Chapter 6 Trigonometric Identities

6.1 Reciprocal, Quotient, and Pythagorean Identities

KEY IDEAS

Trigonometric Identities

A trigonometric identity is a trigonometric equation that is true for *all* permissible values of the variable in the expressions on both sides of the equation.

- **Reciprocal Identities**

  $\csc x = \frac{1}{\sin x}$ $\sec x = \frac{1}{\cos x}$ $\cot x = \frac{1}{\tan x}$

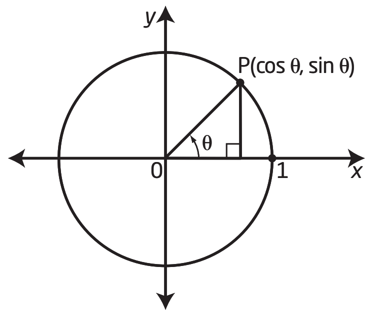
- **Quotient Identities**

  $\tan x = \frac{\sin x}{\cos x}$ $\cot x = \frac{\cos x}{\sin x}$

- **Pythagorean Identities**

  There are three forms of the Pythagorean identity:

  – Form 1: Derived from the Pythagorean theorem, $a^2 + b^2 = c^2$, and applied to a right triangle in the unit circle where angle θ is in standard position. The hypotenuse is 1, the adjacent side is $x = \cos \theta$, and the opposite side is $y = \sin \theta$. Therefore, $\cos^2 \theta + \sin^2 \theta = 1$.



  – Form 2: Divide both sides of form 1 by $\sin^2 \theta$ and apply the quotient and reciprocal identities.

  $$\frac{\cos^2 \theta}{\sin^2 \theta} + \frac{\sin^2 \theta}{\sin^2 \theta} = \frac{1}{\sin^2 \theta}$$
  $$\cot^2 \theta + 1 = \csc^2 \theta$$

  – Form 3: Divide both sides of form 1 by $\cos^2 \theta$ and apply the quotient and reciprocal identities.

  $$\frac{\cos^2 \theta}{\cos^2 \theta} + \frac{\sin^2 \theta}{\cos^2 \theta} = \frac{1}{\cos^2 \theta}$$
  $$1 + \tan^2 \theta = \sec^2 \theta$$

Verification and Use of Trigonometric Identities

- Trigonometric identities can be verified in two ways:
  i) numerically, by substituting specific values for the variable
  ii) graphically, using technology
- Verifying that two sides of an equation are equal for given values, or that they appear equal when graphed, is not sufficient to conclude that the equation is an identity.
- You can use trigonometric identities to simplify more complicated trigonometric expressions.

Working Example 1: Verify a Potential Identity Numerically and Graphically

a) Determine the non-permissible values, in degrees, for the equation $\csc \theta = \frac{\cot \theta}{\cos \theta}$.

b) Numerically verify that $\theta = 45°$ is a solution for the equation.

c) Numerically verify that $\theta = \frac{\pi}{6}$ is a solution for the equation.

d) Use technology to graphically determine whether the equation could be an identity over the domain $-360° < \theta \le 360°$.

Solution

a) Assess each trigonometric function in the equation individually. Are there values for which the denominators are zero or the numerators are undefined? Visualize the graph of each function to help you determine the non-permissible values.

For the left side of the equation, $\csc \theta$, the non-permissible values are ________________.

For the right side of the equation, $\frac{\cot \theta}{\cos \theta}$, the non-permissible values for $\cot \theta$ are ________________________. The denominator is zero when $\cos \theta = 0$. Therefore, the non-permissible values for the denominator are ___________________________.

Combined, the non-permissible values for $\csc \theta = \frac{\cot \theta}{\cos \theta}$ are ___________________________.

b) Substitute $\theta = 45°$ into each side of the equation.

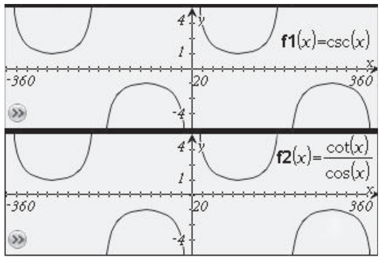
Left Side	Right Side
$\csc \theta$	$\frac{\cot \theta}{\cos \theta}$
$= \csc \mathbf{45°}$	$= \frac{\frac{1}{\boxed{}\ \mathbf{45°}}}{\cos \mathbf{45°}}$
$= \frac{1}{\boxed{}\ 45°}$	$= \frac{\boxed{}}{\boxed{}}$
$= \frac{\boxed{}}{\boxed{}}$	$=$ ________________
$=$ ________________	

This equation ________________ (*is* or *is not*) true for $\theta = 45°$.

c) Substitute $\theta = \frac{\pi}{6}$.

Left Side	Right Side

d) Use technology to graph $y =$ ______________ and $y =$ ______________ with domain $-360° < \theta \leq 360°$.



The two graphs look ____________, so
(*identical* or *different*)

$\csc \theta = \dfrac{\cot \theta}{\cos \theta}$ ____________ be an ____________.
(*could* or *could not*)

Working Example 2: Use Identities to Simplify Expressions.

a) Determine the non-permissible values, in radians, of the variable in the expression $\dfrac{\tan x \cos x}{\sec x \cot x}$.

b) Simplify the expression.

Solution

a) Consider non-permissible values, if any.

Explain why you need to consider the non-permissible values for tan *x* in the numerator, but you do not consider the non-permissible values for cos *x*.

The non-permissible values of tan *x* are ______________.

The non-permissible values of sec *x* are ______________.

The non-permissible values of cot *x* are ______________.

Combined, the non-permissible values for $\dfrac{\tan x \cos x}{\sec x \cot x}$ are ______________________.

b) To simplify the expression, use the reciprocal identity for sec *x* and quotient identities for tan *x* and cot *x* to write trigonometric functions in terms of cosine and sine.

$$\frac{\tan x \cos x}{\sec x \cot x} = \frac{\left(\dfrac{\boxed{}}{\boxed{}}\right) \cos x}{\dfrac{1}{\boxed{}}\;\dfrac{\boxed{}}{\boxed{}}}$$

$$= \frac{\sin x}{\dfrac{\boxed{}}{\boxed{}}}$$

$= \_\_\_\_\_\_ (\_\_\_\_\_\_)$

$= \_\_\_\_\_\_\_\_\_\_$

To see a similar question, refer to Example 2 on pages 293 and 294 of *Pre-Calculus 12*.

