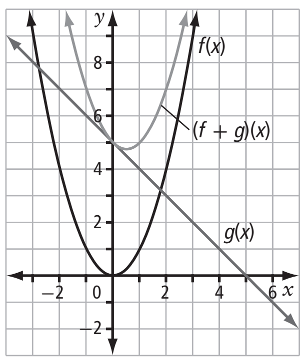
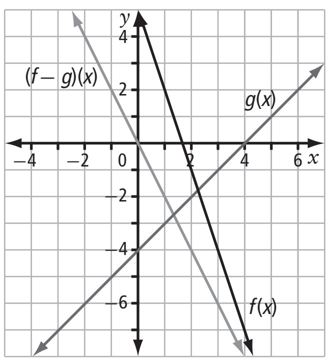
Chapter 10 Function Operations

10.1 Sums and Differences of Functions

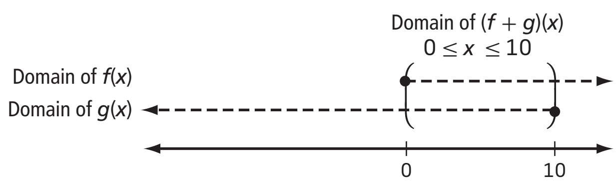
KEY IDEAS

- You can form new functions by performing operations with functions.

Sum of Functions	Difference of Functions
$h(x) = f(x) + g(x)$ or $h(x) = (f + g)(x)$ **Example** $f(x) = x^2$ and $g(x) = -x + 5$ $h(x) = f(x) + g(x)$ $h(x) = \mathbf{x^2} + (\mathbf{-x + 5})$ $h(x) = x^2 - x + 5$ 	$h(x) = f(x) - g(x)$ or $h(x) = (f - g)(x)$ **Example** $f(x) = -2x$ and $g(x) = x - 4$ $h(x) = f(x) - g(x)$ $h(x) = \mathbf{-2x} - (\mathbf{x - 4})$ $h(x) = -3x + 4$ 

- The domain of the combined function formed by the sum or difference of two functions is the domain common to the individual functions.

  Example
  If the domain of $f(x)$ is $\{x \mid x \geq 0, x \in \mathrm{R}\}$ and the domain of $g(x)$ is $\{x \mid x \leq 10, x \in \mathrm{R}\}$, the domain of $(f + g)(x)$ is $\{x \mid 0 \leq x \leq 10, x \in \mathrm{R}\}$.



- The range of a combined function can be determined using its graph.

Working Example 1: Determine the Sum of Two Functions

Consider $f(x) = x + 1$ and $g(x) = 2x - 5$.

a) Determine the equation of the function $h(x) = (f + g)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain and range of $h(x)$.

d) Determine the values of $f(x)$, $g(x)$, and $h(x)$ when $x = -3$.

Solution

a) $h(x) = (f + g)(x)$

$h(x) = f(\_\_\_\_\_\_\_\_) + g(\_\_\_\_\_\_\_\_)$

$h(x) = \mathbf{x + 1} + \_\_\_\_\_\_\_\_$

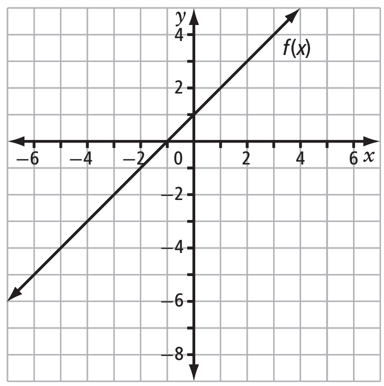
$h(x) = \_\_\_\_\_\_\_\_x + \_\_\_\_\_\_\_\_$

b) Use the slope-intercept form of the equation of a line to graph the functions $g(x)$ and $h(x)$.

For $g(x)$, the slope of the line $y = 2x - 5$ is ________ and the y-intercept is ________.

For $h(x)$, the slope of the line $y = 3x - 4$ is ________ and the y-intercept is ________.

Add the graphs of $g(x)$ and $h(x)$ to the grid shown.



c) The domain of $f(x)$ is $\{x \mid x \in R\}$. The domain of $g(x)$ is $\{x \mid x \in R\}$.

Therefore, the domain of $h(x)$ is ________________________.

The range of $f(x)$ is $\{y \mid y \in R\}$. The range of $g(x)$ is $\{y \mid y \in R\}$.

Therefore, the range of $h(x)$ is ________________________.

d) Substitute $x = -3$ into $f(x)$, $g(x)$, and $h(x)$.

$f(x) = x + 1$	$g(x) = 2x - 5$	$h(x) = 3x - 4$
$f(-3) = \_\_\_\_\_\_\_\_ + 1$	$g(-3) = \_\_\_\_\_\_\_\_$	$h(-3) = \_\_\_\_\_\_\_\_$
$f(-3) = \_\_\_\_\_\_\_\_$	$g(-3) = \_\_\_\_\_\_\_\_$	$h(-3) = \_\_\_\_\_\_\_\_$

This example should help you complete #1 on page 483 of *Pre-Calculus 12*.

Working Example 2: Determine the Difference of Two Functions

Consider $f(x) = x^2$ and $g(x) = 4x - 4$.

a) Determine the equation of the function $h(x) = (f - g)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain and range of $h(x)$.

Solution

a) $h(x) = (f - g)(x)$

$h(x) = f(x) - g(x)$

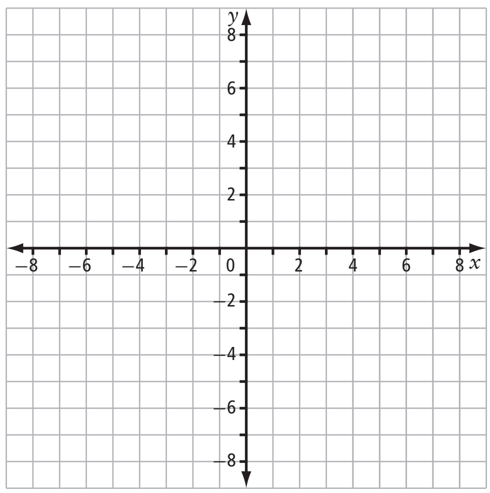
$h(x) = \mathbf{x^2} - ($______________$)$

$h(x) =$ ______________________

b) Generate a table of values using a spreadsheet. Enter four column headings into a spreadsheet: x, $f(x)$, $g(x)$, and $h(x)$. Enter the integer values −4 to 4 into the first column. Enter the formula for x^2 into the second column and copy down to fill in the values. Enter the formula for $4x - 4$ into the third column and copy down. Enter the formula for $x^2 - 4x + 4$ into the fourth column and copy down. Fill in the values in the table below.

Use the spreadsheet graphing function to generate the three graphs. Sketch and label them below.

x	$f(x)$	$g(x)$	$h(x)$
−4			
−3			
−2			
−1			
0			
1			
2			
3			
4			



c) The function $f(x) = x^2$ has domain $\{x \mid$ ______________ $\}$.

The function $g(x) = 4x - 4$ has domain $\{x \mid$ ______________ $\}$.

Therefore, the function $h(x) = x^2 - 4x + 4$ has domain $\{x \mid$ ______________ $\}$.

The function $f(x) = x^2$ has range $\{y \mid$ ______________ $\}$.

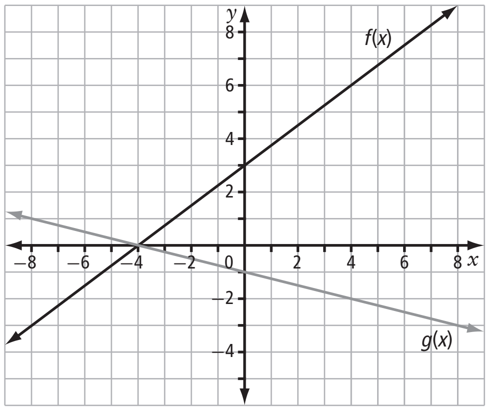
The function $g(x) = 4x - 4$ has range $\{y \mid$ ______________ $\}$.

Therefore, the function $h(x) = x^2 - 4x + 4$ has range $\{y \mid$ ______________ $\}$.

This example should help you complete #2 on page 483 of *Pre-Calculus 12*.

Working Example 3: Determine a Combined Function From Graphs

Use the graphs of $f(x)$ and $g(x)$ to sketch the graph of $h(x) = (f + g)(x)$.



