Trigonometry

Opener

***Pre-Calculus 11*, pages 72–73**

Suggested Timing
30–40 min

Blackline Masters
BLM 2–2 Chapter 2 Prerequisite Skills
BLM U1–1 Unit 1 Project Checklist

Key Terms

initial arm	quadrantal angle
terminal arm	sine law
angle in standard position	ambiguous case
reference angle	cosine law
exact value	

What's Ahead

Section 2.1 of this chapter introduces angles in standard position and reference angles on the Cartesian plane. Students extend their previous knowledge of the sine, cosine, and tangent ratios by determining ratios for angles greater than 90° and sketching them on the Cartesian plane. Exact value solutions are introduced. Section 2.2 extends this knowledge to help students solve trigonometric equations using reference angles. Section 2.3 introduces the sine law, its proof, and how it can be used to solve triangles that are not right triangles. Students identify triangles that can be solved using the sine law and recognize the ambiguous case of the sine law and how it affects problem solving. Section 2.4 introduces the cosine law, its proof, and how it can be used to solve problems.

Planning Notes

As a class, read and talk about the chapter opener. Have students mention situations in which they believe trigonometry can be used, or where they have seen it used in everyday life. Challenge them to consider what careers might use trigonometry. You may wish to mention that the bridge shown on pages 72 and 73 is the Peace River Bridge in Dunvegan, AB. The building shown is Walt Disney Concert Hall, in Los Angeles, California.

Tell students that in this chapter, they will build on their existing knowledge and skills of the primary trigonometric ratios—tangent, sine, and cosine—to explore angles greater than 90° and non-right triangles.

As students progress through the chapter, have them record the Key Terms and develop their own definitions. They can refer to the definitions in the student resource. Have students explain their understanding of the Key Terms.

As a class, highlight the careers related to trigonometry throughout the chapter, and have students explain how trigonometry is used in each career.

Unit Project

You might take the opportunity to discuss the Unit 1 project described in the Unit 1 opener. Throughout the chapter, Project Corner boxes provide information related to the unit project. These features are not mandatory but are recommended because they provide some background for the final report for the Unit 1 project assignment.

If you are going to develop a project rubric with the class, you may want to start now. See pages 91–92 in this Teacher's Resource for information on working with students to develop a class rubric.

Chapter Summary

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. If they have used Foldables™ before, you may wish to have them select a style they found useful to keep their notes in for Chapter 2. Discuss other methods of summarizing information. For example, many students may have used different types of graphic organizers, such as a mind map, concept map, spider map, Frayer model, and KWL chart. Discuss which one(s) might be useful in this chapter.

Encourage students to use a summary method of their choice. Allowing personal choice in this way will increase student ownership in their work. It may also encourage some students to experiment with different summary techniques.

Give students time to develop the summary method they have chosen. Ask them to include some method of keeping track of what they need to work on. Explain the advantage of doing this.

Meeting Student Needs

- Consider having students complete the questions on **BLM 2–2 Chapter 2 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Have students recall what they know about the Pythagorean Theorem and the primary trigonometric ratios. You might have students create a poster that features the trigonometric ratios and the Pythagorean Theorem. Display the poster in the classroom for reference.
- You might use a diagram to help explain how a GPS system works to determine exact locations.
- Invite students to talk about examples of applications of trigonometry that they are familiar with. They might mention designs of winter shelters, drums, tipis, or star blankets.
- You may wish to post the student learning outcomes for the entire chapter in the classroom, colour-coding the outcomes by section in the chapter. Ensure that students understand the outcomes as written, and be prepared to rewrite some into language they understand. Provide students with their own copy. They can then refer to the outcomes as they work through the chapter. This will help them to self-assess their progress and to identify areas of weakness.
- Hand out **BLM U1–1 Unit 1 Project Checklist** to students, which provides a list of all the requirements for the Unit 1 project.

ELL

- Encourage students to create their own math vocabulary dictionary for the Key Terms using written descriptions, diagrams, and examples.

Enrichment

- Students may wish to brainstorm and research careers that involve trigonometry. For example, they might research how a pilot, an astronaut, or a forester uses trigonometry.
- Encourage students to collect interesting examples of trigonometry in action, such as the robotic arm.

Gifted

- Challenge students to explain the limitations of the sine, cosine, and tangent ratios. Have them explore the possibility of extending these ratios to solve problems involving angles greater than 90°. Students might sum sines, cosines, or tangents to determine values for obtuse triangles.

Career Link

You may wish to have students who are interested in learning more about physiotherapists research the career, including the training and qualifications required, and employment opportunities. Have students present their findings orally. Explain how this career connects to the chapter.

Web Link

For information about a career in physiotherapy, go to www.mhrprecalc11.ca and follow the links.

Angles in Standard Position

2.1

***Pre-Calculus 11*, pages 74–87**

Suggested Timing
100–120 min

Materials
- centimetre grid paper
- ruler
- protractor

Blackline Masters
Master 2 Centimetre Grid Paper
BLM 2–3 Chapter 2 Warm-Up
BLM 2–4 Section 2.1 Extra Practice

Mathematical Processes
- ☑ Communication (C)
- ☑ Connections (CN)
- ☑ Mental Math and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☐ Technology (T)
- ☑ Visualization (V)

Specific Outcomes
T1 Demonstrate an understanding of angles in standard position [0° to 360°].
T2 Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–11, 14, 16, 17, 23–25
Typical	#2, 4–7, 9, 10, two of 12–15, 16, 18, 23–25
Extension/Enrichment	#7, 9, 16, 19–25

Planning Notes

Have students complete the warm-up questions on **BLM 2–3 Chapter 2 Warm-Up** to reinforce prerequisite skills needed for this section. If you have posted the outcomes, refer to the outcomes for this section.

As a class, read the opening paragraph and invite students to find examples of angles in classroom objects. Alternatively, have them consider angles outside of the classroom, such as outdoors.

Investigate Exact Values and Angles in Standard Position

The purpose of this investigation is to help students develop a concrete understanding of angles in standard position on the Cartesian plane, and exact values for sine, cosine, and tangent ratios with a reference angle of 30°, 45°, and 60°. Once students make these connections, they explore angles in quadrants II, III, and IV.

Have students work in pairs to complete Part A. When completed, have groups discuss their response to Reflect and Respond #4 and 5 as a class.

Some students may not make the connection that angles can be both positive (rotating counterclockwise) or negative (rotating clockwise) from the positive *x*-axis. Use leading questions about negative angles and how they rotate around the Cartesian plane, such as:
- Where does the initial arm begin when drawing an angle in standard position?
- When rotating the terminal arm in a counterclockwise direction, is the angle positive or negative?
- When rotating the terminal arm clockwise, is the angle positive or negative?
- How could you indicate the difference between rotating the terminal arm counterclockwise and clockwise?

These questions may help students find a second method for drawing angles in standard position in quadrants III and IV.

Have student pairs complete Part B using plain 8.5″ by 11″ paper. For #7b) and c), ask students:
- What kind of triangle is formed by C′DE? How do you know?
- How do you know that triangle DEF is equilateral?

Have students label the angles and side lengths on both sides of the paper triangle. This will help them to answer #4 to 9.

Some students may not make the connection that some side lengths of the triangle in quadrants II, III, and IV are negative. You might want to use the following prompts:
- When the triangle is in quadrant II, what is the length of the side resting on the *x*-axis?
- Is this value positive or negative?
- How will this value affect the trigonometric ratio for cosine and tangent?
- Are there similar concerns when the triangle is in quadrant III? quadrant IV? Explain.

As you circulate, talk with student pairs about how to determine sine, cosine, and tangent ratios for 90° angles. You might pose the following questions:

- When using the 90° angle in your triangle, which side would you label as the hypotenuse? opposite side? adjacent side? Explain why.
- Does this cause confusion for the sine ratio? cosine ratio? tangent ratio? Explain.

As a class, have students discuss their responses to Reflect and Respond #13 to 16. Consider posing leading questions to help students conclude that the side lengths of a triangle change signs when measured left and below the origin. Ask students how this affects the sine, cosine, and tangent ratios.

Challenge students to find ways to create a 45°–45°–90° triangle by paper folding and to explain their solution to the class.

Meeting Student Needs

- Have students organize a list of angles found in classroom objects by type of angle.
- Help students understand that angles can be interpreted as rotations of a ray by having them cut out and tape an arrowhead to the end of their ruler. Next, place the ruler on a sheet of paper and draw a ray with the arrow at one end. Mark the other end of the ray as the point of origin. Place the end of the ruler without the arrow on the point of origin and rotate the ruler. Draw another ray. Add the labels *initial arm* and *terminal arm*.
- Some students may benefit from recalling the primary trigonometric ratios and how to determine them. Encourage students who have difficulty determining the primary trigonometric ratios for angles in quadrants II, III, and IV to label the sides of their paper triangle. In this way, they should be able to state the values of the opposite, adjacent, and hypotenuse sides of the triangle.
- For #9, encourage students to record the exact ratios in a table similar to this one.

θ	Sin θ	Cos θ	Tan θ
30°			
60°			

- For Part A #1, you might enlarge the diagrams and use one page per diagram. Post the diagrams as students work, and invite them to place each diagram in either Group A or B and justify their choice. Then, have students record a definition of an angle in standard position.
- For Part B #6 to 8, prepare a paper example in advance for each step. Label each one. Some students may benefit from referring to the prepared examples to complete the steps.
- For Part B #11a), remind students that the lengths of the sides need to reflect their position on the x-axis and the y-axis. (Some units will become negative.)

ELL

- Direct students to the Did You Know? on page 76 in the student resource to clarify the meaning of *exact measure.*

Common Errors

- Some students may not place the initial arm on the x-axis or the vertex on the origin.

R_x Check diagrams as students work or use prompts about the placement of the vertex, initial arm, and terminal arm during a class discussion.

- Students may make errors in paper folding.

R_x Suggest to students that when folding paper, they firmly crease the paper on each fold.

- Some students may use rounded decimal values instead of exact values.

R_x Reinforce the idea that rounding a decimal value means that it is no longer an exact value.

Web Link

For information about angles in standard position and reference angles, go to www.mhrprecalc11.ca and follow the links.

For a video about the Cartesian coordinate system and angles in standard position, go to www.mhrprecalc11.ca and follow the links.

Answers

Investigate Exact Values and Angles in Standard Position

1. Angles in standard position have the vertex of the angle at the origin and the initial arm on the positive x-axis.

2. B because the vertex is at the origin and the initial arm is on the positive x-axis.

3. For a) and b), place the protractor with the centre on the origin and the base line on the x-axis.

a) Count the angle from the right up and left to mark at 75°. Mark a point and draw a line from the origin through the point.

b) Count the angle up and left and mark a point at 105°. Draw a line from the origin through the point.