Working Example 3: Use the Pythagorean Identity

a) Verify that the equation $\cot^2 x - \csc^2 x = -1$ is true when $x = \frac{\pi}{4}$.

b) Use the quotient identities to express the Pythagorean identity $\cos^2 x + \sin^2 x = 1$ as the equivalent identity $\cot^2 x - \csc^2 x = -1$.

Solution

a) Substitute $x = \frac{\pi}{4}$.

Left Side	Right Side
$\cot^2 x - \csc^2 x$	-1
$= \cot^2 \square - \csc^2 \square$	
$= \frac{1}{\square\ \frac{\pi}{4}} - \frac{1}{\square\ \frac{\pi}{4}}$	
$= \frac{1}{\square} - \frac{1}{\square}$	
= ________	
= ________	

Left Side ________ Right Side
(= or ≠)

The equation $\cot^2 x - \csc^2 x = -1$ is ________ when $x = \frac{\pi}{4}$.
(*true* or *not true*)

b) $\cos^2 x + \sin^2 x = 1$

Multiply both sides by $\frac{1}{\square}$, $x \neq \pi n$, where $n \in \mathrm{I}$.

$$\left(\frac{1}{\square}\right)\cos^2 x + \left(\frac{1}{\square}\right)\sin^2 x = \left(\frac{1}{\square}\right)1$$

$$\frac{\cos^2 x}{\square} + \frac{\sin^2 x}{\square} = \frac{1}{\square}$$

________ + 1 = ________ Use the quotient identities.

________ − ________ = −1 Rearrange the terms.

To see a similar example, refer to Example 3 on page 295 of *Pre-Calculus 12*.

Check Your Understanding

Practise

1. Determine the non-permissible values of x, in radians, for each expression.

a) $\frac{\sin x}{\cos x}$

b) $\frac{\cos x}{\tan x}$

c) $\frac{\cot x}{1 + \sin x}$

d) $\frac{\tan x}{\cos x - 1}$

In parts c) and d), explain whether it is possible to write a single restriction.

2. Simplify each expression to one of the three primary trigonometric functions, $\sin x$, $\cos x$, or $\tan x$.

a) $\cot x \sin x$

b) $\frac{\sec^2 x \cos x}{\csc x}$

c) $\frac{\cot x \tan x}{\csc x}$

3. Simplify. Then, rewrite each expression as one of the three reciprocal trigonometric functions, $\csc x$, $\sec x$, or $\cot x$.

a) $\frac{\csc x}{\sec x}$

b) $\csc x \tan x \sec x \cos x$

c) $\frac{\sin x}{1 - \cos^2 x}$

4. a) Verify that the equation $\frac{\csc x}{\tan x + \cot x} = \cos x$ is true for $x = 60°$ and for $x = \frac{\pi}{6}$.

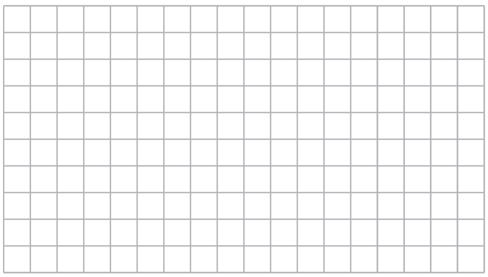
b) What are the non-permissible values of the equation in the domain $0° \leq x < 360°$.

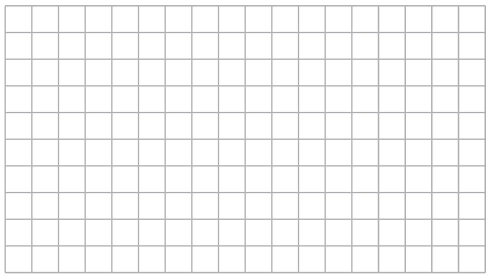
What determines if a value is permissible when it is in the denominator?

5. Consider the equation $\tan x + \frac{1}{\tan x} = \frac{1}{\cos x \sin x}$.

a) What are the non-permissible values, in radians, for this equation?

b) Using technology, graph the two sides of the equation over the domain $0 \leq x \leq 2\pi$. Sketch the graphs below. Could this equation be an identity? Explain.





c) Verify that the equation is true when $x = \frac{\pi}{4}$. Use exact values for each expression in the equation.

Apply

6. When a polarized lens is rotated through angle θ over a second lens, the amount of light passing through both lenses decreases by $(1 - \cos\theta)(1 + \cos\theta)$.

   a) Determine an equivalent expression for this decrease, using only $\sin\theta$.

   $(1 - \cos\theta)(1 + \cos\theta)$

   $= 1 +$ ________ $-$ ________ $- \cos^2\theta$

   $=$ ________ $-$ ________

   $=$ ________

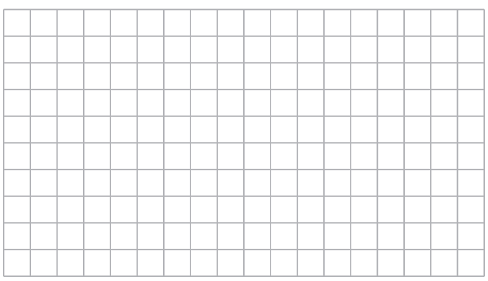
   b) What fraction of light is lost when $x = \frac{\pi}{3}$?

   c) What percent of light is lost when $\theta = 30°$?

7. Compare $y = \cos x$ and $y = \sqrt{1 - \sin^2 x}$ by completing the following.

   a) Verify that $\cos x = \sqrt{1 - \sin^2 x}$ for $x = \frac{\pi}{6}$, $x = \frac{4\pi}{3}$, and $x = \pi$.

