Chapter 11 Permutations, Combinations, and the Binomial Theorem

11.1 Permutations

KEY IDEAS

- The fundamental counting principle states that if one task can be performed in a ways and a second task can be performed in b ways, then the two tasks can be performed in $a \times b$ ways.
- For any positive integer n, n factorial or $n!$ represents the product of all of the positive integers up to and including n.

  $n! = n \times (n-1) \times (n-2) \times \cdots \times 3 \times 2 \times 1$. $0!$ is defined as 1.
- Linear permutation is the arrangement of objects or people in a line. The order of the objects is important. When the objects are distinguishable from one another, a new order of objects creates a new permutation.
- The notation ${}_nP_r$ is used to represent the number of permutations, or arrangements in a definite order, of r items taken from a set of n distinct items. A formula for permutations is

$${}_nP_r = \frac{n!}{(n-r)!}, n \in \mathrm{N}$$

- For permutations with repeating objects, a set of n objects with a of one kind that are identical, b of a second kind that are identical, and c of a third kind that are identical, and so on, can be arranged in $\frac{n!}{a!b!c!\ldots}$ ways.
- To solve some problems, you must count the different arrangements in all the cases that together cover all the possibilities. Calculate the number of arrangements for each case and then add the values for all cases to obtain the total number of arrangements.
- Whenever you encounter a situation with constraints or restrictions, always address the choices for the restricted positions first.

Working Example 1: Arrangements With or Without Restrictions

a) A school cafeteria offers sandwiches made with fillings of ham, salami, cheese, or egg on white, whole wheat, or rye bread. How many different sandwiches can be made using only one filling?

b) In how many ways can five black cars and four red cars be parked next to each other in a parking garage if a black car has to be first and a red car has to be last?

Solution

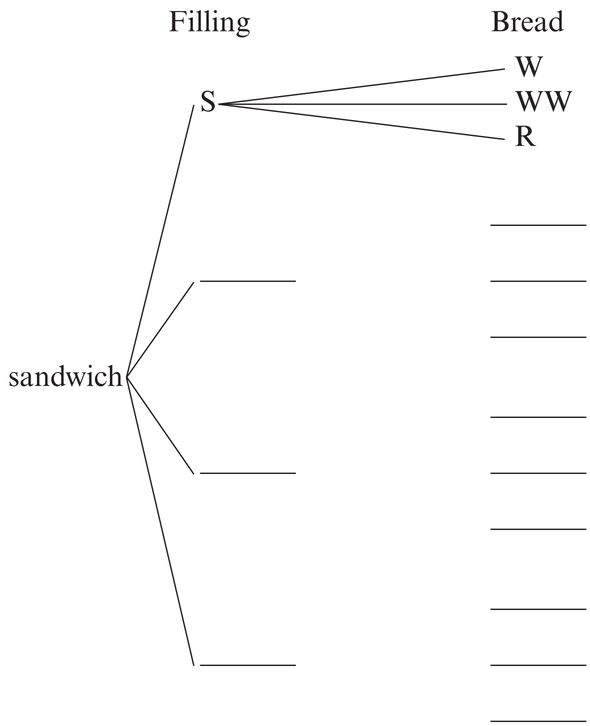
a) Method 1: List Outcomes and Count the Total

Use a tree diagram and count the outcomes, or list all of the sandwich choices in a table.

For the fillings, H represents ________, S represents ________, C represents ________, and E represents ________.

For the bread, W represents ________, WW represents ________, and R represents ________.

Tree Diagram



List

Filling	Bread
S	W

_______ possibilites

Total pathways = ________

There are ________ different sandwiches that can be made.

Method 2: Use the Fundamental Counting Principle

________ × ________

(number of choices for sandwich fillings) (number of choices for bread)

According to the fundamental counting principle there are

(________)(________) = ________ different sandwiches that can be made.

b) Use nine blanks to represent the nine cars parked in a row.

There are restrictions. A black car must be in the first position and a red car must be in the last position. Fill these positions first.

________ ________ ________ ________ ________ ________ ________ ________ ________

There are ________ black cars for the first position.

There are ________ red cars for the last position.

After filling the end positions, there are ________ positions to fill with ________ cars remaining.