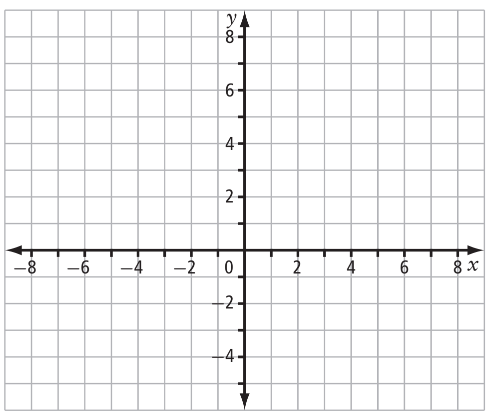
Solution

Method 1: Add the *y*-Coordinates of Corresponding Points

Complete the second and third columns of the table below using the graphs of $f(x)$ and $g(x)$. Add the y-values of $f(x)$ and $g(x)$ at each x-value to determine the points on the graph of $h(x)$.

x	$f(x)$	$g(x)$	$h(x) = (f + g)(x)$
−8			
−4			
0			
4			
8			

Plot the points for $h(x)$ and draw the graph of $h(x)$ on the grid below.



Method 2: Determine the Equations

For the graph of $f(x)$, the y-intercept is ________ and the slope is ________.

Therefore, the equation is $f(x) =$ ________________.

For the graph of $g(x)$, the y-intercept is ________ and the slope is ________.

Therefore, the equation is $g(x) =$ ________________.

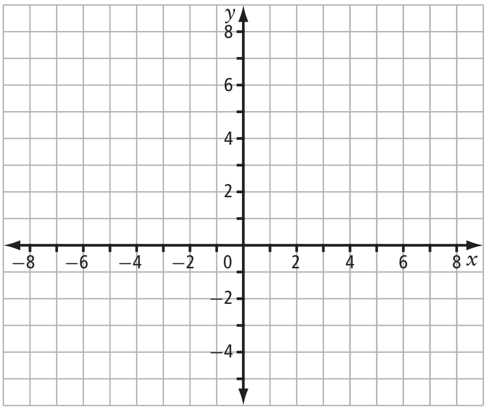
Determine the equation of $h(x)$ algebraically.

$h(x) = f(x) + g(x)$

$h(x) =$ ________________ + ________________

$h(x) =$ ________________

Graph $h(x) = \frac{1}{2}x + 2$ on the grid.



Verify by graphing $f(x)$, $g(x)$, and $h(x)$ on a graphing calculator.

This example should help you complete #7 and #8 on page 484 of *Pre-Calculus 12*.

Working Example 4: Application of the Differences of Two Functions

The yearbook committee prepares the annual book to sell to students. The fixed cost is \$800 and the printing cost for each yearbook is \$15. The committee plans to sell the yearbooks for \$20 each.

a) Write an equation to represent

- the total cost, C, as a function of the number, n, of yearbooks printed
- the revenue, R, as a function of the number, n, of yearbooks sold

b) Graph $C(n)$ and $R(n)$ on the same set of axes. What does the point of intersection represent?

c) Profit, P, is the difference between revenue and cost. Write a function representing $P(n)$. How many yearbooks need to be sold before the committee can start making a profit?

d) What is the domain of $C(n)$, $R(n)$, and $P(n)$?

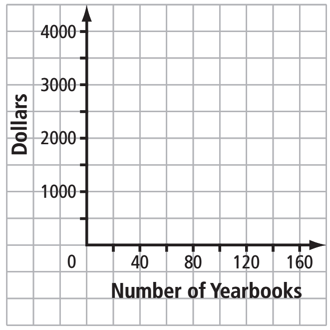
Solution

a) The total cost of printing the yearbooks can be represented by the function

$C(n) = 15n +$ ________.

The revenue can be represented by the function $R(n) =$ ________n.

b) Graph and label $C(n) = 15n + 800$ and $R(n) = 20n$.



The point of intersection of the graphs of $C(n)$ and $R(n)$ is (________, ________). The point of intersection represents the point at which total cost equals the ______________. This is also called the break-________ point. When 160 yearbooks are sold, the revenue is $________.

c) $P(n) = R(n) - C(n)$

$P(n) = \mathbf{20n} -$ ______________

$P(n) =$ ______________

The committee will start making a profit after ________ yearbooks have been sold.

d) The domain for $C(n)$, $R(n)$, and $P(n)$ is $\{n \mid n \geq$ ________, $n \in W\}$.

The range of $C(n)$ is $\{C \mid C$ ______________$\}$.

The range of $R(n)$ is $\{R \mid R$ ______________$\}$.

The range of $P(n)$ is ____________________.

Check Your Understanding

Practise

1. Complete the table.

x	$f(x)$	$g(x)$	$(f + g)(x)$	$(f - g)(x)$
−6	2			−2
−4	4	2		
−2	6		6	
0		−2	6	
2		−4		14
4	12		6	
6		−8		22

2. Let $f(x) = \{(-6, 4), (-4, 2), (-2, 0), (0, -2)\}$ and $g(x) = \{(-2, 9), (-6, 3), (-4, 6), (0, 12)\}$. Determine each of the following.

a) $(f + g)(x) =$

b) $(f - g)(x) =$

3. Let $f(x) = 3x - 2$ and $g(x) = 2x + 1$. Determine the equation of each combined function.

a) $(f + g)(x) =$

b) $(f - g)(x) =$

c) $(g - f)(x) =$

4. Let $f(x) = x^2 - 3x$ and $g(x) = |2x|$. Determine the value of each combined function.

a) $(f + g)(-4) =$

b) $(f - g)(6) =$

c) $(g - f)(1) =$

5. Let $f(x) = 5x^2$ and $g(x) = \sqrt{x^2 - 4}$. Determine each combined function and state its domain.

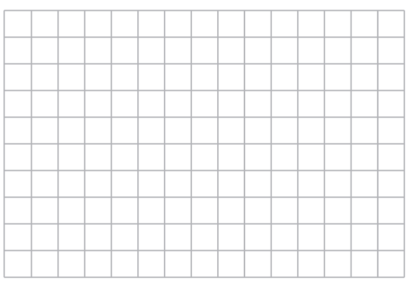
a) $y = (f + g)(x)$ Domain:

b) $y = (f - g)(x)$ Domain:

c) $y = (g - f)(x)$ Domain:

6. Let $f(x) = \sqrt{x}$ and $g(x) = \sin x$. Use graphing technology to graph the following combined functions. Sketch the graphs on the grids provided. State the domain and range of each function.

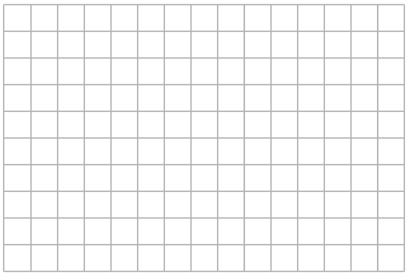
a) $y = (f + g)(x)$



Domain:

Range:

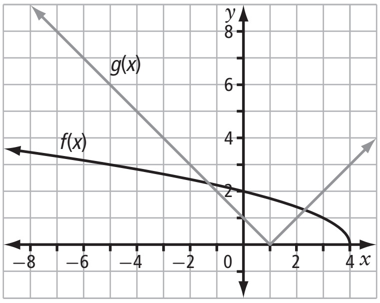
b) $y = (f - g)(x)$



Domain:

Range:

7. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.



a) $(f + g)(-5) =$

b) $(f - g)(3) =$

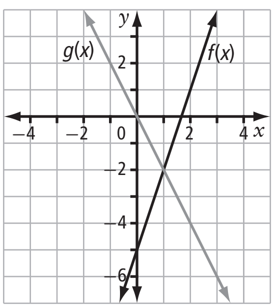
c) $(g - f)(4) =$

d) $(f + g)(0) =$

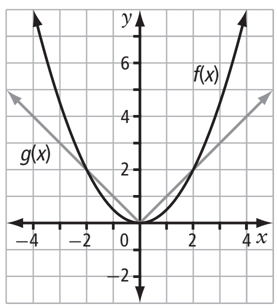
e) Explain why $(f + g)(5)$ cannot be evaluated.