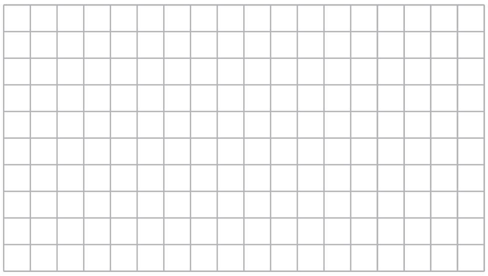
   b) Using technology, graph $y = \cos x$ and $y = \sqrt{1 - \sin^2 x}$ in the same window. Sketch the graphs below.

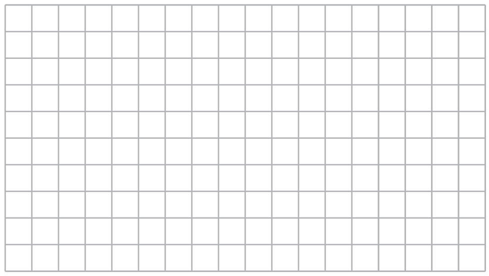


   c) State whether $\cos x = \sqrt{1 - \sin^2 x}$ is possibly an identity. Justify your answer.

8. Simplify $\frac{\cot x + \tan x}{\sec x}$ to one of the three reciprocal trigonometric ratios. What are the non-permissible values of the original expression in the domain $0 \leq x < 2\pi$?

9. a) Use technology to determine, graphically, whether the expression $\csc x - \frac{\cot x}{\sec x}$ appears to be equivalent to $\cos x$ or $\sin x$. Sketch the graphs. Write the identity below.





b) What are the non-permissible values of the identity in part a), expressed in radians?

c) Express $\csc x - \frac{\cot x}{\sec x}$ as the single primary trigonometric ratio that you identified in part a).

10. Simplify $(\cos x - \sin x)^2 - (\sin x - \cos x)^2$.

Connect

11. a) Complete the table for non-permissible values for each trigonometric ratio.

Trigonometric Ratio	Non-Permissible Values (degrees)	Non-Permissible Values (radians)
tan x		
csc x		
sec x		
cot x		

b) Explain why the table does not include sin x and cos x.

c) If sin x or cos x is the only term in the denominator, do you need to consider non-permissible values? Explain.

12. a) Complete the chart with the zeros of each trigonometric ratio in the first column.

Trigonometric Ratio	Zero Values (degrees)	Zero Values (radians)
sin x		
cos x		
tan x		
cot x		

b) When are the zeros non-permissible values?

c) Why are there no zero values for csc x and sec x?

13. Do all identities have non-permissible values? Explain.

6.2 Sum, Difference, and Double-Angle Identities

KEY IDEAS

Sum and Difference Identities

The sum and difference identities are used to simplify expressions and to determine exact trigonometric values of some angles.

- **Sum Identities**

  $\sin (A + B) = \sin A \cos B + \cos A \sin B$

  $\cos (A + B) = \cos A \cos B - \sin A \sin B$

  $\tan (A + B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

Examples:

$\sin (12° + 23°) = \sin 12° \cos 23° + \cos 12° \sin 23°$

$\cos \left(\frac{\pi}{6} + \frac{\pi}{4}\right) = \cos \frac{\pi}{6} \cos \frac{\pi}{4} - \sin \frac{\pi}{6} \sin \frac{\pi}{4}$

$\tan (40° + 25°) = \dfrac{\tan 40° + \tan 25°}{1 - \tan 40° \tan 25°}$

- **Difference Identities**

  $\sin (A - B) = \sin A \cos B - \cos A \sin B$

  $\cos (A - B) = \cos A \cos B + \sin A \sin B$

  $\tan (A - B) = \dfrac{\tan A - \tan B}{1 + \tan A \tan B}$

$\sin (52° - 33°) = \sin 52° \cos 33° - \cos 52° \sin 33°$

$\cos \left(\frac{\pi}{3} - \frac{\pi}{4}\right) = \cos \frac{\pi}{3} \cos \frac{\pi}{4} + \sin \frac{\pi}{3} \sin \frac{\pi}{4}$

$\tan (70° - 35°) = \dfrac{\tan 70° - \tan 35°}{1 + \tan 70° \tan 35°}$

Double-Angle Identities

Double-angle identities are special cases of the sum identities when the two angles are equal.

$\sin 2A = 2 \sin A \cos A$

The double-angle identity for cosine can be expressed in three different forms:

$\cos 2A = \cos^2 A - \sin^2 A$
$\cos 2A = 2 \cos^2 A - 1$
$\cos 2A = 1 - 2 \sin^2 A$

$\tan 2A = \dfrac{2 \tan A}{1 - \tan^2 A}$

Examples:

$\sin \frac{\pi}{4} = 2 \sin \frac{\pi}{8} \cos \frac{\pi}{8}$

$\cos 140° = \cos^2 70° - \sin^2 70°$
$\cos 140° = 2 \cos^2 70° - 1$
$\cos 140° = 1 - 2 \sin^2 70°$

$\tan \frac{\pi}{6} = \dfrac{2 \tan \frac{\pi}{12}}{1 - \tan^2 \frac{\pi}{12}}$

Special Angles and Their Exact Trigonometric Values

Degrees	Radians	sin θ	cos θ	tan θ
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
60°	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

Working Example 1: Simplify Expressions Using Sum, Difference, and Double-Angle Identities

Write each expression as a single trigonometric function.

a) $\cos 32° \cos 50° + \sin 32° \sin 50°$

b) $\dfrac{2 \tan \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}}$

Solution

a) The expression $\cos 32° \cos 50° + \sin 32° \sin 50°$ has the same form as the right side of the difference identity for ________:

________(A − B) = ________ A ________ B + ________ A ________ B

Thus,

$\cos 32° \cos 50° + \sin 32° \sin 50°$ = ________ (32° ________ 50°)

= ________

How can you verify your solution?

b) The expression $\dfrac{2 \tan \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}}$ has the same form as the right side of the ________ identity for tangent: tan 2A = $\dfrac{\square}{\square}$.

