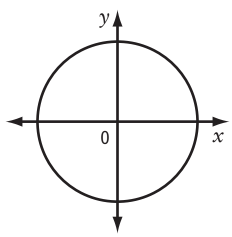
Working Example 1: Solve Simple Trigonometric Equations

Solve each equation over the specified interval.

a) $\sin x = 0.5$, $0° \leq x \leq 720°$ **b)** $\sin 2x = 0.5$, $0° \leq x \leq 720°$

Solution

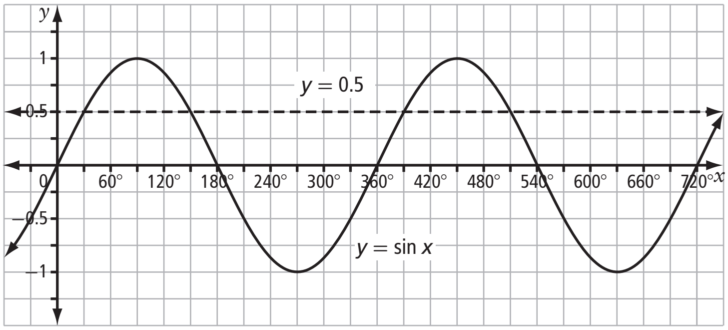
a) Method 1: Use the Unit Circle and Special Triangles



$\theta_R = \sin^{-1}(0.5) =$ ________

The solutions are $x =$ ________, ________, ________, ________, $0° \leq x \leq 720°$.

Method 2: Use a Graph
Graph $y = \sin x$ (Left Side) and $y = 0.5$ (Right Side) on the same set of axes on the domain $0° \leq x \leq 720°$.

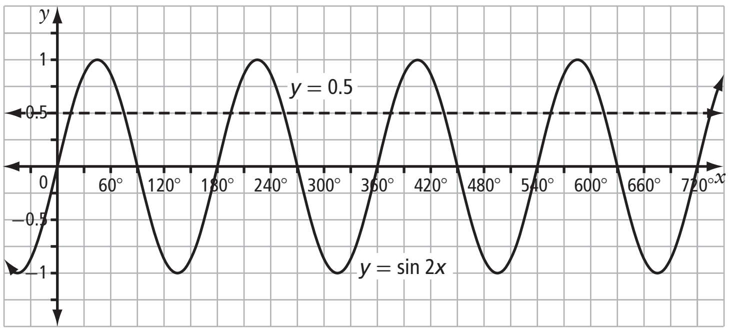


Check that your calculator is set to the correct mode.

Determine the coordinates of the points of intersection of the two functions.

The solutions are $x =$ ________, ________, ________, ________, $0° \leq x < 720°$.

b) Graph $y = \sin 2x$ (Left Side) and $y = 0.5$ (Right Side) on the same set of axes over the domain $0° \leq x \leq 720°$.



How many solutions are there?

The solutions are $x =$ ____, $0° \leq x < 720°$.

Working Example 2: Solve a Trigonometric Equation by Graphing

Solve $3 = 16 \cos [5(x + 1)] - 5$, $0 \leq x \leq \pi$.

Solution

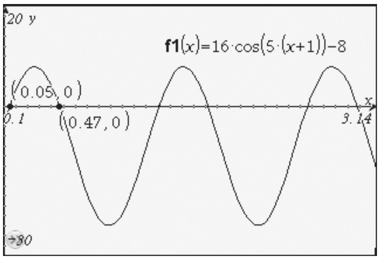
Collect all terms on the same side of the equation.
$3 = 16 \cos [5(x + 1)] - 5$

$0 =$ ____

Substitute y for 0.

$y =$ ____

Graph the related function using graphing technology.



How many solutions are there within the given domain?

The solutions within the first cycle are $x_1 \approx$ ________ and $x_2 \approx$ ________.

Additional solutions are of the form $x + n$ (period).

The solutions are $x \approx$ ______________________________________, $0 \leq x \leq \pi$.

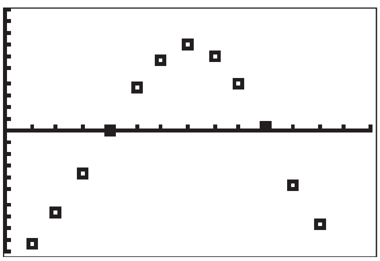
Working Example 3: Model Average Temperature

Write a sinusoidal function that models the average temperature each month in Whitehorse, Yukon, in a given year.