For c) and d), place the protractor upside down, with the centre on the origin and the base line on the x-axis.

c) Since 225° is 45° greater than 180°, start from the negative x-axis, count an angle down and right, and mark a point at 45°. Draw a line from the origin through the point.

d) Since 320° is 140° greater than 180°, count the angle from the negative x-axis down and right from the negative x-axis, and mark a point at 140°. Draw a line from the origin through the point.

4. Angles in standard position have their vertex at the origin and the initial arm on the positive x-axis. The terminal arm rotates in a counterclockwise direction for positive angles and in a clockwise direction for negative angles.

5. Method 1: Starting at the positive x-axis, rotate an angle counterclockwise equal to 130°, 200°, 290°, and 325°, and draw terminal arms for each angle.
Method 2: Using negative angles, 130° has a negative value of $130° - 360° = -230°$. Similarly, $200° = -160°$, $290° = -70°$, and $325° = -35°$. Starting at the positive x-axis, rotate negative angles in a clockwise direction equal to the negative values of −35°, −70°, −160°, and −230°, and draw terminal arms for each angle.

8. c) $\sqrt{3}$

9. a) $\sin 30° = \frac{1}{2}$; $\cos 30° = \frac{\sqrt{3}}{2}$; $\tan 30° = \frac{1}{\sqrt{3}}$

b) $\sin 60° = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{1}{2}$; $\tan 60° = \frac{\sqrt{3}}{1}$

c) No. The opposite and hypotenuse sides are the same, so $\sin 90° = \frac{2}{2} = 1$. But since the opposite and adjacent sides are not well defined, there is no solution for tan 90° or cos 90° using our triangle.

10. b) 60°

11. a) 120° **b)** 300° **c)** 240°

12. 30°, 150°, 330°, 210°

13. When the angle was in quadrant IV, the angle in standard position was $360° - \theta$. Yes, this works for any acute angle.

14. Depending on the quadrant, the sign of the ratios changed. Sine was positive in quadrants I and II but negative in quadrants III and IV. Cosine was positive in quadrants I and IV and negative in quadrants II and III. Tangent was positive in quadrants I and III and negative in quadrants II and IV.

15. a) Yes. The relative size of the sides is always a relationship in which the hypotenuse is twice the side of the shortest side. Also, because of the Pythagorean Theorem, the remaining side is $\sqrt{3}$ times larger than the smallest side.

b) Yes. The hypotenuse = 24.8 cm, the shortest side = 12.4 cm, and the remaining side = 21.5 cm. If the shortest side is 1, then the remaining side is $\frac{21.5}{12.4} = \frac{12.4\sqrt{3}}{12.4} = \sqrt{3}$ and the hypotenuse is $\frac{24.8}{12.4} = 2$.

16. Example: Fold the bottom left corner up to the right until the edges line up. Make a diagonal crease down from the left corner.
$\tan 45° = 1$; $\sin 45° = \frac{1}{\sqrt{2}}$; $\cos 45° = \frac{1}{\sqrt{2}}$

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the investigation. Encourage them to generalize and reach a conclusion about their findings.	• Encourage student pairs to use the terms *initial arm*, *terminal arm*, and *vertex* in their own definition of an angle in standard position. • Some students may benefit from referring to a Cartesian grid on the classroom wall that identifies angles of 0°, 90°, 180°, 270°, and 360°. • You may wish to explain #4 and 5 to the class, before having students complete Part B. It is important for students to have a good understanding of angles in standard position. • As you circulate, check that students who are folding paper have a proper fold from which to draw conclusions. • Consider having students complete #14–16 as a class. Ensure students understand how to obtain exact values.

Link the Ideas

As a class, define angles in standard position and the related terms. Have students record their own notes for each new term. You might ask them to use a Cartesian grid and draw an example of an angle in each quadrant. Ask them if the angles they drew are in agreement with the measures of angles in each quadrant that are shown in the student resource.

Have students talk about reference angles and then record their own notes. Ask students how the reference angle is determined in quadrant III ($\theta_R = \theta - 180°$) and quadrant IV ($\theta_R = 360° - \theta$). Have students produce their own example by choosing their own reference angle in quadrant I and listing the angles in quadrants II, III, and IV. Then, have them compare their example with that of a classmate.

Talk about special right triangles and exact values, and have students record their own notes. Have them draw a 45°–45°–90° triangle and a 30°–60°–90° triangle and label the side lengths as shown on page 79 in the student resource. Have students use their sketches to determine the sine, cosine, and tangent ratios for these special angles. Encourage them to refer to their sketches and ratios as they work on the chapter.

Example 1

This example demonstrates sketching angles in standard position. Encourage students to attempt the solution before working through the given solution. Provide students with an opportunity to discuss any difficulties they encountered.

Have students complete the Your Turn question individually and compare the solution with a classmate.

Example 2

This example demonstrates how to determine a reference angle for a given angle. Encourage students to attempt the solution before working through the given solution. If students have difficulty understanding how to determine the reference angle, use leading questions such as:

- How many degrees are in a straight line?
- How many degrees are in a complete circle?
- What is the difference in degrees between the x-axis and the terminal arm of the angle?

Have students complete the Your Turn question individually and compare the solution with a classmate.

Example 3

This example models how to determine an angle in standard position when a reference angle is reflected in the x-axis and the y-axis.

Refer students to Part B in the investigation on page 76 in the student resource, where they placed their triangle on a Cartesian grid and determined the angles in quadrants II, III, and IV. If they still have difficulty, have them draw and cut out an angle of 40° and then, using this template, place it with the vertex of the angle on the origin of the grid. Have them use a protractor to measure the angle from the positive x-axis. Have them place the template in quadrant II and ask if they got the same result as what is shown in the solution. Have them repeat by placing the template in quadrant IV, and then quadrant III. Each time, ask them to compare their result with the solution shown.

Some students may benefit from having a one-on-one or small-group discussion to help work through the solution.

Have students complete the Your Turn question individually and compare the solution with a classmate.

Example 4

This example models how to solve a problem involving exact distance. Encourage students to draw a diagram to model the situation before solving. Using a protractor and a ruler, you might have them draw a point at the end of the metronome arm and measure the distance from point A to point B. (Since the metronome arm is 10 cm long, they can draw an exact diagram.) Ask if their measurement is the same as 10 cm. Have them reread the example and decide if they agree with the process used to solve the problem.

Some students may benefit from having a one-on-one or small-group discussion to help work through the solution. You might have students revisit this problem during section 2.4 as it can be solved using the law of cosines.

Have students complete the Your Turn question individually and compare the solution with a classmate.

Key Ideas

You might use this feature to help build the chapter review. Have students record their own summary of the Key Ideas. They can store their summary for each section in the same location in order to create a reference for review purposes. Consider allowing students to use their summary for the chapter test.

Meeting Student Needs

- For Link the Ideas, create and post a poster of each quadrant, as shown on page 78 in the student resource, showing the method for determining the reference angle. Emphasize that the reference angle needs to be a positive, acute angle. Students can refer to this poster when working through Example 2.
- For Link the Ideas, produce various sizes of the 30°–60°–90° triangle. Have students measure the length of the side opposite the 30° angle and the hypotenuse to discover that the hypotenuse will always be twice as long as the side opposite the 30° angle.
- For Example 3: Your Turn, some students may benefit from referring to the work they did in Part B of the investigation.

ELL

- Ensure that students add the following terms to their vocabulary dictionary: *initial arm*, *terminal arm*, *angle in standard position*, *reference angle*, and *exact value*. Encourage them to include a verbal description, diagram, and/or example for each term.

- For Example 4, use the photograph in the student resource to help explain what a *metronome* is. Also, clarify the meaning of *pendulum* and *tempo*.

Common Errors

- Some students may not place the initial arm on the x-axis, the vertex on the origin, and/or rotate the terminal arm counterclockwise for positive angles.

Rx Encourage students to build a model with the initial arm on the positive x-axis and a movable arm pinned to the origin. They can rotate the movable arm and explain to a partner the possible values of the angle produced as they rotate it counterclockwise from quadrant I to II, to III, and to IV, and clockwise from quadrant I to IV, to III, and to II.

- Some students may measure reference angles from the y-axis.

Rx Encourage students to refer back to the investigation and the related examples, or use leading questions to help them.

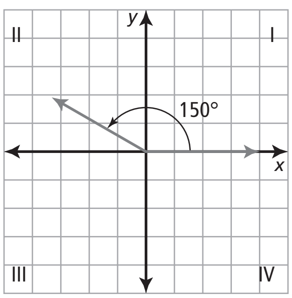
- Students may forget where to place the vertex of the angle when reflecting an angle on an axis.

Rx Encourage students to refer back to the investigation and the related examples, or use leading questions to help them.

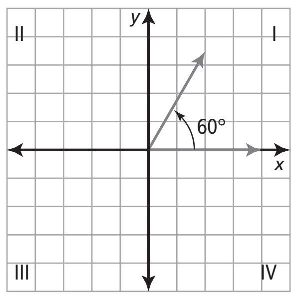
Answers

Example 1: Your Turn

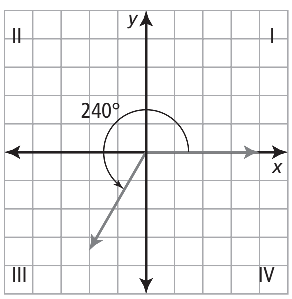
a) II



b) I

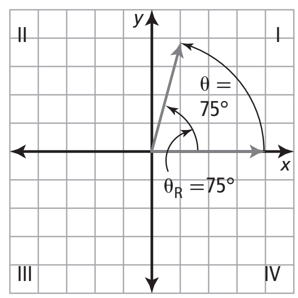


c) III

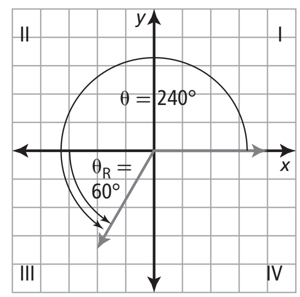


Example 2: Your Turn

a) $\theta_R = 75°$



b) $\theta_R = 60°$



Example 3: Your Turn

a) 120° **b)** 300° **c)** 240°

Example 4: Your Turn

The tip of the arm of the metronome moves a horizontal distance of $10\sqrt{2}$ cm.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Ensure students recognize that they are sketching and not drawing each angle accurately using a protractor. Check that the terminal arm is reasonably located within each quadrant. • Refer students to a visual of the Cartesian plane to help clarify where they should draw the angles.
Example 2 Have students do the Your Turn related to Example 2.	• Some students may find it beneficial to sketch each given angle and observe the location and size of each reference angle. • Coach students to determine the reference angle by looking for the distance in degrees between the terminal arm and the x-axis. • Encourage students to show their calculations for reference angles. This will assist them in the chapter review.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 3 Have students do the Your Turn related to Example 3.	• As students work through the example, some may benefit from recalling reflections and how they affect points on a coordinate grid. • Coach students to determine the reference angle by looking for the distance in degrees between the terminal arm and the *x*-axis. • Encourage students to show their calculations for angles in standard position.
Example 4 Have students do the Your Turn related to Example 4.	• It is important that students are able to clearly differentiate between approximate and exact values. You may wish to have them calculate roots such as $\sqrt{4}$, $\sqrt{5}$, $\sqrt{16}$, and $\sqrt{6}$. Ask which ones are terminating and which are not. • If students have a math journal, encourage them to make a table and list the exact sine, cosine, and tangent values for 30°, 45°, 60°, and 90° angles. • Encourage students to show their calculations in determining exact values.