Therefore,

$\dfrac{2 \tan \frac{\pi}{6}}{1 - \tan^2 \frac{\pi}{6}}$ = ________ $2\left(\frac{\pi}{6}\right)$

= ________ $\dfrac{\square}{\square}$

To see a similar example, refer to Example 1 on page 301 of *Pre-Calculus 12*.

Working Example 2: Simplify Expressions Using Identities

Consider the expression $\dfrac{\sin 2x}{\cos 2x - 1}$.

a) What are the permissible values for the expression?

b) Simplify the expression to one of the three reciprocal trigonometric ratios.

c) How can you verify your answer from part b) in the interval [0, 2π)?

Solution

a) Identify any non-permissible values. The expression $\frac{\sin 2x}{\cos 2x - 1}$ is undefined when ________.

Use a double-angle identity to simplify cos 2*x*: $\cos 2x =$ ________________.

Solve ________________ $= 0$.

Does it matter which double-angle identity for cos 2*x* you select? Explain.

There are two sets of non-permissible values: ________________.

When these two sets of non-permissible values are combined, the permissible values are all real numbers except ________.

To see a second method for determining the permissible values in part a), refer to Example 3 on pages 302 and 303 of *Pre-Calculus 12*.

b) $\frac{\sin 2x}{\cos 2x - 1} = \frac{\boxed{}}{\boxed{} - 1}$

$= \frac{\boxed{}}{\boxed{}}$

$= -\frac{\boxed{}}{\boxed{}}$

$=$ ________

How does the −1 in the denominator help you to determine which double-angle identity for cos 2*x* is best in this situation?

The expression $\frac{\sin 2x}{\cos 2x - 1}$ is equivalent to ________.

c) I can verify my answer by ________________.

Use the method you have described to verify your answer. Does your verification prove your answer? Explain.

To see a similar example, refer to Example 3 on pages 302 and 303 of *Pre-Calculus 12*.

Working Example 3: Determine Exact Trigonometric Values for Angles

Determine the exact value for each expression, stating your answer in the most simplified form.

a) $\cos \frac{7\pi}{12}$

b) $\tan 165°$

Solution

a) Use the ______________ identity for cosine with two special angles that can be ____________________ (*added* or *subtracted*) to get $\frac{7\pi}{12}$.

$$\cos \frac{7\pi}{12} = \cos\left(\frac{\square}{\square} + \frac{\square}{\square}\right)$$

$$=$$

b) Rewrite tan 165° as the sum of special angles:

$\tan 165° = \tan (\_\_\_\_\_\_° + \_\_\_\_\_\_°)$.

Use the tangent ______________ (*sum* or *difference*) identity:

$$\tan (A \_\_\_ B) = \frac{\tan A \_\_\_ \tan B}{1 \_\_\_ \tan A \tan B}.$$

$$\tan (\_\_\_\_\_\_° + \_\_\_\_\_\_°) = \frac{\square + \square}{1 - \square\ \square}$$

$$=$$

Describe a second method that you could use to answer part b).

To see a similar example, refer to Example 4 on pages 304–305 of *Pre-Calculus 12*.

Check Your Understanding

Practise

1. Write each expression as a single trigonometric function.

a) $\cos 87° \cos 22° + \sin 87° \sin 22°$

b) $\sin 72° \cos 46° - \cos 72° \sin 46°$

c) $\dfrac{\tan 28° + \tan 33°}{1 - \tan 28° \tan 33°}$

d) $6 \sin \dfrac{\pi}{10} \cos \dfrac{\pi}{10}$

e) $1 - 2 \sin^2 \dfrac{\pi}{8}$

f) $\dfrac{2 \tan \frac{\pi}{3}}{1 - \tan^2 \frac{\pi}{3}}$

2. Simplify. Then, give an exact value for each expression.

a) $2 \sin \dfrac{\pi}{4} \cos \dfrac{\pi}{4}$

b) $\cos \dfrac{\pi}{3} \cos \dfrac{\pi}{12} + \sin \dfrac{\pi}{3} \sin \dfrac{\pi}{12}$

c) $\dfrac{\tan 80° + \tan 40°}{1 - \tan 80° \tan 40°}$

d) $2 \cos^2 \dfrac{\pi}{2} - 1$

3. Write each as a single trigonometric function.

a) $\sin 80° \cos 40° - \cos 80° \sin 40°$

b) $\cos \frac{2\pi}{3} \cos \frac{\pi}{12} - \sin \frac{2\pi}{3} \sin \frac{\pi}{12}$

c) $\dfrac{\tan \frac{2\pi}{3} - \tan \frac{\pi}{12}}{1 + \tan \frac{2\pi}{3} \tan \frac{\pi}{12}}$

4. Simplify each expression to a single primary trigonometric function.

a) $\dfrac{\cos 2x - 1}{\sin 2x}$

b) $1 - 2 \sin^2 \frac{\theta}{4}$

c) $\frac{1}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2}$

$\frac{1}{2} \sin \frac{\theta}{2} \cos \frac{\theta}{2} = 2$ (________________)

$= \frac{1}{4} \sin 2$(________________)

$=$ ________________

d) $8 \sin^2 2\theta - 4$

$8 \sin^2 2\theta - 4 = -4(1 -$ __________)

$= -4$__________

5. Consider the expression $\dfrac{1 - \cos 2x}{\sin x}$.

a) State the permissible values.

b) Simplify the expression to one of the three primary trigonometric ratios.