Month	Temperature (°C)	Month	Temperature (°C)	Month	Temperature (°C)
1	−18.4	5	6.8	9	7.5
2	−13.4	6	11.9	10	0.6
3	−7.3	7	14.0	11	−9.1
4	0.1	8	12.3	12	−15.7

Solution

Use graphing technology to create a scatter plot of the data.



Let x represent ____________________. Let y represent ____________________.

Use a cosine model, $y = a \cos b(x - c) + d$, where $a < 0$.

$$|a| = \frac{\text{maximum value} - \text{minimum value}}{2}$$

$$=$$

Period: ________

Use the period to determine b, where $\frac{2\pi}{|b|}$ = period.

Determine the phase shift.

At what x-value does the minimum of $y = -\cos x$ occur? At what x-value does the minimum of the table data occur?

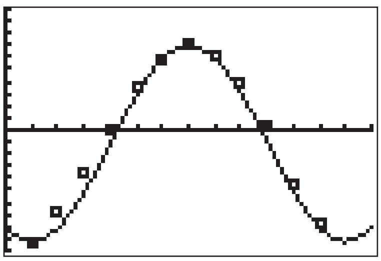
$c =$ ________

$d = \frac{\text{maximum value} + \text{minimum value}}{2}$

$=$

Write your equation: ____

Use graphing technology. Graph the equation with the scatter plot to check that it is a reasonable representation of the data.

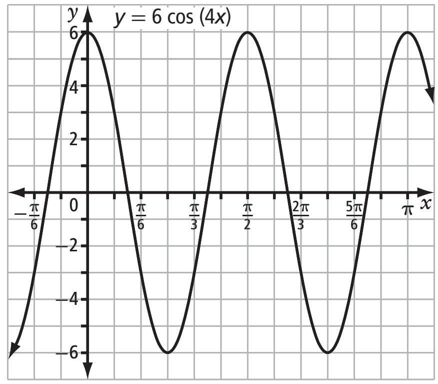


Check Your Understanding

Practise

1. Use the graph of $y = 6 \cos (4x)$ to solve each trigonometric equation.

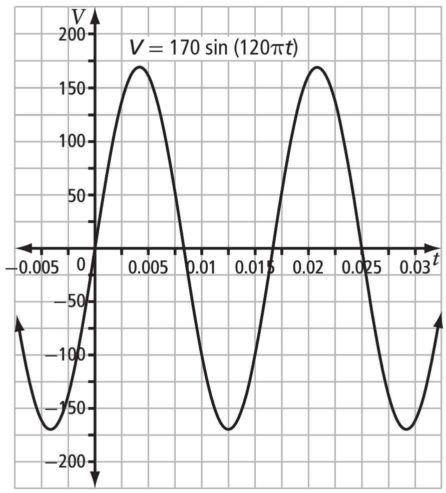
   a) $6 \cos (4x) = 3, 0 \leq x \leq \pi$



   b) $6 \cos (4x) = -6$, general solution in radians

2. Use the graph of $V = 170 \sin (120\pi t)$ to approximate the solutions to each trigonometric equation.

   a) $50 = 170 \sin (120\pi t), 0 \leq x \leq 0.030$



   b) $170 \sin (120\pi t) = -120, 0 \leq x \leq 0.030$

   c) $170 \sin (120\pi t) = 0$, general solution in radians

3. Sound travels in waves. You can see the sinusoidal patterns of sound waves using a device called an oscilloscope.

   a) Orchestra members tune their instruments to $A = 440$ Hz, meaning the sound wave repeats 440 times per second. What is the period of this sound wave, in seconds?

   b) Write a simple sine function representing the waveform of the note $A = 440$.

   $\frac{2\pi}{|b|} = \text{period}$

   c) "Middle C" has a frequency of 261.63 Hz. What sine function could represent middle C?

4. Electricity comes into your home or school as alternating current, which can be modelled by a sinusoidal function. Electrical devices operate at the root mean square voltage, which is $\frac{1}{\sqrt{2}}$ of the peak voltage.

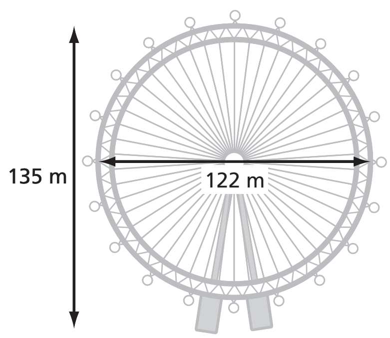
   a) In Canada, many electrical devices require 120 V and 60 Hz. Write a sine function that represents the peak voltage in Canada.