Check Your Understanding

Practise

These questions allow students to build basic skills in trigonometry and require them to practise new skills and/or processes. Students can complete these questions individually or with a partner. After they complete a question, remind students to check the answer. In cases where there is a difference between their answer and the one in the answer key, encourage them to get help from a classmate or from you.

Apply

These questions provide opportunities for students to apply new skills to real-life situations. Some students may become confused about which skill or combination of skills to apply. Encourage them to sketch a diagram whenever possible to help them gain a visual understanding of the problem. Use leading questions to help students make the connection to the correct skill or process that is required for a specific question.

For #11, refer students to Example 4 and ask them how they determined the horizontal distance that the metronome tip moved from point A to point B. Some students may benefit from sketching a diagram for the initial position of the wiper with an angle of 30° in standard position. Ask students:

- For the 30° angle, how can you draw a right triangle? (Extend a line from the end of the wiper blade to the *x*-axis.)
- Knowing the length of the wiper, how can you determine the horizontal distance from the origin to the point at the end of the wiper blade?
- What is the value of x and value of y of the point?
- Does the value of x or the value of y represent a horizontal distance?
- Would you follow the same process for a 150° angle? Explain.

Note that #12 is similar to 11 and 13, and it extends the concepts to the general case. Assign #12 only to students who are comfortable with abstract solutions.

Although #13 is similar to 11, it requires students to determine vertical distances instead of horizontal distances. Encourage them to sketch triangles for both the 30° angle and 60° angle on the same Cartesian plane. Help students determine the ordered pair that represents the end of the boom for both angles by asking the following questions:

- Which side of the triangle represents the vertical height of the boom?
- Since you know the length of the boom, which ratio will you use to find the opposite side of the triangle?
- What is the exact value of sin 30°? of sin 60°?
- What is the vertical height of the 30° angle? the 60° angle?
- What is the difference between these heights?

For #16a), students may wish to determine the angular distance for a passing of one day. Ask students:

- How many days are in an Aztec month?
- How many degrees are in a complete circle?
- How many degrees on the circular calendar represent 1 day? 12 days?

For #16b), students need to reflect the reference angle from quadrant III to quadrant II and then determine the new angle in standard position. Once they know the angle, they can determine the time by dividing the overall angle by the angle for a passing of 1 day. Ask students:

- For part a), what was the angle measure for 12 days?
- Which quadrant is the angle in?
- What is the reference angle for an angle of 216°?
- If you reflect the angle into quadrant II, what is the reference angle in quadrant II?
- How do you determine the angle in standard position if you know the reference angle?

- What is the quadrant II angle in standard position?
- What is the angular measure for a passing of 1 day?
- How can you find the number of days for an angular movement of 144°?

Follow a similar approach for a reflection in quadrant IV.

Extend

These questions require students to extend their knowledge by using new and previously learned skills and processes to solve problems.

For #18a), most students should know how to determine a vertical displacement. Some students may benefit from referring to #13 or completing 13 before attempting 18. When students construct the table, suggest they include columns for the angle, height of arm from its pivot, and overall height of the arm from the base. Ask students:

- Which side of a right triangle does the height represent: hypotenuse, opposite, or adjacent?
- Which side does the length of the robotic arm represent: hypotenuse, opposite, or adjacent?
- Which trigonometric ratio involves the opposite side and the hypotenuse?
- How would you determine the length of the opposite side for angles 0°, 15°, 30°, 45°, 60°, 75°, and 90°?

Then, have students complete the table. Consider allowing students to use a graphing calculator or spreadsheet software.

For #18b), students may not know what relationship to determine. Ask students:

- Is there a common difference between each height as the angles increase by increments of 15°? Explain.
- Do the heights produce an arithmetic sequence? Explain.
- Is there a direct relationship between the angles and the height? Explain.

For #18c), you might have students extend their table to include angles of 105°, 120°, 135°, 150°, 165°, and 180°. Then, have them interpret what they observe.

For #19, ask students:

- What is the sum of two angles that are supplementary?
- What is the angle measure between two lines that are perpendicular?

For #20, students should be able to answer part a), but may have difficulty with the concept of travelling at four revolutions per minute. Ask students:

- What angle will the Ferris wheel travel in one revolution?
- If the wheel rotates at four revolutions per minute, how long does it take to rotate once?
- How many degrees will the wheel rotate in 5 s?
- If Carl's first position forms an angle of 72° in standard position, what will his angle in standard position be after a 5-s rotation? (Have students sketch a diagram and determine Carl's new height after the 5-s rotation.)

Create Connections

Have students work in groups to brainstorm solutions to these problems before developing an individual response. Remind students that they may need to use previous mathematical knowledge to solve the problems.

For #24, consider allowing students to use spreadsheet software.

Project Corner

The Project Corner box provides students with information about prospecting and provides some examples of natural resources that students might consider for their project. Emphasize that these are examples of natural resources and that they may choose something else if they wish. Students might brainstorm other possible natural resources. Encourage them to choose a natural resource (and possibly make a second choice) and begin the research process. Making a decision will help them move forward with their map of the resource location. Students may require computer time to accomplish this.

Meeting Student Needs

- Encourage students to draw diagrams to model problems.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 2–4 Section 2.1 Extra Practice** to students who would benefit from more practice.

ELL

- Some students may not be familiar with the following terms: *landscape plan*, *boom*, *vertical displacement*, *bevel protractor*, *periodic table*, *ritual cycle*, *robotic arm*, *increments*, *revolutions*, *prospecting*, and *prospectors*. Use a combination of descriptions, visuals, and examples to assist in students' understanding.
- For #17, use the Did You Know? that follows the question to explain the term *directions*. Some students may benefit from using a compass to help understand directions.
- For #18c), use the Did You Know? that follows the question to help explain the term *conjecture*.
- For #25, some students may not be familiar with the sport of golf. Have another student who has knowledge of golf explain the game.

Enrichment

- Have students brainstorm why reference angles are valuable in mathematics. Students might indicate situations in which the reference angle is more useful than the obtuse angle. For example, a physiotherapist working on extending the flexibility of a patient's elbow might work on the reference angle to the horizontal rather than the equivalent angle at the elbow.

Gifted

- Challenge students to create a spreadsheet that can determine the reference angle for a variety of angles. Students should show reference angles in all four quadrants. Some students may use "if" and "then" statements, or create a column for each quadrant.

Common Errors

- Students may confuse the trigonometric ratios.

R_x Have students label the hypotenuse, adjacent, and opposite sides on a diagram of a triangle. Have them use the definition of sine, cosine, and tangent ratios to determine which ratios involve the known and unknown sides of a triangle.

- Some students may not label the sides (opposite, adjacent, and hypotenuse) of a right triangle correctly.

R_x Help students recall the sides of a right triangle by having them draw and label a right triangle. Ask students:
  - What is the longest side of a right triangle called?
  - What is the side opposite the known angle of a right triangle called?
  - What is the side next to the known angle of a right triangle called?

- Students may use the incorrect reference angle.

R_x Refer students to the diagram they drew and ask them questions such as the following:
  - What is the measure of the angle from the nearest *x*-axis to the terminal arm?
  - What does this angle represent?

- Students may express an exact value incorrectly.

R_x Remind students that exact values require radical or integer values.

Web Link

For problems related to angles in standard position, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 11, 14, 16, and 17. Students who have no problems with these questions can go on to the remaining questions.	• For #1 to 5, students reinforce their understanding of the location of angles on a Cartesian grid, angles in standard position, and reference angles. Ensure that students are successful with these questions before moving on. • For #6 and 7, students apply their understanding of reference angles to angles in different quadrants. Remind them to measure the reference angle from the terminal arm to the *x*-axis. • For #8, check that students record exact values. Encourage students to sketch diagrams or refer to the diagrams in the Key Ideas to help determine the ratios. Some students may benefit from writing the primary trigonometric ratios first.
Assessment *as* Learning	
Create Connections Have all students complete #23–25.	• Some students may benefit from working with a partner to plan their responses. • For #23, encourage students to use diagrams for the explanation. • For #24, encourage students to use their math journal and record the results and the implication of their conjecture for values of sine, cosine, and tangent.

Trigonometric Ratios of Any Angle

2.2

***Pre-Calculus 11*, pages 88–99**

Suggested Timing
100–120 min

Materials
- centimetre grid paper
- ruler
- protractor

Blackline Masters
Master 2 Centimetre Grid Paper
BLM 2–3 Chapter 2 Warm-Up
BLM 2–5 Section 2.2 Extra Practice
TM 2–1 How to Do Page 99 #30 Using *The Geometer's Sketchpad*®
TM 2–2 How to Do Page 99 #30 Using *GeoGebra*.

Mathematical Processes
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Specific Outcome
T2 Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–11, one of 12–14, 19, 26–28
Typical	#2–9, one of 12–14, 15, 16, 19, 26–29
Extension/Enrichment	#4, 6, 7–9, one of 15–18, 19–21, 26–29

Planning Notes

Have students complete the warm-up questions on **BLM 2–3 Chapter 2 Warm-Up** to reinforce prerequisite skills needed for this section. If you have posted the outcomes, refer to the outcomes for section 2.2.

As a class, read the opening paragraph and invite students to explain what they know about machinery used for resource extraction that is related to the application of trigonometry.

During section 2.2, students will progress from a concrete to an abstract understanding of how mathematics approaches the use of angles greater than 90°. Students should build on their understanding of how to sketch angles in all quadrants and use these sketches to determine sine, cosine, and tangent ratios for any angle from 0° to 360°. Make use of leading questions to keep students involved in thinking about and connecting with the concepts. It is important for students to build a strong understanding, since students who rely on memorization will forget what they have learned.

Investigate Trigonometric Ratios for Angles Greater Than 90°

The purpose of this investigation is to help students develop a concrete connection with the relationships between the signs of trigonometric ratios in quadrants other than quadrant I.

Students should be able to complete this investigation in about 15 min working individually or in pairs. The investigation requires students to use protractors; some students may need a reminder about how to use them.