8. Add a sketch of the combined function indicated to each graph.

a) $y = (f + g)(x)$



b) $y = (f - g)(x)$



Apply

9. If $f(x) = 2x + 3$, $g(x) = -x^2 + 5$, and $h(x) = -x$, determine each combined function.

a) $y = f(x) + g(x) + h(x)$

b) $y = f(x) - g(x) - h(x)$

c) $y = h(x) + f(x) - g(x)$

10. If $h(x) = (f + g)(x)$ and $f(x) = 4x - 7$, determine $g(x)$.

a) $h(x) = x^2 - 2x$

b) $h(x) = 4 - x$

c) $h(x) = -2x^2 + x + 1$

Questions 9 and 10 should help you complete #9, #10, and #11 on page 484 of *Pre-Calculus 12*.

11. A communications company manufactures a mobile phone for \$25 per unit plus a fixed operating cost of \$45 000. The mobile phones are sold for \$100 per unit.

a) Determine a function to represent the cost, C, of producing n units.

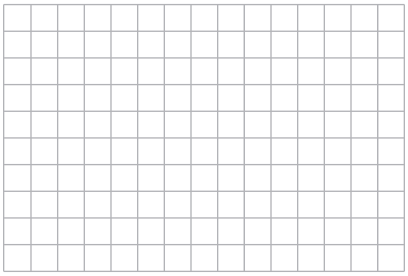
b) Determine a function to represent the revenue, R, from sales of n units.

c) Determine a function to represent the profit, P, from sales of n units.

d) What is the break-even point for the manufacturer of these cell phones?

12. A boat on still water generates a wave modelled by $f(x) = 2 \sin x$. A gust of wind generates a wave modelled by $g(x) = \frac{1}{2} \sin x$, where x is measured in radians.

a) Graph $f(x)$ and $g(x)$.



b) Sketch the graph of $h(x) = (f + g)(x)$ on the same grid as you used in part a).

c) What is the maximum height of the combined waves?

Connect

13. Create two functions, $f(x)$ and $g(x)$, to show that if the domains of $f(x)$ and $g(x)$ have no values in common, it is not possible to add or subtract the functions. Explain why this is.

$f(x) =$ ______________________________ $g(x) =$ ______________________________

14. Is it possible for two different functions, $f(x)$ and $g(x)$, to exist where $(f - g)(x) = (g - f)(x)$? If not, explain why not. If it is possible, give an example of $f(x)$ and $g(x)$.

10.2 Products and Quotients of Functions

KEY IDEAS

- New functions can be formed by performing the operations of multiplication and division with functions.

Product of Functions

$h(x) = f(x) \cdot g(x)$

or

$h(x) = (f \cdot g)(x)$

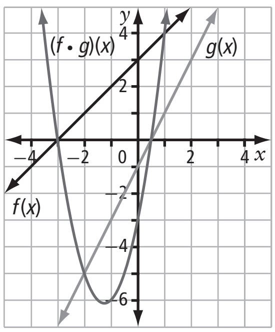
Example

$f(x) = x + 3$ and $g(x) = 2x - 1$

$h(x) = f(x) \cdot g(x)$

$h(x) = \mathbf{(x + 3)(2x - 1)}$

$h(x) = 2x^2 + 5x - 3$



Quotient of Functions

$h(x) = \dfrac{f(x)}{g(x)}$, where $g(x) \neq 0$

or

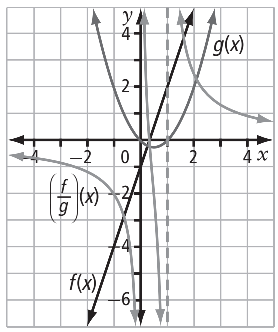
$h(x) = \left(\dfrac{f}{g}\right)(x)$, where $g(x) \neq 0$

Example

$f(x) = 3x - 1$ and $g(x) = x^2 - x$

$h(x) = \dfrac{f(x)}{g(x)}$

$h(x) = \dfrac{\mathbf{3x - 1}}{\mathbf{x^2 - x}}$, where $x \neq 1, 0$



- The domain of a product or a quotient of functions is the domain common to the original functions. The domain of a quotient of functions must have the restriction that the divisor cannot equal zero. That is, for $h(x) = \dfrac{f(x)}{g(x)}$, the values of x are such that $g(x) \neq 0$.
- The range of a combined function can be determined using its graph.

Working Example 1: Determine the Product of Functions

Let $f(x) = x - 3$ and $g(x) = x^2 - 8x + 15$.

a) Determine $h(x) = (f \cdot g)(x)$.

b) State the domain and range of $h(x)$.

Solution

a) $h(x) = (f \cdot g)(x)$

$h(x) = f(x) \cdot g(x)$

$h(x) = (\mathbf{x - 3})(\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)$

$h(x) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

b) The function $f(x) = x - 3$ is a ____________________ function with domain
(*linear* or *quadratic*)

$\{x \mid x$ ________________$\}$.

The function $g(x) = x^2 - 8x + 15$ is a ____________________ function with domain
(*linear* or *quadratic*)

$\{x \mid$ ____________________$\}$.

Therefore, the cubic function $h(x) = x^3 - 11x^2 + 39x - 45$ has domain

____________________ and range ____________________.

Verify by using graphing technology.

Working Example 2: Determine the Quotient of Functions

Let $f(x) = x - 3$ and $g(x) = x^2 - 8x + 15$.

a) Determine the equation of $h(x) = \left(\frac{f}{g}\right)(x)$.

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain and range of $h(x)$.

Solution

a) $h(x) = \left(\frac{f}{g}\right)(x)$

$h(x) = \frac{f(x)}{g(x)}$

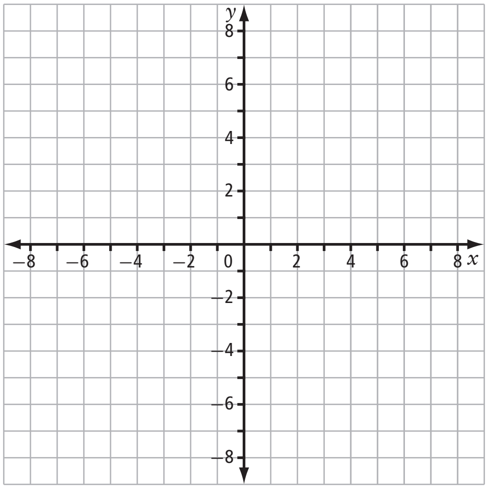
$h(x) = \frac{\mathbf{x - 3}}{\boxed{}}$

$h(x) = \frac{x - 3}{(\boxed{})(x - 3)}$

$h(x) = \frac{1}{\boxed{}}$, $x \neq$ ________, ________

Why are there two non-permissible values?

b) Use graphing technology to graph $f(x)$, $g(x)$, and $h(x)$.
Sketch and label the graphs on the grid below.



c) The function $f(x) = x - 3$ is a ________________ function with domain
(*linear* or *quadratic*)
$\{x \mid$ ________________ $\}$.

The function $g(x) = x^2 - 8x + 15$ is a ________________ function with domain
(*linear* or *quadratic*)
$\{x \mid$ ________________ $\}$.

The domain of $h(x) = \left(\frac{f}{g}\right)(x)$ consists of all values in both the domain of $f(x)$ and $g(x)$ excluding the values for which $g(x) = 0$. Since $g(x)$ does not exist at (________, −0.5) and is undefined at $x =$ ________, the domain is
$\{x \mid x \neq$ ________, ________, $x \in \mathrm{R}\}$.

How are the non-permissible values shown on the graph?

The range of $h(x)$ is $\{y \mid y \neq$ ________, ________, $y \in \mathrm{R}\}$.

This example should help you complete #1, #2, #4, and #6 on page 496 of *Pre-Calculus 12*.

Working Example 3: Application of Products and Quotients of Functions

An electronics store is doing a one-day promotion. Every hour, starting at 9:00 a.m., the price of a 3-D television is reduced by 5%. The price, P, of the television at t hours after 8:00 a.m. can be modelled by the function $P(t) = 1020 - 51t$. The number, N, of televisions sold at time t can be modelled by the function $N(t) = 6t$.

a) Write an expression for the revenue, R, for televisions sold t hours after 8:00 a.m.

b) What is the revenue from 3-D television sales at 12:00 p.m.?

c) The store's cost for each television is \$705. Write an expression for the percent gain or loss in profit at t hours.

d) What is the percent gain or loss at 12:00 p.m.?

Solution

a) The revenue is the product of the number of televisions sold and the

__________________________.

$R(t) = N(t) \cdot P(t)$

$R(t) = (\_\_\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_\_\_)$

$R(t) = \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_$

b) To determine the revenue at 12:00 p.m., replace t in the function $R(t) = 6120t - 306t^2$ with ________.