Use the numbers you have determined to fill in the blanks that represent the nine cars parked in a row.

By the fundamental counting principle, there are

(________)(________)(________)(________)(________)(________)(________)(________)(________)

= ________ ways to park the cars in a row.

For a similar example, see Example 1b) on page 518 of *Pre-Calculus 12*.

Working Example 2: Using Factorial Notation

a) Evaluate ${}_{10}P_6$.

b) Show that 20! − 19! + 18! = (362)18!

c) Solve for n if ${}_nP_2 = 110$.

Solution

a) $${}_{10}P_6 = \frac{\square!}{(\square - \square)!}$$

$$= \frac{\square!}{\square!}$$

$$= \frac{(10)(9)(8)(7)(6)(5)4!}{4!}$$

$$= \text{________}$$

b) 20! − 19! + 18!

Determine a common factor among the terms.

= ________ × ________ × 18! − ________ × ________! + 18!

= 18!(________ × ________ − ________ + ________)

= 18!(________)

c) ${}_nP_2 = 110$

$$\frac{n!}{(n-2)!} = 110$$

$$\frac{\square(\square - \square)(\square - \square)!}{(n-2)!} = 110$$

$$\_\_\_\_(\_\_\_\_ - \_\_\_\_) = 110$$

$$n^2 - n - 110 = 0$$

$$n = \_\_\_\_$$

The solution to ${}_nP_2 = 110$ is $n =$ ________.

To see how algebra or graphing can be used to solve part c), refer to Example 2c) on page 520 of *Pre-Calculus 12*.

Working Example 3: Permutations With Repeating Objects

How many different ten-letter arrangements can you make using the letters of BASKETBALL?

Solution

There are ________ letters in BASKETBALL. There are ________ ways to arrange all the letters.

There are ________ letters ________ that can be arranged in ________ ways.

There are ________ letters ________ that can be arranged in ________ ways.

There are ________ letters ________ that can be arranged in ________ ways.

The number of different ten-letter arrangements is $\frac{\square!}{\square!\square!\square!}$

$= \_\_\_\_$

To see a similar problem, refer to Example 3 on page 521 of *Pre-Calculus 12*.

Working Example 4: Using Cases to Determine Permutations

In how many ways can eight basketball players sit on a bench if either the one centre or both of the two forwards must sit at the end where the coach always sits?

Solution

There are two different cases to be considered.

Case 1: The ________________ sits beside the coach.

Case 2: ________________ sit at the end beside the coach.

Case 1: Centre Sits Beside the Coach

Use a diagram to show the number of choices for the seating arrangement:

Coach _______ _______ _______ _______ _______ _______ _______ _______

Calculate the total number of seating arrangements for this case.

Case 2: Two Forwards Sit Beside the Coach

Use a diagram to show the number of choices for the seating arrangement:

Coach _______ _______ _______ _______ _______ _______ _______

Consider the two forwards as one object.

Calculate the total number of seating arrangements for this case.

To find the total number of seating arrangements, _______ the possibilities from the two cases.

Therefore, there are _______ possible seating arrangements.

To see an example similar to the above, refer to Example 5 on page 523 of *Pre-Calculus 12*.

Check Your Understanding

Practise

1. Use a tree diagram to identify the possible arrangements.
   a) Brett is purchasing a new car. He is given a choice of two upholstery materials (cloth or leather) and four colours (black, red, silver, or taupe).

b) Jireesha wears a uniform to work. She has the option of a blue, pink, or white T-shirt with pants that are black, grey, brown, or beige.

2. Use an organized list or table to identify the possible arrangements.

a) the ways that you can arrange the digits 3, 4, 6, 7, and 8 to form two-digit numbers

b) the ways you can travel from Toronto to Vancouver via Calgary, if you can go from Toronto to Calgary by either plane or train, and you can go from Calgary to Vancouver by bus, plane, train, or car.