As a class, discuss the response to Reflect and Respond #6 to 8. For #7 and 8, take enough time for students to recognize the patterns produced by the ratios in all four quadrants.

Meeting Student Needs

- For #4a), coach students to recognize that the coordinates of the point change to (−3, 4).
- Some students may benefit from doing a concrete activity. If the classroom floor is tiled, push the desks aside and use masking tape to represent the axes. Plot (3, 4) on the floor and use masking tape to "draw" the triangle. Invite a student to stand on (3, 4) and then walk across the *y*-axis to its reflection point C. Use masking tape on the floor to "draw" the triangle. A board protractor can be used to measure the angle.

ELL

- Some students may not be familiar with terms related to oil sands production. Use the photograph of the power shovel to help explain this term.

Common Errors

- Some students may forget that line measurements left of the y-axis are negative and those below the x-axis are negative.

R_x Ask students leading questions:
- What is the value of the first point left of the origin on the x-axis?
- What sign do you use to indicate that the value is negative?
- What is the value of the first point on the y-axis below the origin?
- What sign do you use to indicate that it is negative?

Answers

Investigate Trigonometric Ratios for Angles Greater Than 90°

1. a) Quadrant I

2. a)
$$\begin{aligned} a^2 + b^2 &= c^2 \\ (3)^2 + (4)^2 &= r^2 \\ 25 &= r^2 \\ 5 &= r \end{aligned}$$

b) $\sin \theta = \frac{4}{5}$; $\cos \theta = \frac{3}{5}$; $\tan \theta = \frac{4}{3}$

c) Since $\sin \theta = \frac{4}{5}$, then $\theta \approx 53°$.

3. $\sin \theta = \frac{y}{r}$, $\cos \theta = \frac{x}{r}$, $\tan \theta = \frac{y}{x}$.

4. a) $(-3, 4)$

b) $\sin \angle COB = \frac{4}{5}$; $\cos \angle COB = -\frac{3}{5}$; $\tan \angle COB = -\frac{4}{3}$

5. a) $\angle COB = 127°$

b) $\angle COB + \angle COD = 180°$

6. a) The values of the trigonometric ratios for ∠AOB are the same as for ∠COB except that the signs for the cosine and tangent ratios in ∠COB are negative.

b) The measure of $\overline{OD}$ is negative and it represents the adjacent side of the right triangle ∠COD. Both the opposite and hypotenuse sides are positive, therefore any trigonometric ratio involving the adjacent side will have a ratio that is negative. That is why both cosine and tangent ratios for ∠COB are negative.

7. When point C is reflected over the x-axis, both the adjacent and opposite sides are negative. Since the tangent ratio involves the division of two negative values, the tangent ratio will be positive. Since both the sine and cosine ratios involve the division of a positive and negative value, they will both be negative.

8.

sin θ is [+] cos θ is [−] tan θ is [−]	sin θ is [+] cos θ is [+] tan θ is [+]
sin θ is [−] cos θ is [−] tan θ is [+]	sin θ is [−] cos θ is [+] tan θ is [−]

sine	all
tangent	cosine

The diagram shows the pattern for the ratios that are positive in each quadrant.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the investigation. Encourage them to generalize and reach a conclusion about their findings.	• It may be beneficial to help students recall reflections before beginning. Have them reflect individual points and a shape over both the x-axis and the y-axis. • Students may benefit from working in pairs to explain why the positive and negative signs on the trigonometric ratios vary from quadrant to quadrant. • Have students review their responses to #7 and 8. Some students may benefit from using a mnemonic such as CAST (cos, all, sin, tan) to recall the trigonometric functions that are positive. (The order of the quadrants for CAST is IV, I, II, III.) This will help them with the practice questions.

Link the Ideas

As a class, discuss the Link the Ideas. You might use the opportunity to develop a quadrant sign rule using a mnemonic such as CAST as a class. Make sure to involve students in its development by asking leading questions. Check that students understand the basic trigonometric relationships given for the general case.

Some students may benefit from coaching to understand that the relationship $r = \sqrt{x^2 + y^2}$ is an application of the Pythagorean Theorem. Have students draw a right triangle in the first quadrant of the coordinate plane, in standard position. Place a point P(x, y) anywhere on the hypotenuse. Have them label the sides x, y, and r to represent the adjacent, opposite, and hypotenuse sides, respectively. Ask students:

- What relationship connects the sides of a right triangle?
- Using the sides x, y, and r in the Pythagorean Theorem, what relationship do you get? Solve for r.
- What shape is produced when you rotate the terminal arm for one complete rotation?
- What is the length from the centre of a circle to a point on the circle usually called? Why would the hypotenuse be called r?

Have students use sides x, y, and r to produce the trigonometric ratios for sine, cosine, and tangent in the first quadrant. Then, have them list the signs of the trigonometric ratios in quadrant I. Have students repeat this process for quadrants II, III, and IV. Ask students:

- What do you notice about the signs of the ratios in quadrant I? quadrant II? quadrant III? quadrant IV?
- What patterns, if any, do you observe for the signs in the four quadrants?

Have students summarize their findings using a table.

Example 1

This example models how to determine the trigonometric ratios for an angle. Have students attempt the solution before working through the given solution. Provide coaching using prompts such as the following:

- What quadrant does the point P(−8, 15) lie in?
- Given the point P(−8, 15), sketch a right triangle with the vertex at the origin.
- Which value represents the adjacent side? the opposite side?
- What mathematical relationship can you use to determine the hypotenuse?
- Do you have enough information to determine the exact value for sin θ, cos θ, and tan θ ratios? If so, determine each trigonometric ratio.

Have students complete the Your Turn question individually and discuss the solution with the class.

Example 2

This example models how to determine the exact value of a trigonometric ratio. This implies that students will need to use their 30°–60°–90° and 45°–45°–90° special triangles to solve the problem. Discuss what determining the *exact value* means in any question and what it implies for the solution. Have students work through the given solution before discussing it as a class.

Assign the Your Turn question and then have students discuss the solution as a class.

Example 3

This example models how to determine trigonometric ratios when given one trigonometric ratio. Have students attempt the solution before working through the given solution. If students have difficulty, coach them using prompts such as the following:

- Can you sketch a right triangle in quadrant III with the given information? If so, do so.
- What sides of a right triangle do you know when you are given the cosine ratio?
- How can you determine the unknown side of the right triangle given two sides?
- What mathematical relationship will you use to determine the unknown side?
- Now that you know all three sides, how can you determine the exact values of sin θ and tan θ?

Have students read the green type beside the solution for y. You may wish to have a class discussion on the plus/minus symbol in the solution. This topic will be further explored in section 5.1.

Have students complete the Your Turn question individually and discuss the solution with the class.

Example 4

This example models determining trigonometric ratios of quadrantal angles. Have students work through the solution. Make sure to explain why tan 90° is undefined. You may need to coach students to recall that division by zero is undefined.

Have students complete the Your Turn question individually and discuss the solution with the class.

Example 5

This example models how to solve a trigonometric equation by determining the measure of an angle. Have students use the three-step process provided on page 94 in the student resource. You may wish to have students record the steps in their own words. Whenever students are solving trigonometric equations, they should ask themselves the following questions:

- What quadrants will the angles be in?
- Are there any angular restrictions I should consider? If so, what are they?
- How can I determine the reference angle from the given trigonometric ratio?
- How can I use the reference angle to sketch the angles in the appropriate quadrants?
- How can I use the sketches to determine the angles in standard position?

Have students attempt the solution to part a) using the three-step approach and their self-guided questions. Have them share their solution with the class. Assign students to solve part b), and then discuss the solution as a class.

Some students may not understand the statement $0° \le \theta \le 360°$ and how it will help to determine the possible quadrants used in their solution. Have students explain the angular restrictions and ask them which self-guided questions were useful.

Have students complete the Your Turn question individually and discuss the solution with a classmate.

Example 6

This example models how to determine the measure of an angle. Use a similar approach as for Example 5. Have students use the three-step process and the self-guiding questions to attempt the solution. Walk through the solution as a class. Ask students how the three-step process and the self-guiding questions helped them to solve the problem.

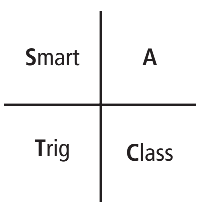
Have students complete the Your Turn question individually and compare the solution with a classmate.

Key Ideas

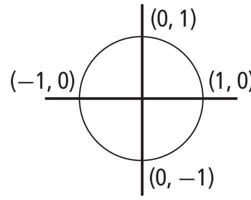
Have students record their own summary of the Key Ideas. Have them store the summary of the Key Ideas for each section in the same location for review purposes.

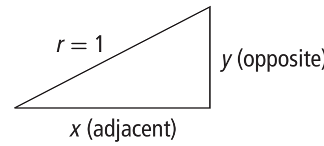
Meeting Student Needs

- Consider developing the formula for the Pythagorean Theorem on the board or an overhead.
- Encourage students to create their own mnemonic to recall which trigonometric ratio is positive in each quadrant. For example, ASTC (**A** **S**mart **T**rig **C**lass). The boldface letters indicate the trigonometric ratio is positive in the quadrant.



- For Example 4, encourage students to draw a unit circle diagram and right triangle to help understand the general case presented. Have them refer to the diagrams for Examples 5 and 6.





- Some students may benefit from recalling that the *cos*$^{-1}$ and *sin*$^{-1}$ buttons on a calculator are used to find the angle given a trigonometric ratio.

Common Errors

- Some students may confuse the trigonometric ratios for sine, cosine, and tangent.

R_x Encourage students to use a formula sheet to help them recall the basic trigonometric ratios.

- Some students may not know how to start the solution.

R_x Encourage students to sketch a diagram and label known sides and angles, whenever possible.

Web Link

For information about trigonometric functions for any angle, and the signs of the trigonometric ratios in each quadrant, go to www.mhrprecalc11.ca and follow the links.