$R(t) = 6120(\_\_\_\_\_\_) - 306(\_\_\_\_\_\_)^2$

$R(t) = \_\_\_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_\_\_$

$R(t) = \_\_\_\_\_\_\_\_\_\_$

The revenue at 12:00 p.m. is ________________.

c) To determine the percent gain or loss, first determine the function for the cost, $C(t)$.

The cost of $6t$ televisions is 705(________).

$C(t) = \_\_\_\_\_\_\_\_\_\_$

The percent gain or loss is the quotient of revenue and cost multiplied by 100. Then, subtract 100 from this total.

$$\text{Percent gain or loss} = \frac{R(t)}{C(t)}(100) - 100$$

$$\text{Percent gain or loss} = \frac{\mathbf{6120}\boldsymbol{t} - \mathbf{306}\boldsymbol{t}^2}{\mathbf{4230}\boldsymbol{t}}(100) - 100$$

Percent gain or loss = ________________________ − 100

Percent gain or loss = ________________________

d) To determine the percent gain or loss at 12:00 p.m., replace t in the Percent-gain-or-loss function from part c) with ________.

The percent gain at 12:00 p.m. is ________________.

See pages 497–498 of *Pre-Calculus 12* for more examples.

Check Your Understanding

Practise

1. Determine $h(x) = f(x) \cdot g(x)$ and $k(x) = \frac{f(x)}{g(x)}$ for each pair of functions.

a) $f(x) = 2x - 1$ and $g(x) = 2x - 3$

$h(x) =$ $k(x) =$

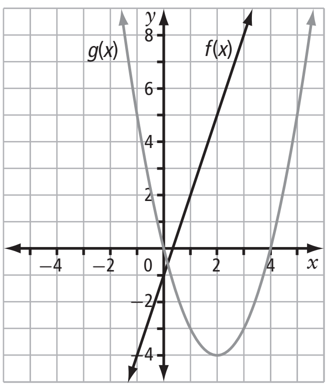
b) $f(x) = 3x$ and $g(x) = -x^2 - 4$

$h(x) =$ $k(x) =$

c) $f(x) = \sqrt{4 - x}$ and $g(x) = 5x - 1$

$h(x) =$ $k(x) =$

2. Use the graphs of $f(x)$ and $g(x)$ to evaluate the following.



a) $(f \cdot g)(-1) =$

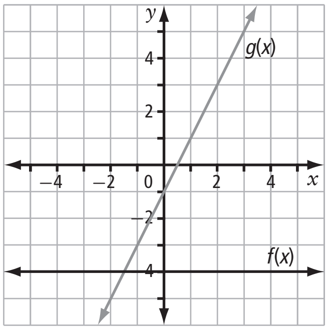
b) $\left(\frac{f}{g}\right)(1) =$

c) $\left(\frac{g}{f}\right)(0) =$

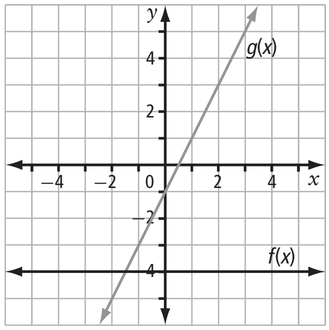
d) $(g \cdot f)(3) =$

3. Add the sketch of each combined function to the set of axes shown.

a) $h(x) = f(x) \cdot g(x)$



b) $k(x) = \dfrac{f(x)}{g(x)}$

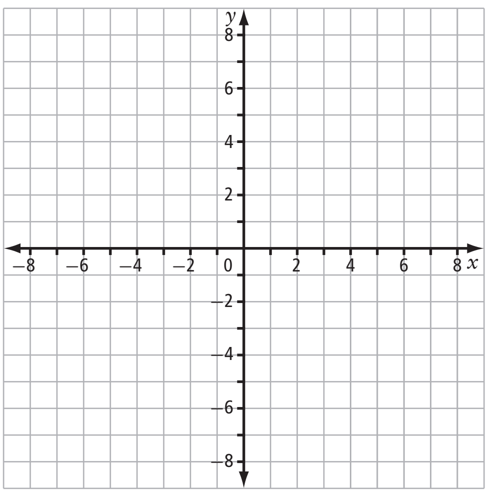


4. Consider the functions $f(x) = 2x - 1$ and $g(x) = (x + 1)^2$.

a) Determine $h(x) = f(x) \cdot g(x)$.

b) Determine $k(x) = \dfrac{f(x)}{g(x)}$.

c) Use graphing technology to graph $f(x)$, $g(x)$, $h(x)$, and $k(x)$. Sketch the graph on the grid below.



d) Determine the domain and range of each combined function.

$h(x)$: Domain: Range:

$k(x)$: Domain: Range:

Apply

5. Evaluate each combined function using $f(x) = (x^2 + 1)$ and $g(x) = -2x$.

a) $(f \cdot g)(5) =$

b) $\left(\frac{f}{g}\right)(-2) =$

c) $\left(\frac{g}{f}\right)(2) =$

6. Given $f(x) = x^2$, $g(x) = 4x - 5$, and $h(x) = \sqrt{x - 1}$, determine each combined function. State any restrictions on x.

a) $y = f(x) \cdot g(x) \cdot h(x)$

b) $y = \dfrac{g(x) \cdot h(x)}{f(x)}$

7. If $h(x) = f(x) \cdot g(x)$ and $f(x) = 2 - x$, determine $g(x)$. State any restrictions on x.

a) $h(x) = x^2 - 2x$

b) $h(x) = \sqrt{x^2 - 4}$

8. If $h(x) = \dfrac{f(x)}{g(x)}$ and $g(x) = -2x$, determine $f(x)$. State any restrictions on x.

a) $h(x) = x^2 + 6x$

b) $h(x) = -x - 3$

Questions 7 and 8 are similar to #7 and #8 on page 496 of *Pre-Calculus 12*.

9. Let $f(x) = 1$ and $g(x) = \sin x$.

a) Write the equation of $h(x)$ as a quotient of functions that is equivalent to $\csc x$.

b) Use graphing technology to graph $f(x)$, $g(x)$, and $h(x)$ on the same axes. Sketch the graphs on the grid below. State any restrictions on x for $h(x)$.

10. Consider $f(x) = \cos x$ and $g(x) = \sqrt{x}$.

a) Use graphing technology to graph $f(x)$, $g(x)$, and $h(x) = f(x) \cdot g(x)$ on the same axes. Sketch the graphs on the grid below.

b) State the domain and range of $h(x)$.

11. An agricultural company calculated that, during harvesting season, the total hours, T, worked per week was the number of hours worked, H, multiplied by the number of employees, E. The function $H(w) = 10 + 0.4w$ represents the hours worked per week and the function $E(w) = 30 - 2w$ represents the number of employees, where w is the number of weeks from the start of harvesting.

a) Represent $T(w)$ as a combined function.

b) What is the total number of hours worked per week after 6 weeks of harvesting?

c) During which week number will the harvesting end? Explain how you determined your answer.

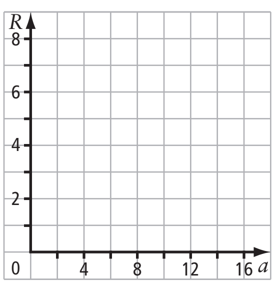
d) State the domain and range of $T(w)$.

12. The total weight, W, in tonnes, of live fish in a fish farm is given by the function $W(a) = 12 + 1.41a$, where a represents the age of the fish, in weeks. The function $F(a) = a^2 + 0.25a$ represents the total weight, in tonnes, of food the fish have been fed when the fish are a weeks old.

a) Write the combined function $R(a)$ that represents the ratio of total weight of food to total weight of fish.

b) What is the total weight of the fish at 15 weeks? How much food has been fed at that age?

c) Sketch the graph of $R(a)$ on the grid below. State the domain and range of $R(a)$.

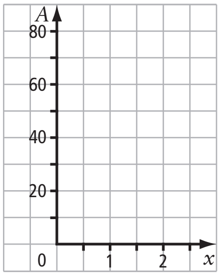


Connect

13. The volume, V, of a cylinder, in cubic inches, is given by $V(x) = \pi(25x^3 - 12.5x^2)$.

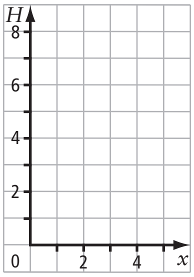
a) Write a function, $A(x)$, to represent the area of the base when the radius is $2.5x$.

b) Graph $A(x)$. State its domain and range in this context.



c) Determine the combined function $H(x)$ to represent the height of the cylinder when the area of the base, $A(x)$, is the function that you wrote in part a).

d) Graph $H(x)$. State its domain and range in this context.