3. Evaluate each expression.

a) ${}_7P_2$

b) ${}_6P_1$

c) ${}_{12}P_9$

d) ${}_8P_4$

4. a) Show that $5! - 3! \neq (5 - 3)!$

Left Side	Right Side
$5! - 3!$	$(5 - 3)!$
=	=

b) Show that $6! - 4! = 29(4!)$

5. What is the value of each expression?

a) $\frac{50!}{3!47!}$

b) $7! - 6!$

c) $\frac{14!}{12!2!}$

d) $10!$

6. In how many ways can you arrange all the letters of each word?

a) schools

b) curriculum

c) winter

d) arrangement

7. Seven students are running in an election for student council president. In how many ways can the seven names be listed on a ballot?

8. Solve for the variable.

a) ${}_nP_4 = 840$

b) ${}_5P_r = 20$

c) ${}_nP_3 = 720$

d) ${}_9P_r = 3024$

Apply

9. Describe the cases you could use to solve each problem, and then solve.

a) How many three-digit odd numbers greater than 300 can you make using the digits 1, 2, 3, 4, 5, and 6? No digits are repeated.

Case 1: Numbers that begin with 3 and end with ________ or ________

Number of choices for first digit	Number of choices for second digit	Number of choices for third digit
________	________	________

Number of possibilities for case 1 = ________.

Case 2: Numbers that begin with 4 and end with ________, ________, or ________

Number of choices for first digit	Number of choices for second digit	Number of choices for third digit
________	________	________

Number of possibilities for case 2 = ________.

Case 3: Numbers that begin with ________ and end with ________ or ________

Number of choices for first digit	Number of choices for second digit	Number of choices for third digit
________	________	________

Number of possibilities for case 3 = ________.

Case 4: Numbers that begin with ________ and end with ________, ________, or ________

Number of choices for first digit	Number of choices for second digit	Number of choices for third digit
________	________	________

Number of possibilities for case 4 = ________.

Total number of possibilities = ________ + ________ + ________ + ________

= ________

b) How many four-letter arrangements beginning with A or B and ending with a vowel can you make using the letters A, B, D, E, I, and U?

Case 1: Four-letter arrangements starting with ________

Number of choices for first letter	Number of choices for second letter	Number of choices for third letter	Number of choices for fourth letter
________	________	________	________

Number of possibilities for case 1 = ________.

Case 2: Four-letter arrangements starting with ________

Number of choices for first letter	Number of choices for second letter	Number of choices for third letter	Number of choices for fourth letter
________	________	________	________

Number of possibilities for case 2 = ________.

Total number of possibilities = ________ + ________

= ________

10. In how many ways can four girls and three boys be arranged in a row each situation?

a) A boy must be at each end of the row.

b) The boys must be together.

c) The girls must be together.

d) The ends of the row must be either both boys or both girls.

11. a) How many seven-letter arrangements can you make using all of the letters L, M, N, O, P, I, and U?

b) Of these, how many begin and end with a vowel?

c) Of these, how many begin with a consonant and end with a vowel?

12. A fitness club plans to issue each member their own five-character ID code. The first character and the third character can be different letters chosen from A through M, inclusive. The other three characters must be different digits from 1 to 9, inclusive. What is the maximum number of members that the club can have?

13. a) How many arrangements using all of the letters of the word PARALLELOGRAM are possible?

b) How many of these arrangements have all of the Ls together?

c) How many of these arrangements have all of the As together?

d) How many of these arrangements have all of the Rs together?

e) How many of these arrangements have all of the Ls, all of the As, and all of the Rs together?

Connect

14. Postal codes in Canada have six characters, with the first, third, and fifth characters being letters, and the other characters being digits.

a) What is the maximum number of unique postal codes in Canada if the letters D, F, I, O, Q, and U are not used and neither W nor Z appears as a first character?

b) How many of the postal codes described in part a) have the same letter repeated three times?

c) How many have the same digit repeated three times?

11.2 Combinations

KEY IDEAS

- A combination is a selection of objects without regard to order.
- The notation ${}_nC_r$ represents the number of combinations of n objects taken r at a time, where $n \geq r$ and $r \geq 0$.
- A formula for combinations is ${}_nC_r = \frac{n!}{(n-r)!r!}$, $n \in N$.
- The number of combinations of n items taken r at a time is equivalent to the number of combinations of n items taken $n - r$ at a time; that is, ${}_nC_r = {}_nC_{n-r}$.
- To solve some problems, count the different combinations in cases that together cover all the possibilities. Calculate the number of combinations for each case and then add the values for all cases to obtain the total number of combinations.