Answers

Example 1: Your Turn

$\sin\theta = -\frac{12}{13}$; $\cos\theta = -\frac{5}{13}$; $\tan\theta = \frac{12}{5}$

Example 2: Your Turn

$\sin 240° = -\frac{\sqrt{3}}{2}$

Example 3: Your Turn

$\sin\theta = -\frac{1}{\sqrt{26}}$; $\cos\theta = -\frac{5}{\sqrt{26}}$

Example 4: Your Turn

	0°	90°	180°	270°
sin θ	0	1	0	–1
cos θ	1	0	–1	0
tan θ	0	undefined	0	undefined

Example 5: Your Turn

$\theta = 225°$ and $315°$

Example 6: Your Turn

$\theta = 234°$ and $306°$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Ensure students recognize that they need to sketch a right triangle. Have them label the units correctly for each side of the triangle. • Have students recall the Pythagorean Theorem and how to use it to solve for the unknown side. Remind students that for these types of questions, they will always draw a right triangle and apply the Pythagorean Theorem. • Encourage students to record the signs for each unit on their diagram. If students do not label the signs on their diagram, remind them to determine the signs of the primary trigonometric ratios in each quadrant. Use a context to explain a negative number sign and explain that a negative distance does not exist.
Example 2 Have students do the Your Turn related to Example 2.	• Remind students that exact values are radicals or integers (and not decimal approximations). • Have students identify the sign of each value prior to calculating to help them make the connection between the value and an appropriate angle. • Encourage students to show their calculations for determining exact values.
Example 3 Have students do the Your Turn related to Example 3.	• Have students record the primary trigonometric ratios and store them for future reference. • Ensure that students know what the values for a given ratio, such as $\cos \theta = -\frac{3}{4}$, represent $\left(\text{i.e., } \frac{\text{adjacent}}{\text{hypotenuse}}\right)$. Some students may benefit from recalling what the values for sine, cosine, and tangent represent before moving on. • It may be beneficial for students to sketch the given angles to help visualize the location and size of the reference angle. • Encourage students to write each trigonometric ratio and clearly show the substitution of values. • Encourage students to show their calculations when determining the reference angle. This will assist them in the chapter review.
Example 4 Have students do the Your Turn related to Example 4.	• Tell students to refer to their notes about the trigonometric ratios. • Have students work through the solution on their own. You might use students' work to assess their understanding and identify those who struggle to label the sketch or plug the given values into the trigonometric ratios. Provide additional coaching to these students. • If students have a math journal, encourage them to record the solution in their journal.
Example 5 Have students do the Your Turn related to Example 5.	• It is important that students are able to identify the exact values for common angles. If they are having difficulty, refer them to previous examples and the tables that they made. • Some students may benefit from recalling how to locate a reference angle by looking for the distance between the terminal arm and the *x*-axis. Have students show how they know the reference angle for the Your Turn. • Have students refer to their math journal for the table that lists the exact sine, cosine, and tangent values for 30°, 45°, 60°, and 90° angles. Remind them to use exact values for the sides (i.e., $\sqrt{3}$) and not a decimal approximation. • Ensure that students understand how to solve trigonometric equations of this type before moving on.
Example 6 Have students do the Your Turn related to Example 6.	• Ensure students are able to identify the reference angle. Some students may benefit from recalling how to locate this angle. • Encourage students to sketch their own diagrams. • Have students show all their calculations.

Check Your Understanding

Practise

These questions allow students to build basic skills in trigonometry and require them to practise new skills and/or processes. Students can complete the questions individually or with a partner. After they complete a question, remind students to check the answer. In cases where there is a difference between their answer and the one in the answer key, encourage them to get help from a classmate or from you.

Tell students to sketch diagrams whenever possible. Encourage them to refer to the worked examples as a guide.

For #2, use the following prompts to coach students:
- What type of answer is expected when you are asked to determine an exact value?
- What is the reference angle? Is it an angle from one of the special triangles? If so, which one?
- What triangle would you draw in the terminal quadrant? Is it one of the special triangles? If so, which one?
- What are the side lengths of the triangle?
- How can you determine the sine, cosine, and tangent ratios using exact values?

For #3, remind students to sketch a right triangle and determine any unknown side lengths sides before writing the trigonometric ratios.

Encourage students to refer to their summary of the Key Ideas to help them with #4.

Coach students by asking what kind of answer is expected in #5. Ask how using a sketch might be helpful.

For each part of #8, coach students by asking them to identify the known sides of the right triangle, given the trigonometric ratio. Ask how using a sketch might be helpful.

For #9, coach students by asking if they can determine the solution quadrants using the sign of the trigonometric ratio.

Apply

These questions provide opportunities for students to apply new skills to real-life situations. Some students may become confused about which skill or combination of skills to apply. Encourage them to sketch a diagram whenever possible to help them gain a visual understanding of the problem. Use leading questions to help students make the connection to the correct skill or process that is required for a specific question. Allow students to work in pairs to solve the problems.

Use the following prompts to coach students for individual questions.

For #12 and 13, ask students what sign of the ratio they would use when determining the reference angle.

For #14, ask students:
- Can the radical expression for the hypotenuse be simplified? If so, how?
- Can the sine ratio be simplified? If so, how?
- Are the sine ratios the same in all cases? Explain.

For #15, ask students:
- Can P(k, 24) exist in more than one quadrant? If so, which quadrants?
- Which side—adjacent, opposite, or hypotenuse—represents 25 units?
- Which ratio can you use to find the reference angle?
- How can you use the point(s) to sketch a right triangle?
- What mathematical relationship can you use to find the unknown sides of the right triangle?
- How can you determine the sine, cosine, and tangent ratios?

For #16, ask students:
- In what quadrant are both cos θ and tan θ positive?
- Is it possible to sketch a triangle that represents the given trigonometric ratios? If so, do so.
- How can you determine the adjacent, opposite, and hypotenuse sides from the given ratios?

For #21, you might allow students to use a graphing calculator or spreadsheet software. Ask students:
- As the angle increases from 0° to 60°, what trend do you notice about the value of the sine ratio? cosine ratio? tangent ratio?
- Are there any angles for which the sine and cosine ratios are equal? Explain.

Extend

These questions require students to extend their knowledge by using new and previously learned skills and processes to solve problems.

Use the following prompts to coach students for individual questions.

For #22, ask students:
- Given the equation $y = 6x$, what value of y do you get when $x = 1$?
- Does this point lie on the line $y = 6x$? Explain.
- If you plot this point as a vertex of a right triangle with another vertex at the origin, how can you use it to determine the side lengths of the right triangle?
- How can you find the sine, cosine, and tangent ratios using these sides?
- How can you follow a similar process to answer part b)?

For #24, ask students:
- If $\cos \theta = a$, is this the same as $\cos \theta = \frac{a}{1}$? Explain.
- How can you use the ratio $\cos \theta = \frac{a}{1}$ to find the adjacent and hypotenuse sides of a right triangle with $\angle\theta$ and the same cosine ratio?
- If you know two sides of a right triangle, can you determine the unknown side in terms of 1 and a? Explain.

For #25, encourage students to sketch a right triangle with an angle of 60° at the origin and vertex at point A. Ask students:
- How can you determine the ordered pair for point A using trigonometric ratios, a hypotenuse of 1 unit, and an angle of 60°?

- What is the ordered pair of point B? point C?
- How can you use points A, B, and C to determine the lengths of AB, AC, and BC?
- If you know the distances for AB, AC, and BC, how can you determine if these side lengths satisfy the Pythagorean Theorem?
- If you know that the three side lengths satisfy the Pythagorean Theorem, what do you know about the angle opposite the hypotenuse?

Create Connections

You might have students work in groups to brainstorm solutions to these problems before developing an individual response. Remind students that they may need to use previous math knowledge to solve the problems.

For #29, coach students using some of the following prompts:
- In what quadrant are both sine and cosine ratios negative?
- Sketch a right triangle in the quadrant and use the sine and cosine ratios to determine the side lengths.
- How can you determine the reference angle if you know the side lengths of a reference triangle?

For #30, which is a Mini Lab, have students work in pairs or small groups. Provide students with **TM 2−1 How to Do Page 99 #30 Using *The Geometer's Sketchpad*®** or **TM 2–2 How to Do Page 99 #30 Using *GeoGebra***. Have students choose a point that has whole number values to make the exploration easier. Students may need some guidance in their construction and some coaching to answer the related questions for each step of the Mini Lab. If the Mini Lab is used for summative assessment, ensure that you present your expectations for the completed work and provide a marking rubric for the assignment. If the Mini Lab is used for formative assessment, meet with the class as a whole and ask a couple of groups to lead a discussion of the results.

Meeting Student Needs

- Encourage students to draw diagrams to model problems. For #15, have students sketch a diagram before attempting the solution.
- Consider allowing students to work in pairs to complete #19 to 21.
- Have students refer to their own list of learning outcomes and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 2−5 Section 2.2 Extra Practice** to students who would benefit from more practice.

ELL

- The language in #17 may be challenging for some students. Consider having English language learners work with a partner who can help them understand the question.
- For #30, have group members help students who have difficulty with the language in the procedure.

Enrichment

- Challenge students to describe what happens to the values of the three trigonometric functions as the unit circle terminal ray is rotated from the positive x-axis to the negative x-axis. Students should show the values of the three sides of the triangle that are created when the ray moves into quadrant II. Note that the values of x become negative but the values of y remain positive in this scenario.

Gifted

- Ask students to show how Pythagorean thinking is connected to determining exact trigonometric values. Students should show Pythagorean special triangles (e.g., 3, 4, and 5; 1, 1, and $\sqrt{2}$; 1, 2, and $\sqrt{3}$) being used to find exact values of trigonometric functions.

Common Errors

- Students may not determine exact values when asked to do so.

R_x Remind students to express exact values as radicals, rational numbers in fraction form, or integers.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 11, one of 12 to 14, and 19. Students who have no problems with these questions can go on to the remaining questions.	• Encourage students to sketch diagrams, even for questions that provide diagrams. • Remind students that the given coordinates will help them draw the legs of the triangle, and that the hypotenuse is the line that returns to the origin. • Some students may benefit from referring to their notes about reference angles in section 2.1. • For #4, 6, and 9, refer students to their notes about the sign rules of the trigonometric ratios. • For #12 to 14, students are required to identify the quadrant in which each point is located. Check that students draw and label a triangle in the correct quadrant for each question. • Since #19 includes an important summary table, consider having students record the answer in their math journal for future reference.
Assessment *as* Learning	
Create Connections Have all students complete #26 to 28.	• For #27 and 28, encourage students to use diagrams to support their explanation. Consider collecting students' responses to these questions and checking for weaknesses in their thinking. Use the feedback to provide coaching to students, as needed.

The Sine Law

2.3

Pre-Calculus 11, **pages 100–113**

Suggested Timing
100–120 min

Materials
- ruler
- protractor

Blackline Masters
BLM 2–3 Chapter 2 Warm-Up
BLM 2–6 Section 2.3 #27 Concept Map
BLM 2–7 Section 2.3 Extra Practice

Mathematical Processes
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Specific Outcome
T3 Solve problems, using the cosine law and the sine law, including the ambiguous case.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–11, 14, 16, 18, 24, 25, 27
Typical	#1a), c), 2a), 3b), 4a), b), 5a), c), e), 6a), c), e), 7, 8a), 10 or 11, two of 12–16, two of 17–19, 21, 24, 25, 27, 28
Extension/Enrichment	#4, 7, 8, 11, 17, 21–28

Planning Notes

Have students complete the warm-up questions on **BLM 2–3 Chapter 2 Warm-Up** to reinforce prerequisite skills needed for this section. If you have posted the outcomes, refer to the outcomes for this section.