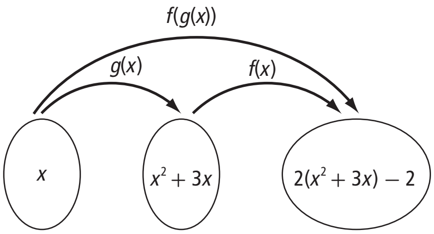


10.3 Composite Functions

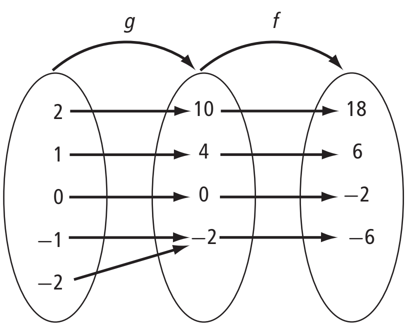
KEY IDEAS

- Composite functions are functions that are formed from two functions, $f(x)$ and $g(x)$, in which the output of one of the functions is used as the input for the other function.
  – $f(g(x))$ is read as "f of g of x"
  – $(f \circ g)(x)$ is another way of writing $f(g(x))$ and is read the same way

For example, if $f(x) = 2x - 2$ and $g(x) = x^2 + 3x$, then $f(g(x))$ is shown in the mapping diagram.



The output for $g(x)$ is the input for $f(x)$.



To determine the equation of a composite function, substitute the second function into the first. To determine $f(g(x))$,

$f(g(x)) = f(\mathbf{x^2 + 3x})$	Substitute $x^2 + 3x$ for $g(x)$.
$f(g(x)) = 2(\mathbf{x^2 + 3x}) - 2$	Substitute $x^2 + 3x$ into $f(x) = 2x - 2$.
$f(g(x)) = 2x^2 + 6x - 2$	Simplify.

To determine $g(f(x))$,

$g(f(x)) = g(\mathbf{2x - 2})$	Substitute $2x - 2$ for $f(x)$.
$g(f(x)) = (\mathbf{2x - 2})^2 + 3(\mathbf{2x - 2})$	Substitute $2x - 2$ into $g(x) = x^2 + 3x$.
$g(f(x)) = 4x^2 - 8x + 4 + 6x - 6$	Simplify.
$g(f(x)) = 4x^2 - 2x - 2$	

Note that $f(g(x)) \neq g(f(x))$.

- The domain of $f(g(x))$ is the set of all values of x in the domain of g for which $g(x)$ is the domain of f. Restrictions must be considered.

Working Example 1: Evaluate a Composite Function

If $f(x) = 4 - x$ and $g(x) = x^2 + x$, determine each value.

a) $f(g(2))$ **b)** $g(f(-3))$ **c)** $g(g(1))$

Solution

a) Method 1: Determine the Value of the Inner Function and Then Substitute

Evaluate the function inside the brackets for $x = 2$. Then, substitute this value into the outer function.

$g(x) = x^2 + x$

$g(2) =$ ________________

$g(2) =$ ________________

Substitute the value of $g(2)$ into $f(x)$.

$f(g(2)) = f(6)$

$f(g(2)) = 4 -$ ________ Substitute 6 for $g(2)$.

$f(g(2)) =$ ________ Evaluate $f(x) = 4 - x$ when x is 6.

Therefore, $f(g(2)) =$ ________.

Method 2: Determine the Composite Function and Then Substitute

$f(g(x)) = f(\mathbf{x^2 + x})$ Substitute $x^2 + x$ for $g(x)$.

$f(g(x)) = 4 - ($________$)$ Substitute $x^2 + x$ into $f(x) = 4 - x$.

$f(g(x)) =$ ___________________________

Substitute $x = 2$ into $f(g(x))$.

$f(g(2)) =$ ________ $-$ ________^{2} $-$ ________

$f(g(2)) =$ ________

Therefore, $f(g(2)) =$ ________.

b) Determine $f(-3)$ and then $g(f(-3))$.

$f(x) = 4 - x$

$f(-3) =$ ___________________________

Substitute $f(-3) = 7$ into $g(x)$.

$g(f(-3)) = g(\mathbf{7})$

$g(f(-3)) =$ ___________________________

Therefore, $g(f(-3)) =$ ________.

c) Determine $g(g(x))$ and then evaluate.

$g(g(x)) = g(\mathbf{x^2 + x})$ Substitute $x^2 + x$ for $g(x)$.

$g(g(x)) = (\_\_\_\_\_\_\_\_)^2 + (\_\_\_\_\_\_\_\_)$ Substitute $x^2 + x$ into $g(x) = x^2 + x$.

$g(g(x)) = \_\_\_\_\_\_\_\_\_\_\_\_$ Expand.

$g(g(x)) = \_\_\_\_\_\_\_\_\_\_\_\_$ Simplify.

Substitute $x = 1$ into $g(g(x))$.

$g(g(1)) = \_\_\_\_^4 + 2(\_\_\_\_)^3 + 2(\_\_\_\_)^2 + \_\_\_\_$

$g(g(1)) = \_\_\_\_\_\_\_\_\_\_\_\_$

Therefore, $g(g(1)) = \_\_\_\_$.

This example should help you complete #1 and #3 on page 507 of *Pre-Calculus 12*.

Working Example 2: Compose Functions With Restrictions

If $f(x) = x - 3$ and $g(x) = \sqrt{x}$, determine the following.

a) $(f \circ g)(x)$

b) $(g \circ f)(x)$

c) Does order matter when composing functions?

d) State the domain of $f(x)$, $g(x)$, $(f \circ g)(x)$, and $(g \circ f)(x)$.

Solution

a) Determine $(f \circ g)(x)$.

$(f \circ g)(x) = f(\sqrt{x})$ Substitute $\sqrt{x}$ for $g(x)$.

$(f \circ g)(x) = \_\_\_\_\_\_ - 3$ Substitute $\sqrt{x}$ into $f(x) = x - 3$.

Therefore, $(f \circ g)(x) = \_\_\_\_\_\_\_\_$.

b) Determine $(g \circ f)(x)$.

$(g \circ f)(x) = g(\_\_\_\_\_\_\_\_)$ Substitute $x - 3$ for $f(x)$.

$(g \circ f)(x) = \sqrt{\square}$ Substitute $x - 3$ into $g(x) = \sqrt{x}$.

Therefore, $(g \circ f)(x) = \_\_\_\_\_\_\_\_$.

c) Order ________ (*does* or *does not*) matter when composing functions because

$\sqrt{x} - 3$ ______ (= or ≠) $\sqrt{x - 3}$.

d) The domain of $f(x)$ is $\{x \mid x \in \mathrm{R}\}$.

The domain of $g(x)$ is $\{x \mid x$ ________________, $x \in \mathrm{R}\}$.

The domain of $(f \circ g)(x)$ is a combination of the restrictions for which $g(x)$ is the domain of $f(x)$. Therefore, the domain is $\{x \mid x$ ________________, $x \in \mathrm{R}\}$.

The domain of $(g \circ f)(x)$ is a combination of the restrictions for which $f(x)$ is the domain of $g(x)$. Therefore, the domain is $\{x \mid x$ ________________, $x \in \mathrm{R}\}$.

This example should help you complete #6 on page 507 of *Pre-Calculus 12*.

Working Example 3: Determine the Composition of Two Functions

Let $f(x) = x^2 + 1$ and $g(x) = 2x$. Determine the equation of each composite function, graph it, and state its domain and range.

a) $y = f(g(x))$ **b)** $y = g(f(x))$ **c)** $y = f(f(x))$ **d)** $y = g(g(x))$

Solution

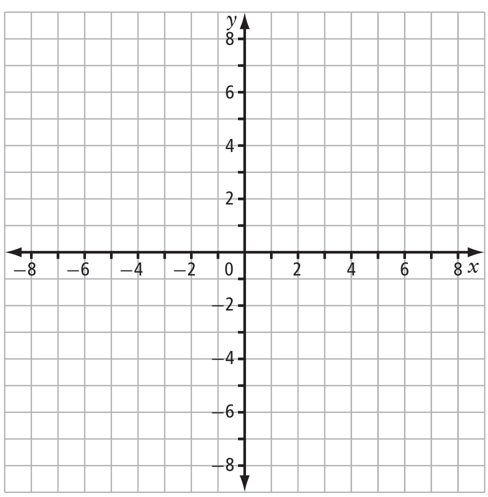
a) Determine $f(g(x))$.