6. Determine the exact value of each trigonometric expression.

a) $\sin 105°$

b) $\cos 165°$

c) $\tan \frac{23\pi}{12}$

d) $\csc \frac{5\pi}{12}$

Apply

7. Simplify $\sin (x + y) + \sin (x - y)$.

8. Angle θ is in quadrant III and $\tan \theta = \frac{7}{24}$. Determine an exact value for each of the following.

a) $\sin 2\theta$

b) $\cos 2\theta$

c) $\tan 2\theta$

9. Angle x is in quadrant II, angle y is in quadrant III, $\cos x = -\frac{5}{13}$, and $\tan y = \frac{4}{3}$. Determine the value of each of the following.

a) $\sin (x + y)$

b) $\cos (x - y)$

c) $\tan (x - y)$

10. **Simplify each expression to the equivalent expression shown.**

a) $\dfrac{\sin 2x}{1 - \cos 2x}$; $\cot x$

b) $\sin (x + y) \sin (x - y)$; $\sin^2 x - \sin^2 y$

11. Simplify each of the following.

a) $\cos \left(\dfrac{3\pi}{4} + x\right) + \sin \left(\dfrac{3\pi}{4} + x\right)$

b) $\cos \left(\dfrac{\pi}{4} - x\right) \sec \dfrac{\pi}{4} - \sin \left(\dfrac{\pi}{4} - x\right) \csc \dfrac{\pi}{4}$

Connect

12. a) Explain how the double-angle identities are related to the sum identities.

b) Explain why there are three forms of the double-angle identity for cosine. How are they related to each other?

6.3 Proving Identities

KEY IDEAS

Guidelines for Proving Identities

- To prove that an identity is true for all permissible values, express both sides of the identity in equivalent forms. One or both sides of the identity must be algebraically manipulated into an equivalent form to match the other side.
- There is a major difference between solving a trigonometric equation and proving a trigonometric identity:
  – *Solving* a trigonometric equation determines the value that makes a particular case true. You perform equivalent operations on both sides of the equation (that is, perform operations across the = sign) to isolate the variable and solve for the variable.
  – *Proving* an identity shows that the expressions on each side of the equal sign are equivalent for *all* values for which the variable is defined. Therefore, you work on each side of the identity independently, and you *do not* perform operations across the = sign.

Tips for Proving Identities

- It is easier to simplify a complicated expression than to make a simple expression more complicated, so start with the more complicated side of the identity.
- Use known identities to make substitutions.
- If a quadratic is present, consider the Pythagorean identity first. It, or one of its alternative forms, can often be used.
- Rewrite the expression using sine and cosine only.
- Multiply the numerator and the denominator by the conjugate of an expression.
- Factor to simplify expressions.

Verifying Identities

- Identities can be verified using a specific value, but this validates that the identity is true for that value only.
- Graphing each side of a possible identity may show the identity might be true, but it does not prove the identity formally.

Working Example 1: Verify Versus Prove That an Equation Is an Identity

a) Verify numerically that $1 - \cos^2 x = \sin x \cos x \tan x$ for some values of x. Work in degrees.

b) Prove that $1 - \cos^2 x = \sin x \cos x \tan x$ for all permissible values of x.

Solution

a) First, determine the non-permissible values. The only function in the equation that has non-permissible values in its domain is ________________. The non-permissible values are ________________.

Verify the identity numerically. Use $x =$ ________.

> Does it matter what choice of angle measure you make to verify the identity numerically? Explain.

Left Side	Right Side
$1 - \cos^2 x$	$\sin x \cos x \tan x$
$= 1 - \cos^2$ ________°	$=\sin$ ________° $\cos$ ________° $\tan$ ________°
$= 1 - \left(\dfrac{\square}{\square}\right)^2$	$=$
$= 1 - \dfrac{\square}{\square}$	
$=$ ________	

Left Side = Right Side

> Describe how you could verify this identity graphically.

b) To prove the identity algebraically, examine both sides of the equation and simplify each side to a common expression.

Left Side $= 1 - \cos^2 x$ Right Side $= \sin x \cos x \tan x$

$=$ $=$

The simplified forms of the two sides ________________ equal. Therefore,
(*are* or *are not*)

$1 - \cos^2 x = \sin x \cos x \tan x$ ____________________________.

To see an example similar to the above, refer to Example 1 on pages 310 and 311 of *Pre-Calculus 12*.

Working Example 2: Prove an Identity Using Double-Angle Identities

Prove that $\frac{1 + \cos 2x}{\sin 2x} = \cot x$ is an identity for all permissible values of x.

Solution

Left Side $= \frac{1 + \cos 2x}{\sin 2x}$

Right Side $= \cot x$

$=$

Which double-angle identity should be selected for cos 2*x*? Explain.

The Left Side ________ Right Side. Therefore, $\frac{1 + \cos 2x}{\sin 2x} = \cot x$ ________ an identity for all permissible values of x.
(= or ≠) (*is* or *is not*)

To see another example similar to the above, refer to Example 2 on page 311 of *Pre-Calculus 12*.

Working Example 3: Prove More Complicated Identities

Prove that $\frac{\cos \theta}{1 - \sin \theta} + \frac{\cos \theta}{1 + \sin \theta} = \frac{2}{\cos \theta}$

Solution

Left Side $= \frac{\cos \theta}{1 - \sin \theta} + \frac{\cos \theta}{1 + \sin \theta}$

Right Side $= \frac{2}{\cos \theta}$

$= \frac{\cos \theta\left(\boxed{}\right) + \cos \theta\left(\boxed{}\right)}{(1 - \sin \theta)(1 + \sin \theta)}$

$=$

Which identity is used to simplify the denominator?

The Left Side ________ Right Side. Therefore, $\frac{\cos \theta}{1 - \sin \theta} + \frac{\cos \theta}{1 + \sin \theta} = \frac{2}{\cos \theta}$. ________ an identity for all permissible values of x.
(*is* or *is not*)

To see a similar example, refer to Example 3 on page 312 of *Pre-Calculus 12*.

Working Example 4: Prove an Identity That Requires Factoring

Prove the identity $\cos^4 x - \sin^4 x = \cos 2x$.

Solution

Left Side $= \cos^4 x - \sin^4 x$

$= (\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_)$

$= (\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_)$

$= \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

How do you factor a difference of squares?

Which identity did you use to simplify the first factor? Which identity did you use to simplify the second factor?

To see another example similar to the above, refer to Example 4 on page 313 of *Pre-Calculus 12*.