As a class, read the opening paragraph and invite students to explain what they know about global positioning systems. Have them provide examples of problems involving oblique triangles. For instance, they might mention building rafters in house construction.

The sine law is the first of two laws used to solve triangles. Students are expected to develop their problem solving skills by developing a process for using the sine law. They should know the conditions required for using the sine law to solve triangles, and understand the process for determining the possible solutions in the ambiguous case. Students need to practise the basic skills and solve contextual problems to develop an understanding of the concepts involved in using the sine law.

Investigate the Sine Law

This investigation allows students to develop the sine law using a simple, straightforward method.

Have students work in pairs but record their own response to all questions. Have student pairs compare their results with another pair before discussing the results as a class.

Meeting Student Needs

- Some students may benefit from referring to the poster about the Pythagorean Theorem and the primary trigonometry ratios associated with a right triangle that they created at the beginning of the chapter.
- Some students may benefit from doing a more concrete activity for the investigation. Draw an oblique triangle on a heavy piece of cardboard, plywood, or MDF. Have students draw the altitude and then measure the angles and sides of the triangle, including the altitude. Have students use the measurements to verify the partial equation for the sine law.
- Prepare several different oblique triangles, each on a separate sheet of paper. Once students complete the investigation, have them work with a partner to repeat the investigation using the measurements on the prepared diagrams.

ELL

- Ensure that students add the term *oblique triangle* to their vocabulary dictionary. Encourage them to include a verbal description, diagram, and/or example.

Common Errors

- Students may make computational errors.

R_x Encourage students to check each other's work for errors in substitution, multiplication, or division.

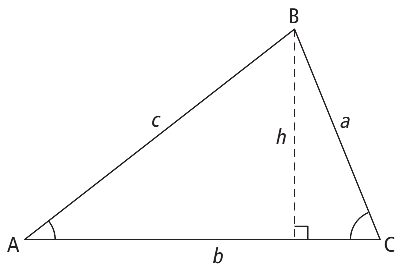
- Students may make measurement errors.

R_x Have students measure twice for each side and angle. Also, have students express the side to sine angle ratio to the nearest whole number.

Answers

Investigate the Sine Law

2.



3. $\sin A = \frac{h}{c}$ and $\sin C = \frac{h}{a}$. Both have the same opposite side, h.

4. $h = c \sin A$ and $h = a \sin C$

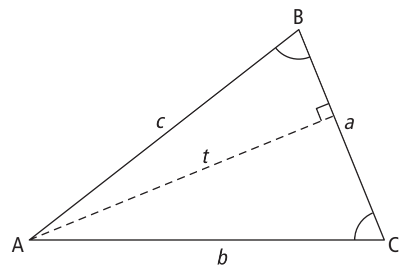
5. a) $c \sin A = a \sin C$

b) $\frac{\cancel{c}\sin A}{a\cancel{c}} = \frac{\cancel{a}\sin C}{\cancel{a}c}$; therefore, $\frac{\sin A}{a} = \frac{\sin C}{c}$ or $\frac{a}{\sin A} = \frac{c}{\sin C}$

6. a) The sine law connects the sine of an angle to the opposite side.

b) To use the sine law, you need to know at least one set of sides and angles (e.g., C, c) and the side or angle of another set.

7.



Since $\sin B = \frac{t}{c}$ and $\sin C = \frac{t}{b}$, then $t = c \sin B$, $t = b \sin C$, and $c \sin B = b \sin C$. Therefore, $\frac{c}{\sin C} = \frac{b}{\sin B}$.

8. Since $\frac{a}{\sin A} = \frac{c}{\sin C}$ and $\frac{c}{\sin C} = \frac{b}{\sin B}$,

then $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$.

9. No. If a given triangle does not include at least one pair with a known angle and its opposite side, then the sine law cannot be used.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the investigation. Encourage them to generalize and reach a conclusion about their findings.	• Some students may find it beneficial to complete #1 to 3 as a class, which would provide a model for students to use to complete #4 and answer 6 and 7. • Some students may benefit from reactivating their skills in setting up equations and applying substitution, both of which could be used in #3. • Consider having students complete #9 and exchange responses with a classmate. This would give students an opportunity to work through a second oblique triangle and check their own understanding by reviewing each other's work.

Link the Ideas

The sine law proof is similar to the activity that students completed in the investigation. The proof is written in a more formal manner. Students should become comfortable reading and understanding this type of proof. Have students read through the proof on their own, and then describe to a classmate, in their own words, the given information and the target of the proof. Then, have them read the proof out loud to each other and note the construction and the connection made as a result of the construction. In this case, the construction is a line perpendicular to BC and the connections that result from the construction are the two equations for h. The sine law can be proved by connecting these two equal expressions for h.

Students should become familiar with reading and understanding formal proofs to help them in their future mathematical studies. At this point in their studies, it is sufficient for students to be comfortable reading and understanding a formal proof.

Example 1

This example models how to determine an unknown side length of a triangle using two methods. Have students work through both methods with a partner. Give students an opportunity to report on any difficulties they encountered.

Have student pairs complete the Your Turn question. Have one partner use right triangles and the trigonometric ratios and the other use the sine law. Have them compare solutions and explain which method they prefer and why.

Example 2

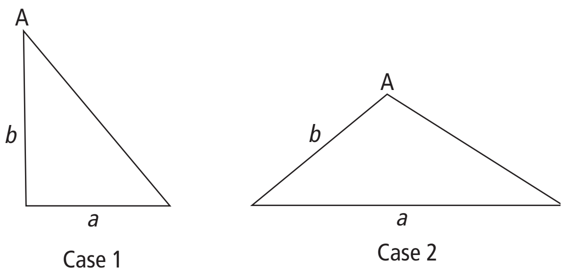
This example models using the sine law to determine an unknown angle measure. Have students work through the solution with a partner.

Have students work in pairs to solve the Your Turn question. Discuss the solution as a class.

Example 3

The ambiguous case may confuse some students. Have them consider the given angle in two cases. Use diagrams such as the following in which ∠A is acute and ∠A is obtuse.

Define ∠A as the given angle, *a* as the opposite side, and *b* as the other given side.



Have students consider the given angle in these two cases. For each, ask if the given angle is acute or obtuse.

Case 1: The given ∠A is acute.
Ask if the opposite side is larger than, smaller than, or equal to the other given side.

- If it is larger or equal, then there is one solution.
- If it is smaller, ask: What is the relationship between the opposite side and the product of the other given side and the sine of the given angle?
  - If the opposite side is larger, then there are two solutions.
  - If it is smaller, then there are no solutions.

Case 2: The given ∠A is obtuse.
Ask if the opposite side is larger, smaller than, or equal to the other given side.

- If it is smaller or equal, then there is no solution.
- If it is larger, then there is one solution.

Have students construct a table listing the information for an acute and an obtuse angle, the relationship between side *a* (representing the opposite side) and side *b* (other given side), and the relationship between *a* and the product of *b* sin A. Have them indicate the solutions that are possible in each case.

Ambiguous Case and the Sine Law

Given Angle	Sides	Product	Solutions
acute	$a \geq b$	$a = b \sin A$	one solution
	$a < b$	$a < b \sin A$	no solution
		$a > b \sin A$	two solutions
obtuse	$a \leq b$		no solution
	$a > b$		one solution

Example 3 demonstrates using the sine law in an ambiguous case. Students should be able to determine what condition leads them to consider an ambiguous case.

Have students work through the solution in pairs or small groups. Have them identify the given information that leads them to consider the ambiguous case. Once they have determined the necessary relationship between the sides and angles, ask them to determine the value of *b* sin A for the given triangle and compare that value to the length of side *a*. Have them refer to the table they created for the ambiguous case and ask what type of solution they expect given the relationship between side *a* and *b* sin A.

After using the sine law to determine ∠B, ask students:

- How might you determine the second possible measure of ∠B?
- In what quadrants do you expect the sine ratio to be positive?
- Using the original measure of ∠B as a reference angle, how can you find the second possible angle?

Have students work in pairs to solve the Your Turn. Use some of the following prompts to assist students:

- What conditions would lead you to consider the ambiguous case?
- Do any of these conditions exist in the Your Turn? Explain.
- How can you determine the number of possible solutions?
- After you solve for one possible angle, how can you determine the second angle, if it exists?

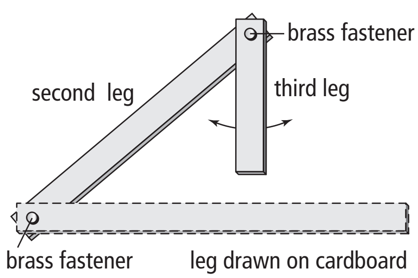
As a class, discuss the solution.

Key Ideas

Have students record their own summary of the Key Ideas. Have them store the summary of the Key Ideas for each section of the chapter in the same location for review purposes.

Meeting Student Needs

- Invite students to work in groups to create a poster featuring an oblique triangle and the sine law. Alternatively, have students add the sine law to the poster about the primary trigonometric ratios they created at the beginning of the chapter.
- Remind students that *solving a triangle* means to determine all unknown sides and angles.
- Hand out the oblique triangles prepared for the investigation and have students use the measurements to check the sine law. Ask them if the sine law is accurate.
- For the ambiguous case, when the given ∠A is acute, students may wish to use different lengths of craft sticks to model possible lengths of side *a*, to help understand the observations on page 107 in the student resource. Some students may benefit from a concrete representation, such as using craft sticks to model poles for a tipi.
- Consider using the following manipulative to help visual and concrete and kinesthetic learners explore conditions for the ambiguous case of the sine law. On a piece of cardboard, draw one leg of a triangle (using a dotted line). Have students attach a second leg of predetermined length to the drawn line using a brass fastener that goes through the cardboard. Have students measure the angle at this vertex. Secure the brass fastener so that the angle is fixed. Attach a third leg to the second one using a brass fastener. The angle between these two legs should be flexible. Have students manipulate the third leg to determine if a triangle can be made. Then, have them use the sine law and the information about the ambiguous case to calculate the possibility mathematically.



- Have students explain why Example 1 does not represent an ambiguous case.

ELL

- Ensure that students add the term *ambiguous case* to their vocabulary dictionary. Encourage them to include a verbal description and examples.
- For Example 1, clarify the meaning of *transit*. You may wish to refer to the Web Link in the next column for information on how a transit works.

Common Errors

- Some students may not recognize when they can use the sine law to solve problems.

R_x When solving oblique triangles, tell students to look for a sine pair (knowing an angle and its opposite side). They should ask themselves:
  - Do I know two angles of the triangle? If I do, I can find the third angle using the sum of angles of a triangle = 180°.
  - Do I know an angle and its opposite side? If I do, I can use the sine law.