$f(g(x)) = f(\mathbf{2x})$

$f(g(x)) = ($________$)^2 + 1$

$f(g(x)) =$ ________________

Sketch the graph of $f(g(x))$ on the grid below.



The domain is $\{x \mid x$ ________________$\}$.
The range is
$\{y \mid y$ ____________________________$\}$.

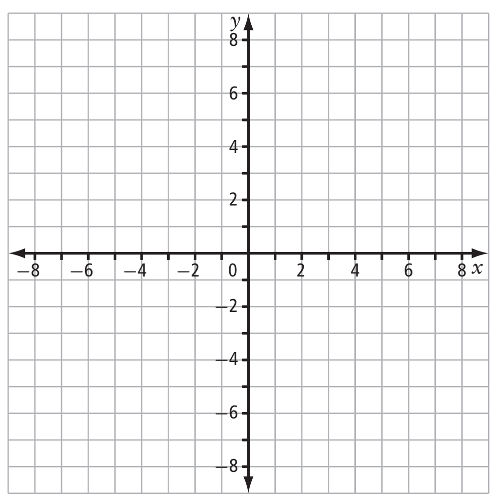
b) Determine $g(f(x))$.

$g(f(x)) = g(\mathbf{x^2 + 1})$

$g(f(x)) = 2($________________$)$

$g(f(x)) =$ ____________________________

Sketch the graph of $g(f(x))$ on the grid below.



The domain is $\{x \mid x$ ________________$\}$.
The range is
$\{y \mid y$ ____________________________$\}$.

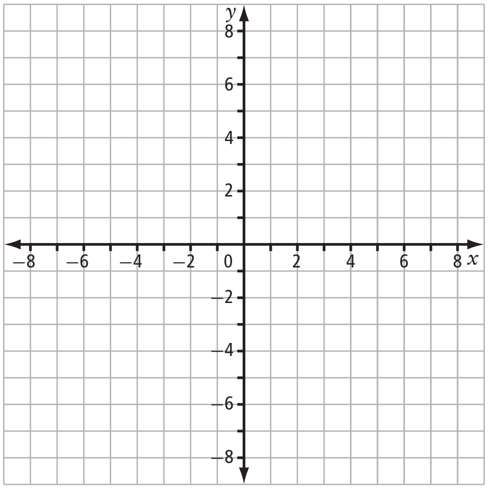
c) Determine $f(f(x))$.

$f(f(x)) = f(\mathbf{x^2 + 1})$

$f(f(x)) = (\_\_\_\_\_\_\_\_\_\_\_\_)^2 + 1$

$f(f(x)) =$ ____________________

Sketch the graph of $f(f(x))$ on the grid below.



The domain is $\{x \mid x$ ____________$\}$. The range is $\{y \mid y$ __________________$\}$.

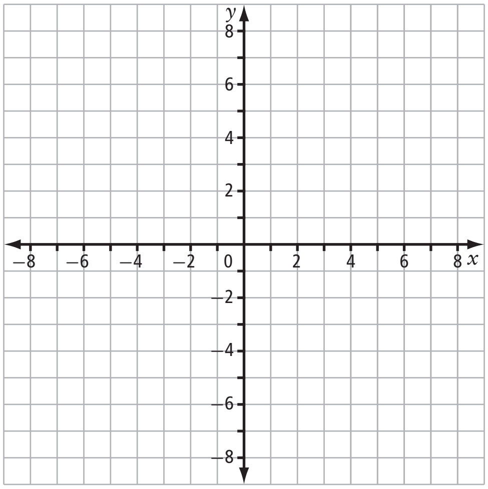
d) Determine $g(g(x))$.

$g(g(x)) = g(\mathbf{2x})$

$g(g(x)) =$ ____________

$g(g(x)) =$ ____________

Sketch the graph of $g(g(x))$ on the grid below.



The domain is $\{x \mid x$ ____________$\}$. The range is $\{y \mid y$ __________________$\}$.

Working Example 4: Determine the Original Functions From a Composition

Determine two functions, $f(x)$ and $g(x)$, where $h(x) = f(g(x))$.

a) $h(x) = \sqrt{3x - 2}$

b) $h(x) = (x - 4)^2 + 3(x - 4) + 4$

c) $h(x) = x^2 - 6x + 9$

Solution

a) Let $g(x) = 3x - 2$. Then, work backward to determine $f(x)$.

$h(x) = \sqrt{3x - 2}$

$f(g(x)) = \sqrt{\boxed{}}$ Replace $3x - 2$ with $g(x)$.

Therefore, $f(x) = \sqrt{x}$.

The two functions are $f(x) =$ ____________ and $g(x) =$ ____________.

b) Look for a function that may be common to more than one term in $h(x)$. The expression $(x - 4)$ occurs in two terms.

Let $g(x) = x - 4$. Then, work backward to determine $f(x)$.

$h(x) = (x - 4)^2 + 3(x - 4) + 4$

$f(g(x)) = (\_\_\_\_\_\_\_\_\_\_\_\_)^2 + 3(\_\_\_\_\_\_\_\_\_\_\_\_) + 4$ Replace $x - 4$ with $g(x)$.

Therefore, $f(x) =$ ____________________.

The two functions are $f(x) =$ ____________________ and

$g(x) =$ ____________________.

c) Factor $h(x) = x^2 - 6x + 9$.

$h(x) =$ ____________________

Let $g(x) = x - 3$. Then, work backward to determine $f(x)$.

$h(x) = (\_\_\_\_\_\_\_\_\_\_\_\_)^2$

$h(g(x)) = (\_\_\_\_\_\_\_\_\_\_\_\_)^2$ Replace $x - 3$ with $g(x)$.

Therefore, $f(x) =$ ____________.

The two functions are $f(x) =$ ____________ and $g(x) =$ ____________.

This example should help you complete #7 on page 507 of *Pre-Calculus 12*.

Working Example 5: Application of Composite Functions

The temperature as you descend a mine shaft is a function of the depth below the surface. An equation expressing the relationships is $T = 0.01d + 20$, where T is the temperature, in degrees Celsius, and d is the depth, in metres.

a) If you go down a mine shaft at a rate of 4 m/s, express the temperature as a function of the time, t, in seconds.

b) What is the temperature after 1 min of travelling down the shaft?

Solution

a) Since the rate of travel is 4 m/s, the distance, in metres, travelled in t seconds is

$d =$ ____________. Since the temperature, T, is a function of depth, d, you can compose the two functions.

$T(d) = 0.01d + 20$

$T(d(t)) = 0.01(\_\_\_\_\_\_\_\_\_\_\_\_) + 20$

$T(d(t)) =$ ____________ $+ 20$

The temperature expressed as a function of time is $T(d(t)) =$ ____________.

b) To determine the temperature after 1 min of travel, substitute $t =$ ________ (time in seconds) in the composite function.

$T(d(t)) = 0.04t + 20$

$T(d(60)) = 0.04($________$) + 20$

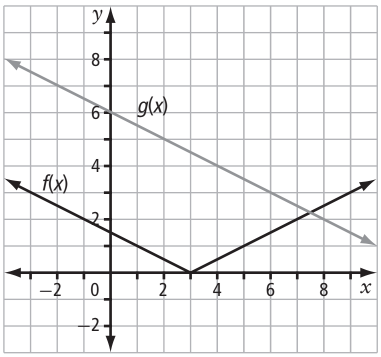
$T(d(60)) =$ ________

The temperature after one minute of travelling down the shaft is ________________.