   To get the amplitude of the wave, multiply the required voltage by $\sqrt{2}$ (≈ 1.4).

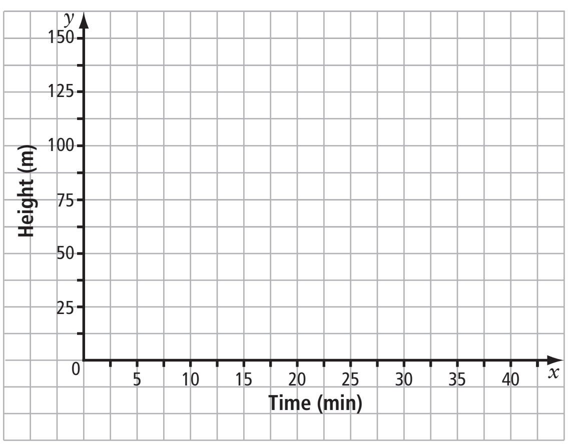
   b) In Europe (and in most of Asia, Africa, and parts of South America), many appliances require 220 V and 50 Hz. Write a sine function that represents the peak voltage in Europe.

Apply

5. The London Eye has diameter 122 m and height 135 m. It takes approximately 30 min for one rotation of the wheel. Passengers board at the bottom of the ride. The ride moves slowly enough that it is usually not necessary for the wheel to stop to let passengers on or off.



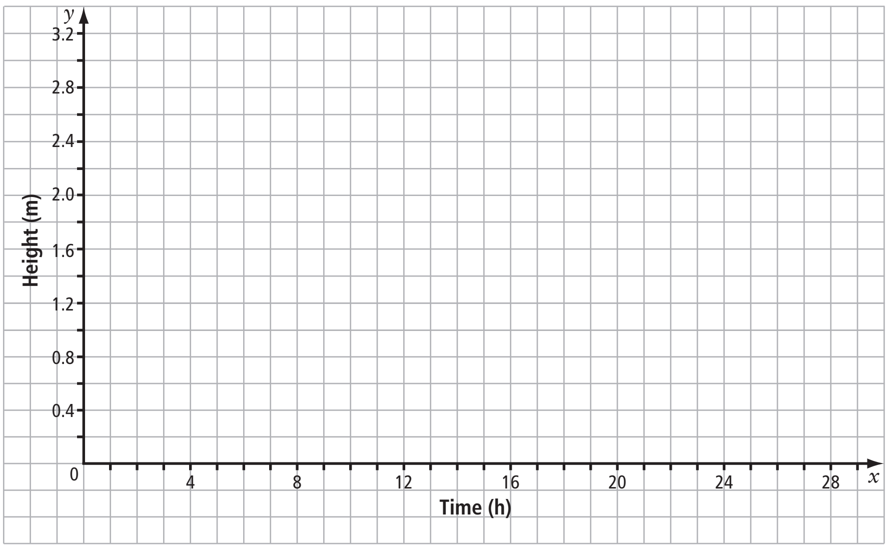
   a) Sketch a sinusoidal function representing the height of a passenger riding the London Eye. What assumptions do you have to make?



b) Write a sinusoidal function that represents the height of a passenger riding the London Eye. Over what domain is the function valid?

6. One particular afternoon, the tide in Victoria, BC, reached a maximum height of 3.0 m at 2:00 p.m. and a minimum height of 0.2 m at 8:00 p.m.

a) Sketch a sinusoidal function based on these data. What assumptions do you have to make?



b) Write a sinusoidal function that represents the tide in Victoria, BC, on this day. Over what domain is the function valid?

7. Write a sinusoidal function that models the average temperature in Brandon, Manitoba. Use graphing technology to verify that your function is a good representation of the data.