Check Your Understanding

Practise

1. Factor and simplify each rational trigonometric expression.

a) $\dfrac{\cos x - \sin^2 x \cos x}{\cos^2 x}$

b) $\dfrac{\cos^2 x - 3 \cos x - 10}{8 + 4 \cos x}$

c) $\dfrac{3 \sec x + 6 \sec x \sin x}{4 \sin^2 x - 1}$

2. Use factoring to help to prove each identity for all permissible values of x.

a) $\dfrac{\sin x + \sin^2 x}{\cos x + \sin x \cos x} = \tan x$

b) $1 - \tan x = \dfrac{\cos^2 x - \sin^2 x}{\cos^2 x + \sin x \cos x}$

c) $\dfrac{3\cos^2 x + 5\cos x - 2}{9\cos^2 x - 1} = \dfrac{\cos x + 2}{3\cos x + 1}$

3. Use a common denominator to express the rational expressions as a single term.

a) $\dfrac{\cos x}{\sin x} + \sec x$

b) $\dfrac{1}{1 - \cos x} - \dfrac{1}{1 + \cos x}$

c) $\dfrac{\cos x}{1 + \sin x} + \dfrac{\sin x}{\cos x}$

4. Prove each identity.

a) $\frac{1 - \sin^2 x}{\cos x} = \frac{\sin 2x}{2 \sin x}$

b) $\frac{\csc^2 x - 1}{\csc^2 x} = \cos^2 x$

Left Side $= \frac{\csc^2 x - 1}{\csc^2 x}$

$= \frac{\boxed{}}{\boxed{}} = \frac{1}{\boxed{}}$

$= 1 -$ ________________

$=$ ________________

Right Side $=$ ________________

c) $(\cos x - \sin x)^2 = 1 - \sin 2x$

5. Match each expression on the left with an equivalent expression on the right. Justify your answer.

a) $\sin x \cot x$

b) $1 - 2 \sin^2 x$

c) $(\sin x + \cos x)^2$

d) $\sec^2 x$

A $\sin^2 x + \cos^2 x + \tan^2 x$

B $1 + 2 \sin x \cos x$

C $\cos x$

D $2 \cos^2 x - 1$

Apply

6. **a)** Use technology to verify graphically that $\tan x = \dfrac{\sin x + \sin 2x}{\cos 2x + 1 + \cos x}$ could be an identity. Sketch the graphs below.

b) Prove the identity using double-angle identities.

c) Determine any non-permissible values.

7. Prove each identity.

a) $\sec x = \dfrac{2(\cos x \sin 2x - \sin x \cos 2x)}{\sin 2x}$

$$\text{Right Side} = \frac{2(\cos x \sin 2x - \sin x \cos 2x)}{\sin 2x}$$

$$= \frac{2\boxed{}(2 \sin x \cos x) - \boxed{}(2 \cos^2 x - 1)}{2 \sin x \cos x}$$

$$= \frac{2\boxed{} - 2\boxed{} + \sin x}{\sin x \cos x}$$

$$=$$

$$\text{Left Side} = \sec x$$

b) $\sec x = \frac{2 \csc 2x \tan x}{\sec x}$

c) $\tan 2x - \sin 2x = 2 \tan 2x \sin^2 x$

d) $\frac{1 + \tan x}{1 + \cot x} = \frac{1 - \tan x}{\cot x - 1}$

8. Prove each identity.

a) $\sin (45° + x) + \sin (45° - x) = \sqrt{2} \cos x$

Left Side $= \sin (45° + x) + \sin (45° - x)$

$= (\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_)$

$= 2 \_\_\_\_\_\_ \cos x$

$= 2 \frac{\square}{\square} \cos x$

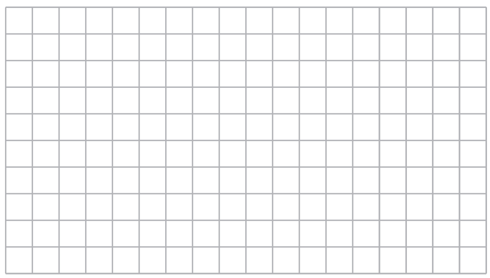
$= \_\_\_\_\_\_ \cos x$

Right Side $= \_\_\_\_\_\_$

b) $\sin (x + \pi) = -\cos \left(x + \frac{3\pi}{2}\right)$

9. Consider the equation $\sin^4 x - \cos^4 x = 2 \sin^2 x + 1$.

a) Graph each side of the equation. Sketch your graphs below. Could the equation be an identity? Explain.



b) Either prove that the equation is an identity or find a counterexample to show that it is not an identity.

10. Prove that $2 \sin (x + y) \cos (x + y) = \sin 2x \cos 2y + \sin 2y \cos 2x$.

Connect

11. Consider the equation $\sin^4 x + \cos^2 x = \sin^2 x + \cos^4 x$.

a) Using technology, graph each side of the equation. Could the equation be an identity?

b) Either prove that the equation is an identity or find a counterexample to show that it is not an identity.

c) Explain the difference between the verification you made in part a) and the proof you provided in part b).

6.4 Solving Trigonometric Equations Using Identities

KEY IDEAS

Solving Trigonometric Equations

Solving a trigonometric equation means to find all the possible angle values that make the equation true within the given or restricted domain. When the domain is not restricted, you provide a general solution.

Strategies for Solving Trigonometric Equations

Description	Example
Isolate the trigonometric ratio, if possible.	$2 \sin x = 1$ $\sin x = \frac{1}{2}$
Factor and then set each factor equal to 0.	Common factoring: $\sin x \tan x + \sin x = 0$ $\sin x (\tan x + 1) = 0$ $\sin x = 0$ or $\tan x = -1$ Difference of squares: $\sin^2 x - 1 = 0$ $(\sin x - 1)(\sin x + 1) = 0$ $\sin x = 1$ or $\sin x = -1$ Trinomial factoring: $2 \sin^2 x - \sin x - 1 = 0$ $(2 \sin x + 1)(\sin x - 1) = 0$ $\sin x = -0.5$ or $\sin x = 1$
Simplify the given equation or change the given equation to one common ratio (such as $\sin x$ or $\cos x$) by using one or more of the following: – reciprocal identities – quotient identities – Pythagorean identities – double-angle identities Then, solve.	$\cos 2x - 2 \sin x + 3 = 0$ $(1 - 2 \sin^2 x) - 2 \sin x + 3 = 0$ Replace $\cos 2x$ with $1 - 2 \sin^2 x$. $-2 \sin^2 x - 2 \sin x + 4 = 0$ Simplify. $\sin^2 x + \sin x - 2 = 0$ $(\sin x + 2)(\sin x - 1) = 0$ Factor. $\sin x = -2$ or $\sin x = 1$ Solve. It is important to consider all possible solutions to ensure that they are not non-permissible values. In this example, the root $x = -2$ is rejected because the minimum value for $\sin x$ is -1.