- Some students may not recognize the ambiguous case.

R_x When using the sine law, tell students to always consider the possibility of an ambiguous case. They should ask themselves: Do I know two sides and an opposite angle (SSA)? If I do, what relationship exists between the side opposite the given angle (a) and the product of the other side and the sine of the angle (b sin A)?
  - Is $a < b$ sin A? If so, there is no solution.
  - Is $a = b$ sin A? If so, there is one solution.
  - Is $a > b$ sin A? If so, there are two solutions.

- Students may forget to check for a possible second solution in the SSA case.

R_x Point out to students that there are two angles (one acute and one obtuse) that have the same sine value if the sine law is used to obtain a second angle of the triangle.

Web Link

For free programs for the TI-83/84 family of calculators, go to www.mhrprecalc11.ca and follow the links. Teachers and students may find the following programs valuable for this chapter: Triangle Solver [trisolver.zip], Law of Cosines (show work) [cos.zip], and Law of Sines (show work) [sin.zip].

For a description of how a transit works, go to www.mhrprecalc11.ca and follow the links.

Answers

Example 1: Your Turn

The distance from Pudluk's friend's cabin to the store is 3.1 km.

Example 2: Your Turn

The measure of ∠N, to the nearest degree, is 80°.

Example 3: Your Turn

$a > b \sin A$, but $a > b$; therefore, there is only one solution.
If $a > b \sin A$ and $a < b$, then there would be two solutions.
In acute △ABC, ∠A = 39°, ∠B = 26.7°, ∠C = 114.3°, a = 14 cm, b = 10 cm, c = 20.3 cm

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• When solving for the unknown side of an oblique triangle, be aware that some students may prefer Method 1, since the process is repetitive and familiar. Challenge students who prefer to use the Pythagorean Theorem to attempt the sine law and verify their answer. • It may be necessary to help students recall how the angle and the side opposite it correspond to the sine law. • Remind students to round the answer only at the end of the solution.
Example 2 Have students do the Your Turn related to Example 2.	• When using the sine law to solve for an unknown angle, students may benefit from recalling which angles and sides form the ratios of the sine law. Remind students that they can have only one unknown in two pairs of ratios. • For students who prefer using the Pythagorean Theorem, ask them to verbalize where they will draw the altitude. You may need to model a diagram showing an altitude coming from vertex M to LN. • Remind students to round answers only at the end.
Example 3 Have students do the Your Turn related to Example 3.	• Some students may be confused by the term *ambiguous*. Refer them to the explanation on page 104 in the student resource, which also provides a summary of the three types of solutions. Encourage students to record their own summary in their math journal for reference. • Students may find it easier to begin with the case in which there are two possible triangles. Work through an example as a class before having students work in pairs to solve an additional problem. Ask student pairs to show their work and provide both possible solutions. • Remind students to round answers only at the end.

Check Your Understanding

Practise

These questions allow students to reinforce their skills in solving obtuse triangles using the sine law, including those that involve the ambiguous case. Encourage students to always check for the ambiguous case by asking: Does the given information involve two angles and a side (ASA) or two sides and an angle (SSA)? Only information for two sides and an angle (SSA) may lead to the ambiguous case. If students think that a problem involves the ambiguous case, they should list the relationships between the opposite side a and the other given side b, as well as the relationship between the opposite side a and the product of $b \sin A$. Encourage them to refer to the table they created for the ambiguous case and the sine law to determine the number of solutions.

Students can complete these questions individually or with a partner. After they complete a question, remind students to check the answer. In cases where there is a difference between their answer and the one in the answer key, encourage them to get help from a classmate or from you.

Apply

These questions provide opportunities for students to apply the sine law to real-life situations and require them to follow a process sketching diagrams that includes identifying known sides and angles, determining if the sine law is appropriate, and checking for the ambiguous case to determine the number of solutions.

Have students work individually to attempt questions before comparing solutions with a partner. You might use the following prompts to coach students for specific questions.

For #12, ask students:

- Does this involve an ambiguous case? Explain.
- Do you know a sine pair of a known angle and its opposite side? If so, what is it?
- How can you use this sine pair to determine the angle the chain makes with the beam?

For #17, you may wish to suggest that students complete part b) first. Sketching the diagram may help them work through their answer to part a).

For #19, ask students how they could use Nasir al-Din al-Tusi's formal proof to help complete the table.

Extend

These questions require students to extend their knowledge by using new and previously learned skills and processes to solve problems.

Have students work in pairs to brainstorm solutions. Involve students in finding different and interesting solutions. You might use the following prompts to coach students for specific questions.

For #20, ask students:
- What information do you know?
- What is the outcome of the proof?
- What is the relationship between an angle and its opposite side?
- Does this problem allow for more than one such relationship? Explain.

For #21, ask students:
- What is the formula for the area of a triangle?
- Do you know the height of the triangle? How can you determine its height?

For #22, ask students:
- For two solutions, what is the relationship between the opposite side and the other given side?
- For two solutions, what is the relationship between the opposite side and the product of the other given side and the sine of the given angle?
- What are these relationships for one solution? no solution?

For #23, ask students how they can set up oblique triangles to determine the farthest distance that the streetlight shines from lampposts B, C, and D.

Create Connections

For #27, provide students with **BLM 2–6 Section 2.3 #27 Concept Map** to use as a template.

For #28, note that you could use the Mini Lab earlier in the section. Have students work in pairs or small groups. Students may need some guidance in their construction and some coaching to answer the related questions for each step of the Mini Lab. For step 6, ask students:
- What happens to the sine ratio of an angle as the angle increases in size?
- What angle gives the largest value for the sine ratio?
- What happens to the sine ratio as the angle increases from 90° to 180°?
- How does this affect your conjectures from steps 2 to 5?

If you use the Mini Lab for summative assessment, ensure that you present your expectations for the completed work and provide a marking rubric. If you use the Mini Lab for formative assessment, meet with the class as a whole and ask a couple of groups to lead a discussion of the results.

Project Corner

The Project Corner box introduces triangulation, which is a method of determining exact locations on a map. Explain how to use bearings. Tell students to consider three points around their source site that they could use to triangulate their position. Remind them to use the map provided.

Meeting Student Needs

- Encourage students to redraw given diagrams and sketch diagrams to model problems.
- For #4, some students may benefit from recalling that the sum of the angles in a triangle is 180°.
- Some students will need extra practice to identify the ambiguous case, including the cases about how many triangles can be formed. Flash cards are helpful; place the case on one side and the number of triangles on the other side. Create one flash card for each of the six cases.
- Allow students to work on the Apply questions in pairs. Assign each pair one question. Have them present the solution to the class. Consider using the presentation for assessment purposes.
- For the Project Corner, you might develop an example of triangulation related to students' interests, such as the location of fishing and hunting lodges.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 2–7 Section 2.3 Extra Practice** to students who would benefit from more practice.

ELL

- Some students may not be familiar with the following terms: *hot-air balloon*, *stadium*, *angle of inclination*, *chandelier*, *monument*, *cairn*, *golden triangle*, *golden ratio*, and *triangulation*. Use a combination of descriptions, visuals, and examples to assist in student understanding.
- The language in #16 to 18 may be challenging to some students. Consider not assigning these questions to English language learners, or have them work with a partner who can help them understand the questions.
- For #18, use the visual in the student resource to clarify the meaning of *radio tower*, *antenna*, *pulleys*, and *cable set up*.
- For #28, have group members assist students who have difficulty with the language in the procedure.

Enrichment

- Challenge students to consider a scenario in which there are two sightings of a meteoroid entering Earth's atmosphere. Ask them:
  - How could you use the sine law to determine the path of the meteoroid? (Students should indicate that they can use the sine law to determine unknown sides given the angles and position of the observations relative to each other and to the meteoroid.)
  - Why might it be difficult to predict the landing point of the meteoroid based on radar information? (The effect of air friction on the meteoroid changes the path it takes. Also, the burning away of mass during high-velocity travel through the atmosphere changes the flight characteristics of the meteoroid.)

Gifted

- GPS technology is a great advantage for navigation in the North. Ask students:
  - Explain why navigation is difficult in the North. (Navigation is problematic due to the distances involved, the weather, and the issues for compasses that get close to the magnetic North Pole.)
  - How is the sine law useful for determining the position of a receiver using GPS technology? (Since GPS navigation relies on the intersection of distance values from satellites, it is important to be able to determine distances and angles over the full 360° compass.)

Common Errors

- Students may set up the sine law ratios incorrectly.

R_x Have students sketch the triangle and label all sides and angles. They can highlight an angle and its opposite side, forming one sine ratio. Have them repeat for the remaining pairs of ratios.

- Students may get the incorrect number of solutions in ambiguous cases.

R_x Have students record the measure of the opposite side *a* next to the measure of the other given side *b*, and then determine the product of the other given side and the sine of the given angle or *b* sin A. Then ask them to determine which is larger—*a* or *b* and, as well, *a* or *b* sin A. Encourage students to refer to the table they created for the ambiguous case and the sine law to determine the number of solutions.

Web Link

For information about the sine law and the ambiguous case, including examples, practice questions, and animated solutions, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 11, 14, 16, and 18. Students who have no problems with these questions can go on to the remaining questions.	• For #1 to 4, students apply the sine law to solve for an unknown angle or side. Encourage students to draw a diagram for each question. Have them label the sides of each triangle using appropriate variables to help them write the sine law ratios. • For #5, students are required to sketch and label a diagram before solving. They might draw a single $\triangle ABC$ with vertices and sides labelled. This could serve as their reference for each question. • For #7 and 8, students solve triangles involving ambiguous cases. Refer students to Link the Ideas, Example 3, and their notes. Consider having students work with a partner to answer and/or compare solutions. • Encourage students to sketch diagrams for the contextual problems. • For #18, have students sketch individual triangles to help them focus on the relationships between the angles and opposite sides. • Remind students to round the answer only at the end of the solution.
Assessment *as* Learning	
Create Connections Have all students complete #24, 25, and 27.	• For #24, students show their understanding of the sine law by explaining why it cannot be used to solve the given triangles. If students struggle with deciding why the sine law cannot be used, suggest that they proceed to use the sine law to determine unknown measures. This may prompt their thinking. • For #25, students partially develop the sine law using a right triangle. Allow students to work with a partner to explain their thinking. As a class, discuss the response and clarify any misunderstandings. • For #27, students use a concept map to show their understanding of the conditions necessary for using the sine law to solve oblique triangles. The concept map is a useful summative tool for students and a useful Assessment *as* Learning tool for you. Use the concept map that students produce to identify any missing or unclear concepts, especially for ambiguous cases.