	Jan 1	Feb 2	Mar 3	Apr 4	May 5	Jun 6	Jul 7	Aug 8	Sep 9	Oct 10	Nov 11	Dec 12
°C	–18.3	–15.8	–7.9	3.5	10.8	16.0	18.9	17.4	11.8	5.1	–5.3	–13.7

Connect

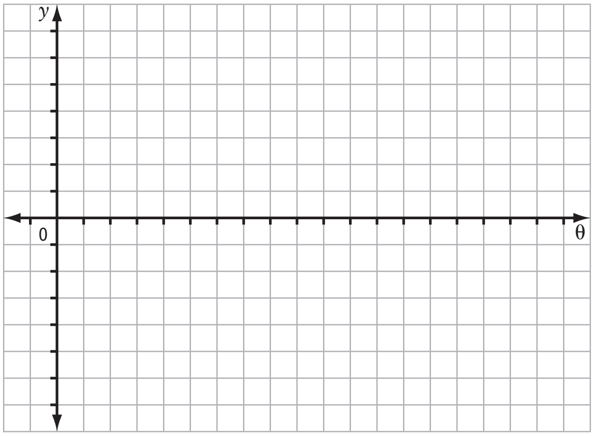
8. Using examples from class, from your textbook, and from this workbook, brainstorm a list of situations that can be modelled using a sinusoidal function in the form $y = a \sin b(x - c) + d$ or $y = a \cos b(x - c) + d$. Next to each item, list any helpful information for constructing the model. One example has been provided to help you get started.

Situation	Notes
Circular motion	• $\lvert a \rvert$ = radius of the circle

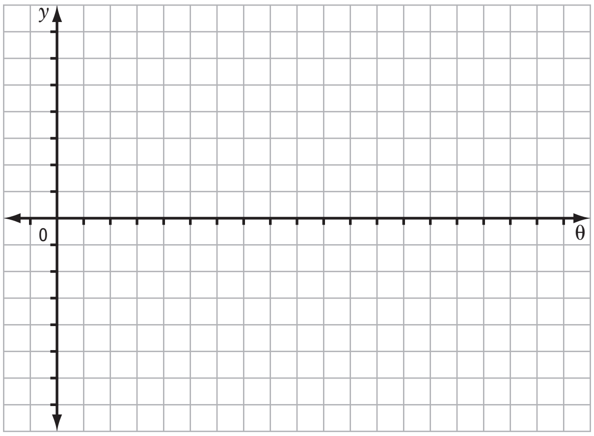
Chapter 5 Review

5.1 Graphing Sine and Cosine Functions, pages 149–157

1. Graph at least two cycles of $y = 3\cos\left(\frac{1}{2}\theta\right)$. State the amplitude and period in degrees.



2. Graph at least two cycles of $y = -0.5\sin(2\theta)$. State the amplitude and period in radians.



3. Without graphing, determine the amplitude and period, in radians and in degrees, of each function.

a) $y = 2\sin 3x$

b) $y = \frac{1}{3}\cos x$

c) $y = \frac{3}{4}\cos 2x$

d) $y = -4\sin\frac{2}{3}x$

5.2 Transformations of Sinusoidal Functions, pages 158–166

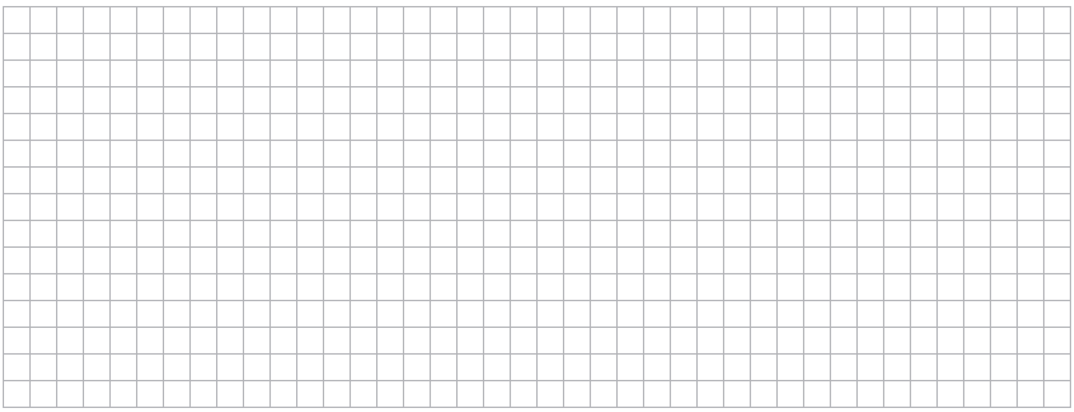
4. Determine the amplitude, period, phase shift, and vertical displacement with respect to $y = \sin x$ or $y = \cos x$ for each function.