Working Example 1: Combinations and the Fundamental Counting Principle

Eight female students and nine male students are running for six offices on the student council executive team.

a) How many selections are possible?

b) How many selections are possible if the executive team must have three females and three males?

c) One of the male students is named David. How many six-member selections consisting of David, one other male, and four females are possible?

Solution

a) This is a combination problem because it involves choosing ________ students out of ________ and the ________________ is not important.

Substitute $n =$ ________ and $r =$ ________ into ${}_nC_r = \frac{n!}{(n-r)!r!}$:

$$ {}_{\square}C_{\square} = \frac{\square!}{(\square - \square)!\square!} $$

$$ = \frac{\square!}{\square!\square!} $$

$$ = \text{\_\_\_\_\_\_\_\_} $$

There are ________ possible ways of selecting the executive team.

b) There are ${}_{\square}C_{\square}$ ways of selecting three female students and ${}_{\square}C_{\square}$ ways of selecting three male students. Using the fundamental counting principle, the number of ways of selecting three females and three males is

$${}_{\square}C_{\square} \times {}_{\square}C_{\square}$$

$$= \frac{\square!}{(\square - \square)!\square!} \times \frac{\square!}{(\square - \square)!\square!}$$

$= \_\_\_\_\_\_\_\_ \times \_\_\_\_\_\_\_\_$

$= \_\_\_\_\_\_\_\_$

There are ________ ways to select an executive team consisting of three females and three males.

c) There is ________ way to select David. There are ________ males remaining, so there are ${}_{\square}C_{\square}$ or ________ choices for the second male.

There are ${}_{\square}C_{\square}$ ways to select four females.

$${}_{\square}C_{\square} = \frac{\square!}{(\square - \square)!\square!}$$

$$= \frac{\square!}{\square!\square!}$$

$= \_\_\_\_\_\_\_\_$

There are ________ × ________ × ________ = ________ ways to select the six-member team that includes David, one other male, and four females.

To see a similar problem, refer to Example 1 on pages 530 and 531 of *Pre-Calculus 12*.

Working Example 2: Combinations With Cases

An emergency response team of four people is selected from a group of expert volunteers consisting of two doctors, three surgeons, three firefighters, and four nurses. In how many ways can the team be chosen if it must contain

a) one of each type of volunteer?

b) at least three nurses?

Solution

a) There must be one of each: doctor, surgeon, fireman, and nurse.

There are ${}_{2}C_{1}$ ways of selecting a doctor.

$$
\begin{aligned}
{}_{2}C_{1} &= \frac{2!}{(2-1)!1!} \\
&= \frac{2!}{1!1!} \\
&= \_\_\_\_\_\_\_\_
\end{aligned}
$$

There are ${}_{\square}C_{\square}$ ways of selecting a surgeon.

$$
\begin{aligned}
{}_{\square}C_{\square} &= \frac{\square!}{(\square - \square)!\square!} \\
&= \frac{\square!}{\square!\square!} \\
&= \_\_\_\_\_\_\_\_
\end{aligned}
$$

There are ${}_{\square}C_{\square}$ ways of selecting a firefighter.

$$
\begin{aligned}
{}_{\square}C_{\square} &= \frac{\square!}{(\square - \square)!\square!} \\
&= \frac{\square!}{\square!\square!} \\
&= \_\_\_\_\_\_\_\_
\end{aligned}
$$

There are ${}_{\square}C_{\square}$ ways of selecting a nurse.

$$
\begin{aligned}
{}_{\square}C_{\square} &= \frac{\square!}{(\square - \square)!\square!} \\
&= \frac{\square!}{\square!\square!} \\
&= \_\_\_\_\_\_\_\_
\end{aligned}
$$

The total possible ways of selecting the team so there is one doctor, one surgeon, one firefighter, and one nurse is ________ × ________ × ________ × ________ = ________.

b) There must be at least three nurses. *At least three* means there could be ________

or ________ nurses.