Check Your Understanding

Practise

1. Use the graphs of f and g to evaluate the functions.



a) $f(g(6))$

b) $f(g(-2))$

c) $g(f(3))$

d) $g(f(7))$

2. If $f(x) = x + 6$ and $g(x) = -3x + 5$, determine each of the following.

a) $f(g(-4))$

b) $g(f(0))$

c) $g(g(8))$

d) $f(f(-1))$

3. Determine $(f \circ g)(x)$ and $(g \circ f)(x)$ for each pair of functions.

a) $f(x) = \sqrt{x + 4}$ and $g(x) = x^2$

$(f \circ g)(x) =$ $\qquad\qquad$ $(g \circ f)(x) =$

b) $f(x) = |x - 4|$ and $g(x) = 3 - x$

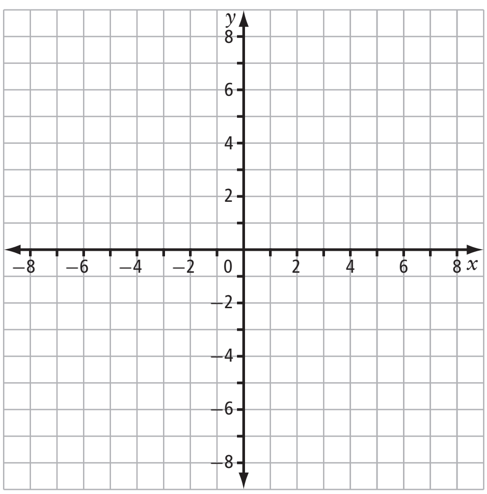
$(f \circ g)(x) =$ $\qquad\qquad$ $(g \circ f)(x) =$

c) $f(x) = \frac{1}{x}$ and $g(x) = x + 3$

$(f \circ g)(x) =$ $\qquad\qquad$ $(g \circ f)(x) =$

4. For $f(x) = x^2 - 2$ and $g(x) = 4x + 1$, sketch the graph of each composite function. Determine the domain and range of the composite function.

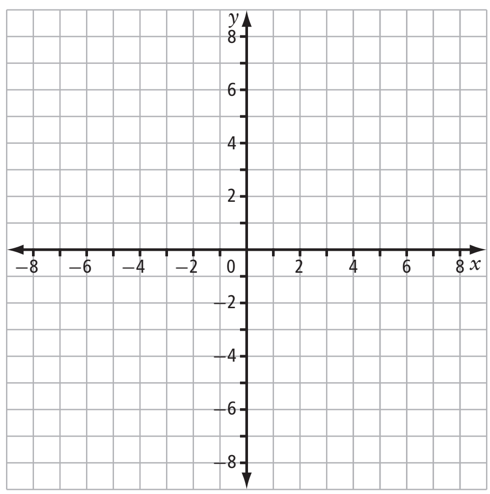
a) $y = f(g(x))$



Domain:

Range:

b) $y = g(f(x))$



Domain:

Range:

This question is similar to #6 on page 507 of *Pre-Calculus 12*.

5. If $h(x) = (f \circ g)(x)$, determine $g(x)$.

a) $h(x) = \frac{1}{(x-2)^2}$ and $f(x) = \frac{1}{x^2}$

b) $h(x) = \sqrt{x^2 - 2}$ and $f(x) = \sqrt{x}$

c) $h(x) = x^2 - 10x + 25$ and $f(x) = x^2$

6. For $f(x) = \sqrt{x - 4}$ and $g(x) = 3x + 1$, determine each composite function and state its domain.

a) $(f \circ g)(x)$

b) $(g \circ f)(x)$

Apply

7. Consider $f(x) = x + 2$ and $g(x) = \frac{x^2 + x - 2}{x - 1}$.

a) Determine $f(g(x))$.

b) Determine $g(f(x))$.

c) What restrictions are placed on the domain of each composite function?

8. Consider $h(x) = \frac{1}{x}$ and $k(x) = x^2$.

a) How are $h(k(x))$ and $k(h(x))$ related?

b) What restrictions are placed on the domain of the composite functions?

9. The cost, C, of manufacturing x engines is given by the function $C(x) = 75x + 900$. The number of engines produced in t hours is given by $x(t) = 10.5x$.

a) Determine $C(x(t))$.

b) What does $C(x(t))$ represent?

c) Determine the cost incurred after 8 h of production.

d) Determine the time that must elapse for the cost to be $33 975.

10. The suggested retail price for a snowmobile is p dollars. The manufacturer offers a $1200 factory rebate, and the dealership offers a 10% discount.

a) Write a function, R, in terms of p that gives the cost of the snowmobile after receiving the rebate from the factory.

b) Write a function, D, in terms of p that gives the cost of the snowmobile after receiving the dealership discount.

c) Determine $(R \circ D)(p)$. What does it represent?

d) Determine $(D \circ R)(p)$. What does it represent?

e) Determine $(R \circ D)(10\ 750)$ and $(D \circ R)(10\ 750)$. Which gives the lower cost for the snowmobile? Explain.

11. A sales representative for a flooring company is paid a monthly salary plus a bonus of 5% of monthly sales, x, over \$50 000. Let $f(x) = x - 50\,000$ and $g(x) = 0.05x$. If x is greater than \$50 000, which represents the sales representative's bonus: $f(g(x))$ or $g(f(x))$? Explain.

12. Determine two functions, $f(x)$ and $g(x)$, such that $(f \circ g)(x) = h(x)$.

a) $h(x) = (2x + 1)^2$

b) $h(x) = \sqrt{9 - x}$

Connect

13. Do the following to determine whether the composition of functions follows the associative property.

a) Create three linear (non-horizontal) functions, $f(x)$, $g(x)$, and $h(x)$.

b) Show whether $((f \circ g) \circ h)(x)$ is equal to $(f \circ (g \circ h))(x)$.

14. Do the following to determine whether the composition of functions follows the commutative property.

a) Create two functions, $f(x)$ and $g(x)$.

b) Show whether $(f \circ g)(x)$ is equal to $(g \circ f)(x)$. What restrictions are placed on the composite functions?

Chapter 10 Review

10.1 Sums and Differences of Functions, pages 325–334

1. Given $f(x) = 2x - 1$ and $g(x) = x^2 + 4$, determine each of the following.

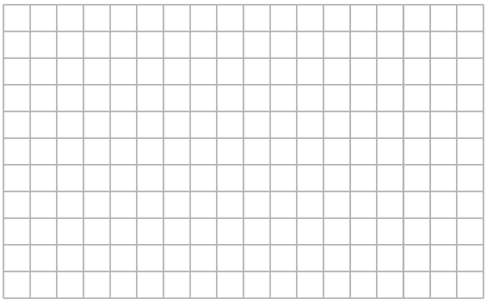
a) $(f + g)(-3) =$

b) $(f - g)(4)$

2. Let $f(x) = \sqrt{x + 6}$ and $g(x) = 4x^2 - 1$.

a) Determine $h(x) = f(x) + g(x)$.

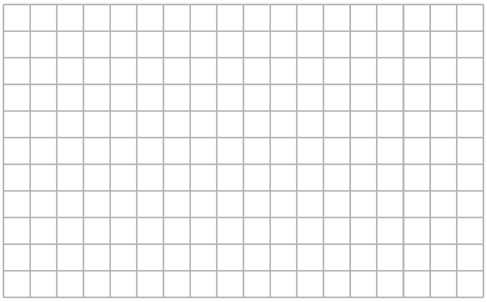
b) Use graphing technology to graph $y = h(x)$. Sketch the graph on the grid.



c) State the domain of $h(x)$. Use the graph to approximate the range of $h(x)$.

d) Determine $k(x) = f(x) - g(x)$.

e) Use graphing technology to graph $y = k(x)$. Sketch the graph on the grid.



f) State the domain of $k(x)$. Use the graph to approximate the range of $k(x)$.

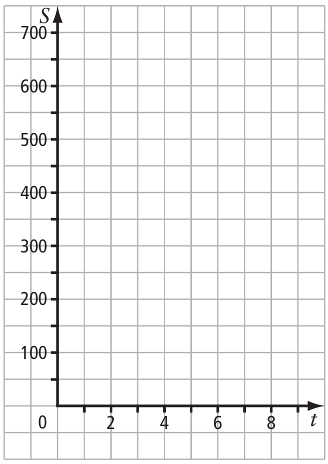
3. If $h(x) = (f - g)(x)$ and $f(x) = -x + 6$, determine $g(x)$.

a) $h(x) = 4x^2 - 12x + 9$

b) $h(x) = \sqrt{x} + x - 6$

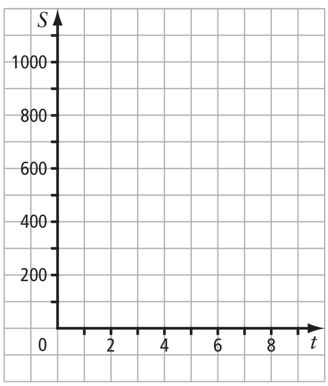
4. Extreme Sports has two store locations. Between the years 2007–2012, the sales, S_1, in thousands of dollars, at the first location decreased according to the function $S_1(t) = 750 - 0.6t^2$, where t represents the number of years after the year 2000. During the same six-year period, the sales in the second store, S_2, in thousands of dollars, increased according to the function $S_2(t) = 335 + 0.8t$, where t represents the number of years after the year 2000.

a) Graph $S_1(t)$ and $S_2(t)$ on the same set of axes.



b) Write a combined function that represents the total sales of the two stores.

c) Graph the combined function.



d) Have the total sales been increasing or decreasing? Explain.