a) $y = 5 \sin \frac{1}{4}\left(x + \frac{\pi}{3}\right) - 1$

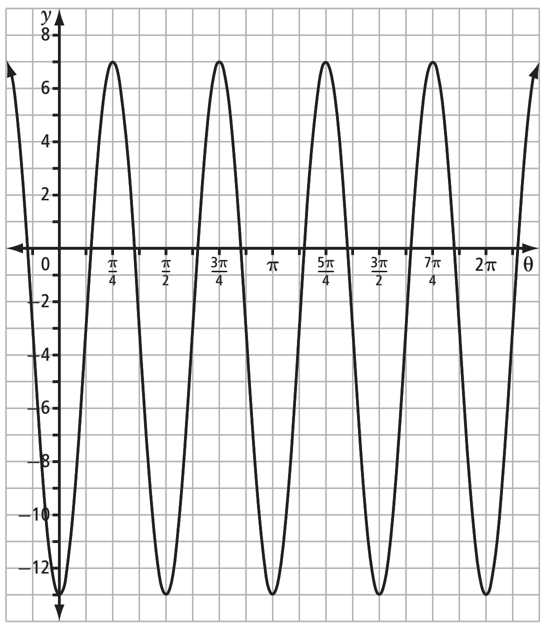
b) $y = -\frac{1}{2} \cos 2(x - \pi) - 3$

c) $y = 3 \cos 4(x + 50°) + 6$

5. Graph at least two cycles of $y = \sin 2\left(x + \frac{\pi}{12}\right) - 0.4$.

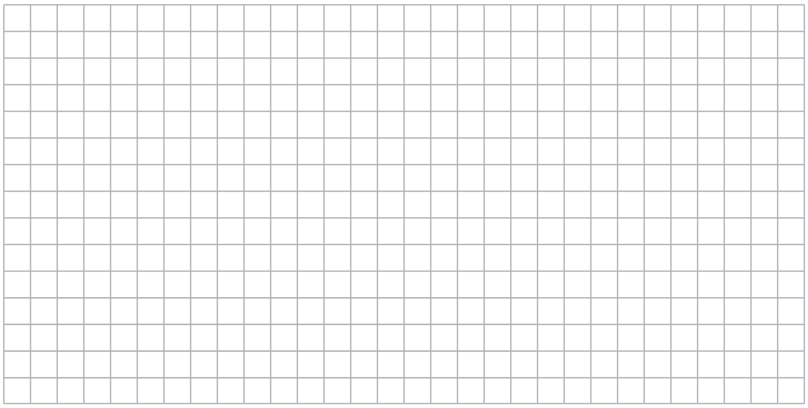


6. Write two equations of the form $y = a \cos b(\theta - c) + d$ that represent the function shown below.



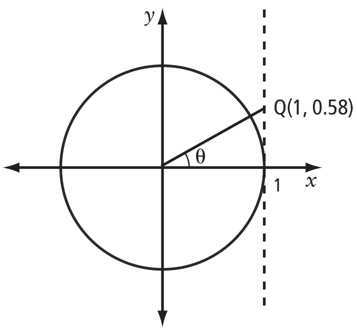
5.3 The Tangent Function, pages 167–174

7. Graph $y = \tan x$ over the domain $-\frac{3\pi}{2} \leq x \leq \frac{5\pi}{2}$.

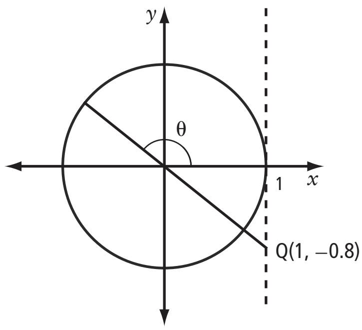


8. For each diagram, determine tan θ and the value of θ, in degrees. If necessary, round your answer to the nearest tenth.

a)



b)



5.4 Equations and Graphs of Trigonometric Functions, pages 175–182

9. Write a sinusoidal function to model the average temperature in Nanaimo, BC.

	Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
°C	1.9	3.6	5.3	8.1	11.8	14.9	17.3	17.2	14.2	9.4	5.1	2.8

10. Solve each equation by graphing.

a) $\sin 2x = 0, 0 \leq x \leq 2\pi$

b) $\cos\left(x + \frac{\pi}{2}\right) + 1 = 0, 0 \leq x \leq 2\pi$

c) $\sin 2(x - 30°) + 0.5 = 0$, general solution in degrees

Chapter 5 Skills Organizer

Complete the table for each trigonometric function.

	$y = a \sin b(\theta - c) + d$	$y = a \cos b(\theta - c) + d$	$y = \tan \theta$
Sketch			
Key points			
Zeros			
Maximum values			
Minimum values			
Period			
Amplitude			
Domain			
Range			