Case 1: ________ nurses and ________ other member

There are $\square C \square$ ways of selecting three nurse and $\square C \square$ ways of selecting ________ other member.

$$\square C \square \times \square C \square = \frac{\square!}{(\square - \square)!\square!} \times \frac{\square!}{(\square - \square)!\square!}$$

$$= \frac{\square!}{\square!\square!} \times \frac{\square!}{\square!\square!}$$

$$= \underline{\qquad\qquad}$$

There are ________ ways of selecting a team with ________ nurses.

Case 2: ________ nurses and ________ other members

There are $\square C \square$ ways of selecting ________ nurses and $\square C \square$ ways of selecting ________ other members.

$$\square C \square \times \square C \square = \frac{\square!}{(\square - \square)!\square!} \times \frac{\square!}{(\square - \square)!\square!}$$

Recall that 0! is defined as 1.

$$= \frac{\square!}{\square!\square!} \times \frac{\square!}{\square!\square!}$$

$$= \underline{\qquad\qquad}$$

There is ________ way of selecting a team with ________ nurses.

The total number of ways of selecting the team so that there are at least three nurses is

________ + ________ = ________.

To see a similar problem, refer to Example 2 on page 532 of *Pre-Calculus 12*.

Working Example 3: Simplifying Expressions and Solving Equations With Combinations

a) Express as factorials and simplify $\dfrac{{}_nC_7}{{}_{n-1}C_5}$.

b) Solve for n if $3({}_nC_3) = {}_{n+1}C_4$.

Solution

a)

$$\frac{{}_nC_7}{{}_{n-1}C_5} = \frac{\square!}{(\square - \square)!\,\square!} \div \frac{(n-1)!}{(n-1-\square)!\,\square!}$$

$$= \left(\frac{\square!}{(\square - \square)!\,\square!}\right)\left(\frac{(\square - \square)!\,\square!}{(n-1)!}\right)$$

$$= \frac{n(n-\square)!}{(n-7)!(7)(6)(\square!)} \times \frac{(n-\square)(n-7)!(\square!)}{(n-1)!}$$

$$= \underline{\hspace{2cm}} \times \underline{\hspace{2cm}}$$

$$= \underline{\hspace{2cm}}$$

b)

$$3({}_nC_3) = {}_{n+1}C_4$$

$$3\left(\frac{\square!}{(\square - \square)!\,\square!}\right) = \frac{(n+1)!}{(n+1-\square)!\,\square!}$$

$$3\left(\frac{\square!}{(n-3)!\,\square!}\right) = \frac{(n+1)!}{(n-3)!\,\square!}$$

$$\frac{\square!}{2!} = \frac{(n+1)!}{4!}$$

$$\frac{\square!}{2} = \frac{(\square)n!}{24}$$

$$12 = \underline{\hspace{2cm}}$$

$$\underline{\hspace{2cm}} = n$$

To see a similar problem, refer to Example 3 on page 533 of *Pre-Calculus 12*.

Check Your Understanding

Practise

1. Decide whether each of the following is a combination or a permutation problem. Briefly describe why. You do not need to solve the problem.

a) A coin is tossed nine times. In how many ways could the result be six heads followed by three tails?

b) In how many ways can three coins be selected from nine?

c) A ski team has 12 members. In how many ways can 5 be selected to compete in the finals?

d) How many numbers less than 400 can be made using the digits 2, 3, 4, 5, and 6?

2. Describe the difference between ${}_8P_5$ and ${}_8C_5$, and then evaluate each one.

3. Evaluate.

a) ${}_7C_2$

b) ${}_7P_2$

c) ${}_9C_3$

d) ${}_{10}C_4$

4. From 12 employees, in how many ways can you

a) select a group of eight?

b) assign six different jobs?

5. a) How many combinations are there of the letters A, B, C, D, and E taken three at a time?

b) List all the combinations of A, B, C, D, and E taken three at a time.

c) How many three-letter permutations are there of the letters A, B, C, D, and E?

d) How is the number of combinations related to the number of permutations?

6. Solve for n.

a) ${}_{n}C_{2} = 15$

b) ${}_{n-1}C_{2} = 6$

c) ${}_{n+1}C_{n-1} = 15$

d) ${}_{n+2}C_{n} = 10$

7. Identify the cases you would use to solve each problem. Do not solve.

a) How many numbers less than 800 can you make using any number of the digits 2, 3, 4, 7, 8, and 9?

b) In how many ways can a six-person team be selected from four grade 11 students and six grade 12 students if the six-person team must have four members from either grade?