Checking Trigonometric Equations

- The algebraic solution can be verified graphically.
- Check that solutions for an equation do not include non-permissible values from the original equation.

Working Example 1: Solve by Substituting Trigonometric Identities and Factoring

Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $\sin 2x + \sin x = 0$

b) $6 \sin^2 x = \cos x + 4$

Solution

a) $\sin 2x + \sin x = 0$

> Explain why a double-angle formula is needed.

2________ ________ + $\sin x = 0$

$\sin x$ (________ + ________) = 0

$\sin x = 0$ or ____________ = 0

$x =$ ______ or $x =$ ______

$\cos x = -\frac{\square}{\square}$

$x =$ ______ or $x =$ ______

State the non-permissible value(s) for this equation, if any: __________________________.

The solutions over the given domain are ____.

b) $6 \sin^2 x = \cos x + 4$

> Which term in the equation might lead you to consider using the Pythagorean identity?

6(______ − ________) − $\cos x - 4 = 0$

6 − 6________ − $\cos x - 4 = 0$

−6________ − $\cos x$ + ________ = 0

6________ + $\cos x$ − ________ = 0

(________________)(________________) = 0

________________ = 0 or ________________ = 0

$\cos x =$ ________________ $\cos x =$ ________________

$x =$ ______ or $x =$ ______ $x \approx$ ______ or $x \approx$ ______

State the non-permissible value(s) for this equation, if any: __________________________.

The solutions over the given domain are ____.

To see a similar example, refer to Example 1 on page 317 of *Pre-Calculus 12*.

Working Example 2: Solve an Equation With a Quotient Identity Substitution

Solve the equation $\sin^2 x = \tan x \cos x$ algebraically in the domain $0° \leq x < 360°$.

Solution

$\sin^2 x = \tan x \cos x$

$\sin^2 x = \frac{\square}{\square} \cos x$

$\sin^2 x =$ ________

________________ $= 0$

$\sin x = 0$ or $\sin x =$ ________

For $\sin x = 0$, $x =$ ________° or $x =$ ________°.

How could you verify your answer graphically?

For $\sin x =$ ________, $x =$ ________°.

State the non-permissible value(s) for this equation, if any: ______________________________.

The solutions over the given domain are ______________________________.

Working Example 3: Determine the General Solution for a Trigonometric Equation

Solve $\cos 2x = \sqrt{3} \cos x - 1$ algebraically. Give the general solution in radians.

Solution

Which double-angle identity should you select for $\cos 2x$? Explain.

$\cos 2x = \sqrt{3} \cos x - 1$

________________ $= \sqrt{3} \cos x - 1$

________________ $- \sqrt{3} \cos x + 1 = 0$

________________ $- \sqrt{3} \cos x = 0$

$\cos x$ (________________) $= 0$

$\cos x = 0$ or ________________ $= 0$

$\cos x =$ ________

For $\cos x = 0$, $x =$ ________________, where $n \in I$.

For $\cos x =$ ________, $x =$ ________________, and $x =$ ________________, where $n \in I$.

State the non-permissible value(s) for this equation, if any: ______________________________.

The general solution is $x =$ ____, where $n \in I$.

To see a similar example, refer to Example 3 on page 319 of *Pre-Calculus 12*.

Working Example 4: Determine the General Solution Using Reciprocal Identities

Algebraically solve $2 \csc^2 x + \csc x - 1 = 0$. Give general solutions expressed in radians.

Solution

How does factoring help to solve this equation?

$$2 \csc^2 x + \csc x - 1 = 0$$

(____________)(____________) = 0

____________ = 0 or ____________ = 0

csc x = ______ csc x = ______

So, $\frac{1}{\boxed{}}$ = ____________ or $\frac{1}{\boxed{}}$ = ____________.

sin x = ______ sin x = ______

For sin x = ______ there is no solution.

For sin x = ______, x = ____________, where $n \in$ I.

The non-permissible values are ____________________.

The solution is ____________________, where $n \in$ I.

Check Your Understanding

Practise

1. Solve each equation algebraically over the domain $0° \leq x < 360°$.

a) $2 \sin x = \sqrt{3}$

b) $2 \cos x - 1 = 0$

c) $\tan x - 1 = 0$

d) $\cot x + 1 = 0$

2. Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $4 \sin^2 x - 1 = 0$

b) $4 \cos^2 x = 3$

c) $\tan^2 x - 3 = 0$

d) $3 \csc^2 x - 4 = 0$

3. Solve each equation algebraically over the domain $0 \leq x < 2\pi$. Use technology to verify your solution graphically.