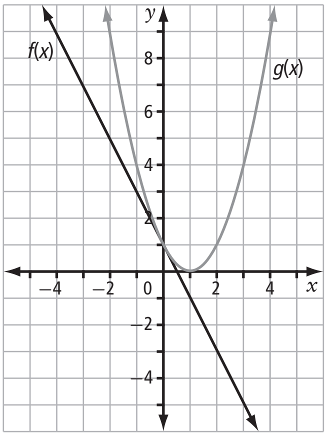
10.2 Products and Quotients of Functions, pages 335–344

5. Let $f(x) = 1 - 2x$ and $g(x) = x^2 + 3$. Determine each combined function and state any restrictions on x.

a) $h(x) = f(x) \cdot g(x)$

b) $k(x) = \dfrac{g(x)}{f(x)}$

6. Use the graphs of $f(x)$ and $g(x)$ to determine the following.



a) $(f \cdot g)(0)$

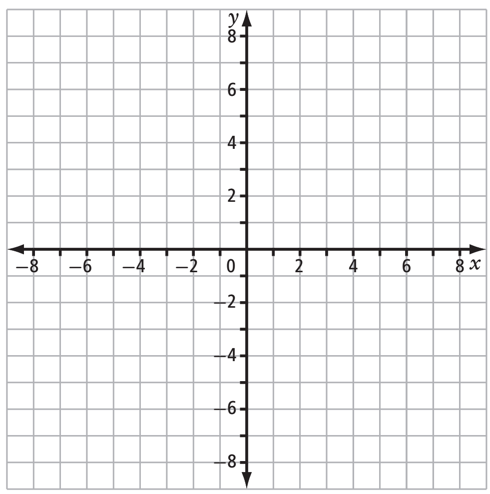
b) $(f \cdot g)(-1)$

c) $\left(\dfrac{f}{g}\right)(2)$

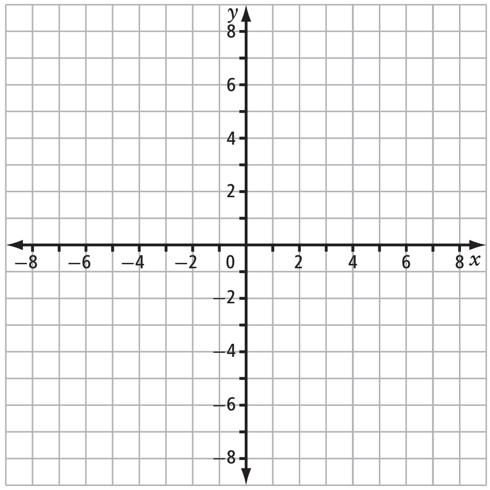
d) $\left(\dfrac{f}{g}\right)(-2)$

7. Consider $f(x) = \dfrac{1}{x - 1}$ and $g(x) = x$.

a) Determine $h(x) = (f \cdot g)(x)$. Then, sketch the graph of $y = h(x)$ and state its domain.



b) Determine $k(x) = \left(\frac{f}{g}\right)(x)$. Then, sketch the graph of $y = k(x)$ and state its domain.



8. If $h(x) = f(x) \cdot g(x)$ and $f(x) = 2x - 3$, determine $g(x)$.

a) $h(x) = 2x^2 - 5x + 3$

b) $h(x) = 2x(\sin x) - 3(\sin x)$

c) $h(x) = -2x^3 + 3x^2$

9. Let $f(x) = \sin x$ and $g(x) = \cos x$.

a) Sketch the graphs of $f(x)$ and $g(x)$.

b) Sketch the graph of $y = \frac{g(x)}{f(x)}$.

c) State the domain and range of the combined function.

d) Use your knowledge of trigonometric identities to state the equation of the function $y = \frac{g(x)}{f(x)}$ as a single trigonometric function.

10.3 Composite Functions, pages 345–355

10. Let $f(x) = x - 3$ and $g(x) = 1 - x^2$. Determine each of the following.

a) $(f \circ g)(x)$

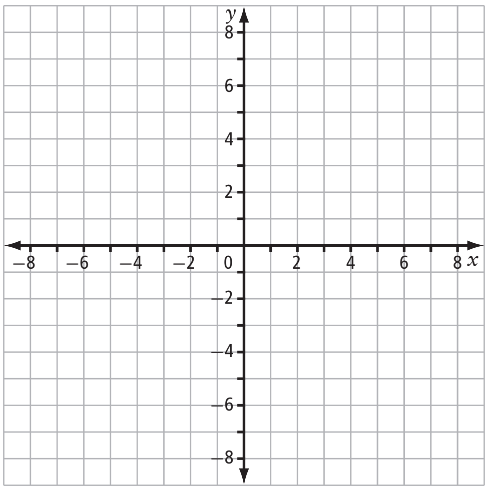
b) $(g \circ g)(x)$

c) $(f \circ g)(-3)$

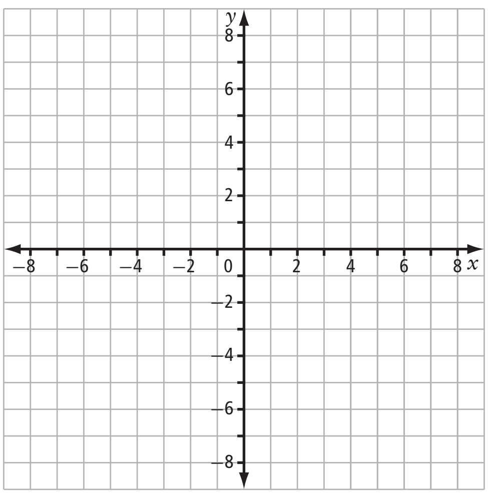
d) $(g \circ g)(2)$

11. Let $f(x) = x^2 - 9$ and $g(x) = \sqrt{x}$.

a) Sketch the graph of $y = f(g(x))$ and state its domain and range.



b) Sketch the graph of $y = g(f(x))$ and state its domain and range.



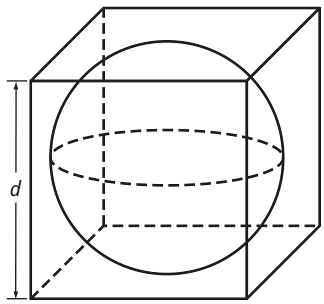
12. Given that $h(x) = (f \circ g)(x)$, determine $g(x)$.

a) $h(x) = \sqrt{9 - x}$ and $f(x) = \sqrt{x}$

b) $h(x) = \frac{12}{(7x - 2)^2}$ and $f(x) = \frac{12}{x^2}$

c) $h(x) = 4x^2 - 20x + 25$ and $f(x) = x^2$

13. The side length, d, of a cube that contains a sphere depends on the radius, r, of the sphere. Assume that the faces of the cube are tangent to the sphere.



a) Write the side length of the cube as a function of the radius of the sphere.

b) Write the volume of the cube as a function of the radius of the sphere.

c) What is the volume of a cube that contains a sphere of radius 7.5 cm?

14. The number of tonnes, n, of wood pulp produced at Al Pac each day is a function of the number of hours, t, the assembly line is in operation that day and is given by $n(t) = 550t - 2t^2$. The cost, C, of producing the wood pulp is a function of the number of tonnes produced and is given by $C(n) = 7.5n + 1195$.

a) Write the equation of the function $(C \circ n)(t)$ that gives the cost of producing wood pulp in terms of the number of hours, t, the assembly line is in operation on a given day.

b) What was the cost of producing wood pulp on a day when the assembly line was in operation for 18 h?

c) How many hours was the plant in operation on a day when the total cost of production was $91 555?

Chapter 10 Skills Organizer

Complete each Frayer model template for combining functions.

Adding/Subtracting Functions

Definition	Examples and Non-Examples
Visual and Numerical Representation	Real-Life Application

Multiplying/Dividing Functions

Definition	Examples and Non-Examples
Visual and Numerical Representation	Real-Life Application

Composing Functions

Definition	Examples and Non-Examples
Visual and Numerical Representation	Real-Life Application