8. Show that each statement is true.

a) ${}_{10}C_3 = {}_{10}C_7$

b) ${}_{12}C_4 = {}_{11}C_4 + {}_{11}C_3$

Apply

9. In how many ways can the coach of the debating club select a team from seven grade 11 students and eight grade 12 students if the team has

a) four members?

b) four members, only one of whom is in grade 11?

c) four members, at least two of whom are in grade 12?

10. A bag contains nine white marbles and seven green marbles. In how many ways can you draw groups of six marbles if

a) exactly four must be green?

b) at least four must be green?

11. Show that $5({}_{n}C_{5}) = n({}_{n-1}C_{4})$.

12. Solve for n.

a) ${}_{n+2}C_{3} = {}_{n+1}C_{2}$

b) ${}_{n+1}C_{5} = {}_{n}C_{4}$

13. Three rooms in a university residence contain three beds, two beds, and five beds, respectively. In how many ways can ten students be assigned to these rooms?

14. The roster of a hockey team consists of ten forwards, five defenders, and two goalies. How many different teams can a coach select if the coach must select three forwards, two defenders, and one goalie?

Connect

15. Twenty people on a sightseeing tour are to travel on a double-decker bus that can hold 12 passengers on the main level and 8 on the upper level. If four of the passengers refuse to sit on the upper level and five want to sit only on the upper level, in how many ways can the passengers be seated if arrangements are not considered?

11.3 The Binomial Theorem

KEY IDEAS

- Pascal's triangle is a triangular array of numbers with 1 in the first row, and 1 and 1 in the second row. Each row begins and ends with 1. Each number in the interior of any row is the sum of the two numbers above it in the preceding row.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1

- In the expansion of the binomial $(x + y)^n$, where $n \in \mathrm{N}$, the coefficients of the terms are identical to the numbers in the $(n + 1)$th row of Pascal's triangle.

Binomial	Pascal's Triangle in Binomial Expansion	Row
$(x + y)^0$	1	1
$(x + y)^1$	$1x + 1y$	2
$(x + y)^2$	$1x^2 + 2xy + 1y^2$	3
$(x + y)^3$	$1x^3 + 3x^2y + 3xy^2 + 1y^3$	4
$(x + y)^4$	$1x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + 1y^4$	5

- You can also determine the coefficients represented in Pascal's triangle using combinations.

Pascal's Triangle	Combinations
1	$_0C_0$
1 1	$_1C_0$ $_1C_1$
1 2 1	$_2C_0$ $_2C_1$ $_2C_2$
1 3 3 1	$_3C_0$ $_3C_1$ $_3C_2$ $_3C_3$
1 4 6 4 1	$_4C_0$ $_4C_1$ $_4C_2$ $_4C_3$ $_4C_4$
1 5 10 10 5 1	$_5C_0$ $_5C_1$ $_5C_2$ $_5C_3$ $_5C_4$ $_5C_5$

- Use the binomial theorem to expand any power of a binomial, $(x + y)^n$, where $n \in \mathrm{N}$. Each term in the binomial expansion has the form $_nC_k(x)^{n-k}(y)^k$, where $k + 1$ is the term number. Thus, the general term of a binomial expansion is $t_{k+1} = {}_nC_k(x)^{n-k}(y)^k$.
- Important properties of the binomial expansion $(x + y)^n$ include the following:
  - Write binomial expansions in descending order of the exponent of the first term in the binomial.
  - The expansion contains $n + 1$ terms.
  - The number of objects, k, selected in the combination $_nC_k$ can be taken to match the number of factors of the second variable. That is, it is the same as the exponent on the second variable.
  - The sum of the exponents in any term of the expansion is n.

Working Example 1: Expand Binomials

a) Use Pascal's triangle to expand $(a + b)^7$.

b) Identify patterns in the expansion of $(a + b)^7$.

Solution

a) The coefficients for the terms of the expansion $(a + b)^7$ occur in the ________ row of Pascal's triangle.