a) $\sin^2 x - \sin x = 0$

b) $\cos^2 x + \cos x = 0$

c) $\tan x + \tan^2 x = 0$

d) $\cos^2 x + 2 \cos x = 0$

4. Solve each equation algebraically over the domain $0 \leq x < 2\pi$. Verify your solution graphically.

a) $\sin 2x - 1 = \cos 2x$

b) $\sqrt{2} \sin^2 x = \tan x \cos x$

c) $\cos 2x = \cos^2 x$

$$\cos 2x = \cos^2 x$$

$$2\,\_\_\_\_\_\_\_\_\_\_ - 1 = \cos^2 x$$

$$\_\_\_\_\_\_\_\_\_\_ = 1$$

$$\_\_\_\_\_\_\_\_\_\_ = 1 \text{ or } \_\_\_\_\_\_\_\_\_\_ = -1$$

$x =$ __________ or $x =$ __________

d) $\cos 2x = 2 \sin^2 x$

e) $\sin 2x \tan x = 1$

5. Rewrite each equation in terms of sine or cosine only. Then, solve algebraically for $0 \leq x < 2\pi$.

a) $\sin^2 x - \cos^2 x = \frac{1}{2}$

b) $2 \sin^2 x - 3 \cos 2x = 3$

c) $3 \cos 2x + \cos x + 1 = 0$

d) $3 + \sin x = 5 \cos 2x$

Apply

6. Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $8 \sin^2 x - 6 \sin x + 1 = 0$

b) $\cos x + 1 = 2 \sin^2 x$

c) $2 \cos^2 x - 3 \cos x + 1 = 0$

d) $\sin^2 x + 2 \sin x - 3 = 0$

e) $2 \tan^2 x = 3 \tan x - 1$

f) $\sin x = -\cos 2x$

7. a) Solve $2 \sin^2 x = -3 \cos x$ algebraically over each domain. Verify your answers graphically.

i) $0 \leq x < 2\pi$ **ii)** $-2\pi \leq x < 2\pi$ **iii)** $-\pi \leq x < \pi$

b) Describe the relationship between the domain and the number of solutions.

c) What is the general solution for this equation?

8. Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $\csc^2 x - \csc x - 2 = 0$

b) $2 \sec^2 x + \sec x - 1 = 0$

c) $3 \csc^2 x - 5 \csc x - 2 = 0$

d) $\sec^2 x + 5 \sec x + 6 = 0$

9. Assume that $\cos x = -\frac{2}{3}$ and $\cos x = \frac{1}{4}$ are the solutions of a trigonometric equation. What are the values of B and C if the equation is of the form $12 \cos^2 x + \text{B} \cos x + \text{C} = 0$?

10. Solve $\sin x \cos 2x + \sin x = 0$ algebraically over the domain of real numbers. Give your answer(s) in radians.

11. Solve the equation $\sin 2x = -\sqrt{2} \cos x$ algebraically. Give the general solution expressed in radians.

Connect

12. Explain if it is possible to solve the equation $\sin^2 x - 5 \sin x + 3 = 0$ by factoring?

13. Determine the mistake that Brooke made in the following work. Then, complete a correct solution.

Solve $\cos 2x = -\cos x$. Express your answers in degrees.

Solution

cos 2x = –cos x

cos 2x + cos x = 0

cos x (cos x + 1) = 0

cos x = 0 or cos x = –1

x = 90° + 180°n or x = 180° + 360°n, where n ∈ I

Chapter 6 Review

6.1 Reciprocal, Quotient, and Pythagorean Identities, pages 188–196

1. Determine the non-permissible values of x, in radians, for each expression.

a) $\dfrac{\sec x}{\sin x}$

b) $\dfrac{\cos x}{\csc x}$

c) $\dfrac{\sec x}{1 + \cos^2 x}$

2. Simplify each expression to one of the three primary trigonometric functions: $\sin x$, $\cos x$, or $\tan x$.

a) $\dfrac{\cos x \csc x}{\sec x \cot x}$

b) $\dfrac{\cot x \tan x}{\csc x}$

3. Simplify. Then, rewrite each expression as one of the three reciprocal trigonometric functions: $\csc x$, $\sec x$, or $\cot x$.

a) $\cot x \sec x$

b) $\dfrac{\cos x}{(1 - \sin x)(1 + \sin x)}$

4. a) Verify that the equation $(\sec x + \tan x) \cos x - 1 = \sin x$ is true for $x = 30°$ and for $x = \frac{\pi}{3}$.

b) What are the non-permissible values of the equation in part a) in the domain $0° \leq x < 360°$?

6.2 Sum, Difference, and Double-Angle Identities, pages 197–204

5. Write each expression as a single trigonometric ratio. Then, give an exact value for the expression.

a) $\cos^2 15° - \sin^2 15°$

b) $\sin 35° \cos 100° + \cos 35° \sin 100°$

c) $1 - 2\sin^2 75°$

6. Determine the exact value of each trigonometric expression.

a) $\sin\left(-\frac{\pi}{12}\right)$

b) $\cos \frac{\pi}{12}$

c) $\cos 105°$

d) $\sin \frac{23\pi}{12}$

7. Angle θ is in quadrant II and $\sin \theta = \frac{7}{25}$. Determine an exact value for each of the following.

a) $\sin 2\theta$

b) $\cos 2\theta$

c) $\tan 2\theta$

6.3 Proving Identities, pages 205–214

8. Prove $\sin(\pi - x) - \tan(\pi + x) = \dfrac{\sin x(\cos x - 1)}{\cos x}$.

9. Consider the equation $\sin^2 x + \tan^2 x + \cos^2 x = \sec^2 x$.

a) Graph each side of the equation. Could the equation be an identity? Explain.

b) Either prove that the equation is an identity or find a counterexample to show that it is not an identity.

10. Prove each identity.

a) $\cos x \tan^2 x = \sin x \tan x$

b) $\sin 2x = \tan x + \tan x \cos 2x$

6.4 Solving Trigonometric Equations Using Identities, pages 215–223

11. Rewrite each equation in terms of sine or cosine. Then, solve algebraically for $0 \leq x < 2\pi$.

a) $2 \cos^2 x - \sin x = -1$

b) $\sin^2 x = 2 \cos x - 2$

c) $\cos x + \cos 2x = 0$

12. Solve each equation algebraically over the domain $0 \leq x < 2\pi$.

a) $2 \cos^2 x + 3 \cos x + 1 = 0$

b) $\sin^2 x + 3 \sin x + 2 = 0$

c) $\sin^2 x + 5 \sin x + 6 = 0$

d) $\cos^2 x + 3 \cos x + 2 = 0$

13. Solve the equation $2 \cos^2 x = 1 - \sin x$ algebraically. Give the general solution expressed in radians.

Chapter 6 Skills Organizer

Use the following organizer to summarize what you have learned about trigonometric identities.

Key Identities	Strategies for Proving
Trigonometric Identities	
Examples	Non-Examples