The ________ row of Pascal's triangle is 1 7 21 35 35 21 7 1.

$(a + b)^7 =$

b) Some patterns are as follows:

i) There are ________ terms in the expansion of $(a + b)^7$.

ii) The powers of a ______________ from ________ to ________ in successive terms of the expansion.
(*increase* or *decrease*)

iii) The powers of b ______________ from ________ to ________.
(*increase* or *decrease*)

iv) Each term is of degree ________ (the ________ of the exponents for a and b is ________ for each term).

v) The coefficients are ______________, and they begin with ________ and end with ________.

To see a similar example, refer to Example 1 on page 539 of *Pre-Calculus 12*.

Working Example 2: Use the Binomial Theorem

a) Use the binomial theorem to expand $(3a - 4b)^5$.

b) What is the third term in the expansion of $(5a^2 + 2)^6$?

c) In the expansion of $\left(a^3 - \frac{2}{a}\right)^7$, which term, in simplified form, contains a^5?

Solution

a) Since $n = 5$, write the binomial theorem for $(x + y)^5$.

$(x + y)^5 = {}_5C_0(x)^5(y)^0 +$ ________ $+$ ________ $+$ ________ $+$ ________ $+$ ________

Since it is necessary to expand $(3a - 4b)^5$, substitute $x =$ ________

and $y =$ ________ into the expression.

What is the sign of y?

$(3a - 4b)^5 = {}_5C_0($________$)^5($________$)^0 +$ ________ $+$ ________ $+$ ________

$+$ ________ $+$ ________

Now, simplify to get the final expansion.

b) The coefficients in the expansion of $(5a^2 + 2)^6$ involve the pattern

${}_6C_0$, ________, ________, ________, . . .

The coefficient of the third term involves ${}_6C_2$.

In the general term, $t_{k+1} = {}_nC_k(x)^{n-k}(y)^k$, substitute

$x =$ ________, $y =$ ________, $n =$ ________, and $k =$ ________.

$t_{\square} =$

The third term in the expansion of $(5a^2 + 2)^6$ is ________________.

c) Determine the first three terms of the expanded binomial. Simplify the variable part of each term to find the pattern.

In the binomial expansion, substitute $x =$________, $y =$ ________, and $n =$ ________.

$\left(a^3 - \frac{2}{a}\right)^7 =$

The pattern shows that the exponents for a are decreasing by ________ in each successive term. The next term contains ________, the term following that contains ________, and so on. Therefore, the ________ term contains a^5.

Refer to Example 2 on pages 540 and 541 of *Pre-Calculus 12* for a similar problem.

Check Your Understanding

Practise

1. Some rows from Pascal's triangle are shown. What is the next row in each case?

a) 1 4 6 4 1

b) 1 6 15 20 15 6 1

c) 1 9 36 84 126 126 84 36 9 1

2. Express each row of Pascal's triangle using combinations. Leave each term in the form ${}_nC_r$.

a) 1 3 3 1

b) 1 5 10 10 5 1

c) 1 6 15 20 15 6 1

3. Express each indicated term in the given row of Pascal's triangle as a combination. Leave your answers in factorial form.

a) 1 4 6 4 1

b) 1 7 21 35 35 21 7 1

c) 1 9 36 84 126 126 84 36 9 1

4. How many terms are in the expansion of each expression?

a) $(m - n)^7$

b) $(2x + 1)^6$

c) $\left(\frac{y}{2} - y^2\right)^4$

d) $(a^3 + 2)^5$

5. Use Pascal's triangle to expand these binomials from #4.

a) $(m - n)^7$

b) $(2x + 1)^6$

c) $(a^3 + 2)^5$

6. Use the binomial theorem to expand each of the following.

a) $(x - 2)^6$

b) $(a - 2b)^4$

c) $(1 + x^2)^6$

Apply

7. Determine the simplified value of the specified term.

a) the sixth term of $(a + b)^{10}$

b) the tenth term of $(1 + x)^{13}$

c) the fifth term of $\left(x^2 - \frac{1}{x}\right)^{10}$

d) the third term of $\left(x^2 + \frac{x}{2}\right)^7$

e) the middle term of $(2x - y)^8$

8. Expand and simplify.

a) $\left(a - \frac{1}{a}\right)^5$

b) $\left(x - \frac{2}{x^2}\right)^5$

9. Use the binomial theorem to determine the first four terms of each.

a) $\left(2a - \frac{1}{a}\right)^9$

b) $\left(2x - \frac{3}{x}\right)^8$

10. Express each expansion in the form $(a + b)^n$, where $n \in \mathbb{N}$.

What do the patterns of the signs of the terms tell you about the sign of *b*?

a) ${}_5C_0x^5 - {}_5C_1x^4y + {}_5C_2x^3y^2 - {}_5C_3x^2y^3 + {}_5C_4xy^4 - {}_5C_5y^5$

$a =$ ________, $b =$ ________, and $n =$ ________. Substitute into $(a + b)^n$:

b) ${}_6C_0x^6 - {}_6C_1(2x^5) + {}_6C_2(4x^4) - {}_6C_3(8x^3) + {}_6C_4(16x^2) - {}_6C_5(32x) + {}_6C_6(64)$

$a =$ ________, $b =$ ________, and $n =$ ________. Substitute into $(a + b)^n$:

11. Determine the general term, in simplified form, in the expansion of $(a^2 + 1)^7$.

12. In the expansion of $\left(x - \frac{1}{x}\right)^6$, determine the term containing x^4.

13. Expand $(x^2 + 1)\left(x + \frac{1}{x}\right)^5$ and simplify. Use the binomial theorem.

Connect

14. Use the binomial theorem to determine the value of $(1.02)^9$, to four decimal places.

Chapter 11 Review

11.1 Permutations, pages 364–373

1. Evaluate each expression.

a) ${}_{10}P_7$

b) ${}_{7}P_4$

2. In how many ways can you arrange all the letters of each word?

a) province

b) Canada

3. Solve for the variable.

a) ${}_{n}P_3 = 60$

b) ${}_{7}P_r = 42$

4. The number of different permutations using all of the letters of a word is given by $\frac{9!}{2!3!}$. What word could have this number of arrangements of its letters?

5. In how many ways can five girls and three boys be arranged in a row if

a) a boy must be at each end of the row?

b) the boys must be together?

c) the girls must be together?

11.2 Combinations, pages 374–382

6. Decide whether each of the following is a combination or a permutation problem. Briefly explain why. You do not need to solve the problem.

a) A *Reach for the Top* team has eight members. In how many ways can they be seated in a row?

b) A *Reach for the Top* team has eight members. In how many ways can four be selected for a competition?

7. Describe the difference between ${}_9P_6$ and ${}_9C_6$.

8. Solve for *n*.

a) ${}_nC_2 = 10$

b) ${}_{n+1}C_{n-1} = 28$

9. From a deck of 52 cards, how many different four-card hands could be dealt that include a card from each suit?

10. A committee of students and teachers is being formed to study the issue of student parking. Fifteen staff members and 18 students have expressed an interest in serving on the committee. In how many ways can a five-person committee be formed if

a) it must include two teachers?

b) it must include at least two students and one teacher?

11.3 The Binomial Theorem, pages 383–389

11. Two rows from Pascal's triangle are shown. What is the next row in each case?

a) 1 1

b) 1 5 10 10 5 1

12. Express each row of Pascal's triangle using combinations. Leave each term in the form ${}_nC_r$.

a) 1 4 6 4 1

b) 1 8 28 56 70 56 28 8 1

13. State the number of terms in the expansion of each expression. Then, expand using Pascal's triangle.

a) $(2x + 3y)^4$

b) $(a^2 - 4)^5$

14. Use the binomial theorem to expand each of the following.

a) $\left(x + \frac{1}{x}\right)^4$

b) $\left(2x - \frac{1}{x}\right)^6$

15. Determine the simplified value of the specified term.

a) the fourth term of $(4 - 2a)^7$

b) the fifth term of $(a + b)^{25}$

Chapter 11 Skills Organizer

Complete the chart to summarize the concepts from this chapter.

	Permutations	Combinations	Binomial Theorem	Pascal's Triangle
Used For				
Related Definitions and Vocabulary				
Related Formulas				
Special Situations				
Examples				