The Cosine Law

2.4

***Pre-Calculus 11*, pages 114–125**

Suggested Timing
100–120 min

Materials
- ruler
- protractor
- compass

Blackline Masters
BLM 2–3 Chapter 2 Warm-Up
BLM 2–8 Section 2.4 Extra Practice

Mathematical Processes
- ☑ Communication (C)
- ☑ Connections (CN)
- ☑ Mental Math and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☑ Technology (T)
- ☑ Visualization (V)

Specific Outcomes
T3 Solve problems, using the cosine law and the sine law, including the ambiguous case.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1a), c), 2a), 3a), b), 4a), c), e), 5, 7–9, 12–14, 18, 24, 30–32
Typical	#3, 4b), d), two of 5–10, 15, two of 18, 19, and 23, 30–32
Extension/Enrichment	#3, 4b), d), 5, 6, 10, 15, 16, 24, 25, 28–33

Planning Notes

Have students complete the warm-up questions on **BLM 2–3 Chapter 2 Warm-Up** to reinforce prerequisite skills needed for this section. If you have posted the outcomes, refer to the outcomes for this section.

As a class, read the opening paragraph and invite students to describe recent activities involving Canadarm 2 and how trigonometric ratios are involved.

The cosine law is the second of two laws used to solve triangles. Students are expected to develop their problem solving skills by developing a process for using the cosine law. They should know the conditions required for using the cosine law and when they cannot use the law. Students need to practise the basic skills and solve contextual problems to develop an understanding of the concepts involved in using the cosine law.

Investigate the Cosine Law

This investigation introduces an intuitive development of the cosine law. Students are directed to begin with a right triangle and then extend their understanding of the Pythagorean Theorem to obtuse triangles. They begin with a specific example and then expand their understanding from the example to the general case.

Have students work in pairs but record their own response to all questions. This will allow them an opportunity to build an understanding of the concepts and brainstorm solutions.

Since this investigation expects students to do a construction of the obtuse triangle, the solutions calculated for $a^2 + b^2 - 2ab \cos C$ may be close but not exact to the value of c^2. Have students use a compass to construct the line segments $\overline{AC}$ and $\overline{BC}$. They should use a ruler to measure the distance $\overline{AC} = 4$ cm, and then measure $\overline{BC}$ to be 3 cm. This may help to make the construction more accurate. Have students express their answers to the nearest whole number and compare the results for c^2 and $a^2 + b^2 - 2ab \cos C$.

Students who have difficulty with the obtuse triangle may need coaching. Use prompts such as the following:
- Does the Pythagorean Theorem hold true for obtuse triangles? Explain.
- Is the solution $a^2 + b^2$ greater than or less than the value of c^2?
- To form an equality, do you need to add an extra value or subtract an extra value from $a^2 + b^2$?
- What is the value of $2ab \cos C$?
- When you subtract the value of $2ab \cos C$ from $a^2 + b^2$, does the answer approximate the value of c^2? Explain.
- What conditions could cause these values to be close but not exactly the same?

For #5, coach students to draw a triangle with an obtuse angle at vertex C and repeat the investigation to explore the relationship between c^2 and $a^2 + b^2 - 2ab \cos C$. Afterward, ask if obtuse triangles support the same relationship as acute triangles do.

For #6 and 7, coach students by asking what side and/or angle measures are required to use the sine law and to use the cosine law.

For #8, coach students by asking how they used the Pythagorean Theorem in the investigation and how this relationship supports the cosine law.

As a class, have students discuss their response to Reflect and Respond #6 to 8.

Meeting Student Needs

- Consider preparing and labelling a triangle with specific measurements. After students complete the investigation, have them use the angles in the triangle and prove that the formula works for any given angle.

Common Errors

- Students may make errors in measurement.

R_x Explain how parallax (an apparent change in the position of an object that is viewed along different lines of sight) affects measurements. Stress the importance of looking straight down over the ruler or protractor.

- Students may make transcription and calculation errors.

R_x Explain how to reduce these errors by labelling each known side and angle on a sketch, listing each value, recording the cosine formula, and using direct substitution.

Answers

Investigate the Cosine Law

1. b) $a^2 = 9$ cm^2; $b^2 = 16$ cm^2; $c^2 = 25$ cm^2 **c)** $a^2 + b^2 = c^2$ **d)** 90°

2. Example:
b) $a = 5$ cm, $b = 4$ cm, $c = 6$ cm
c) $a^2 = 25$ cm^2; $b^2 = 16$ cm^2; $c^2 = 36$ cm^2
d) $a^2 + b^2 > c^2$

3. a) 0 **b)** 5

4. The value $2ab \cos C$ when subtracted from $a^2 + b^2$ equals c^2. Or, $c^2 = a^2 + b^2 - 2ab \cos C$.

5. Example: Yes. The logic from step 4 holds true for an obtuse angle.

6. The cosine law requires that two sides and the angle between them are known or that all the sides are known.

7. a) The sine law requires that an angle and its opposite side are known. This triangle does not have this sine pair.
b) Using the cosine law and the formula $a^2 = b^2 + c^2 - 2bc \cos A$, $a \approx 16.6$ cm.

8. The cosine law uses an extension of the Pythagorean Theorem for oblique triangles.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the investigation. Encourage them to generalize and reach a conclusion about their findings.	• Consider having students complete #1 to 4 with a partner or as a class. • You might demonstrate #5 as a class and use the opportunity to outline the reasons for using the cosine law when neither the sine law nor the Pythagorean Theorem is appropriate. This should prompt sufficient discussion for students to answer #6 and 7 on their own. • Consider having students exchange their work for #7 with a partner and check each other's response. This might help students to clarify their thinking.

Link the Ideas

The Link the Ideas offers students another opportunity to examine a formal proof. Students should use this opportunity to prepare for formal proofs in future mathematics studies. Have students read through the section and the proof, respond to the prompts in the student resource, and then participate in a class discussion.

Some students may benefit from coaching. Ask students:

- Do you agree with the relationships listed for x and a^2 in the proof? Explain.
  - If $a^2 = h^2 + x^2$, does $h^2 + x^2 = a^2$?
  - If $x = a \cos C$, does $2bx = 2b(a \cos C) = 2ab \cos C$?
- Do you agree with the proof as recorded in the student resource? Explain.

Example 1

This example demonstrates using the cosine law to determine a distance. Have students attempt the solution before working through the given solution. Students will have a greater investment in the given solution after attempting the question. Give students an opportunity to raise any concerns in a class discussion.

Have students solve the Your Turn and compare the solution with a classmate. Doing so helps students make immediate connections with the skills and processes required to solve problems similar to the worked example.

Example 2

This example demonstrates two methods for applying the cosine law to determine an angle: substitution and rearranging the formula. Use a similar approach as for Example 1 by having students attempt the solution before working through the one given. Then, walk through the solution as a class.

Have students work in pairs and encourage them to solve the Your Turn using both methods. Have them compare the solutions before discussing the solutions as a class.

Example 3

This example models solving a triangle for unknown angles and a side. Have students attempt the solution before working through the one given. Then, walk through the solution as a class.

Have students solve the Your Turn and compare the solution with a classmate before discussing the solution as a class.

Key Ideas

Have students record their own summary of the Key Ideas. Have them store the summary of the Key Ideas for each chapter section in the same location for review purposes.

Meeting Student Needs

- Invite students to work in groups to create a poster featuring an oblique triangle and the cosine law.
- For the formula for the cosine law, show students that the beginning letter matches the given angle. For example, $\boldsymbol{m}^2 = n^2 + v^2 - 2nv \cos \mathbf{M}$.
- Provide several questions for students to practise setting up the formula and then solving for cos X (where X is any letter).

ELL

- For Example 2, use the photograph in the student resource to clarify the meaning of *suspension bridge* and *triangular braces*.

Common Errors

- Some students may square the terms and subtract the product of $2bc$ only, not recognizing that $2bc \cos A$ is one term.

R_x Remind students about the order of operations.

Web Link

For information about the cosine law, go to www.mhrprecalc11.ca and follow the links.

For information about when and how to use the sine law and the cosine law to solve oblique triangles, go to www.mhrprecalc11.ca and follow the links.

Answers

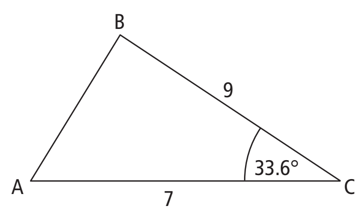
Example 1: Your Turn

The distance AB is 40.1 m.

Example 2: Your Turn

The measure of the angle opposite the 18-m side is 55°.

Example 3: Your Turn



$c = 5.0$; $\angle A = 95.7°$; $\angle B = 50.7°$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• For solving an unknown side of an oblique triangle, students may find it helpful to draw and label the sides of the triangle using the corresponding variables, and then substitute the values into the formula for the cosine law. If students prefer to use the Pythagorean Theorem, ask them to identify how they would solve for the length of side *a*. • Some students may benefit from recalling how an angle and its opposite side fit into the cosine law. • Remind students not to round until the end of the solution.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 2 Have students do the Your Turn related to Example 2.	• For solving an unknown angle, encourage students to draw their own diagram and identify which angles and sides will be used in applying the cosine law. • Encourage students to review the diagrams and use the side lengths to estimate the size of the angles. Coach students to identify that the largest angle is opposite the longest side. • Encourage students to try both methods. Some students may erroneously subtract before multiplying; remind them of the order of operations. • Coach students to use the second function key for cosine on their calculator. • Remind students not to round until the end of the solution.
Example 3 Have students do the Your Turn related to Example 3.	• As students work to determine the unknown measures, point out that some problems can be solved by using the cosine law initially for a calculation and then using the sine law. • Have students sketch and label a diagram of the triangle in the Your Turn. Ask what they would solve for, using the cosine law. After determining this unknown side or angle, ask what else they could use to further solve the triangle, and what law they could apply. • Some students may benefit from recalling how to use the second function key for cosine.

Check Your Understanding

Practise

These questions allow students to practise solving obtuse triangles using the two forms of the cosine law.

Encourage students to ask themselves the following questions when solving obtuse triangles:

- What is the given information for each triangle?
- What am I trying to determine?
- With the given information, should I use the sine law or the cosine law?
- Which form of the cosine law do I use to solve for an unknown side? an unknown angle?

Have students complete these questions individually or with a partner. After completing a question, remind students to check the answer. In cases where there is a difference between their answer and the one in the answer key, encourage them to get help from a classmate or from you.

Apply

These questions provide opportunities for students to apply the cosine law to real-life situations to determine an unknown side given SAS information and an unknown angle given SSS information.

Have students work individually to solve the problems before comparing solutions with a classmate. You might use the following prompts to coach students for specific questions.

For #5, ask students:

- What information is required to use the sine law for solving an unknown angle? an unknown side?
- What information is required to use the cosine law for solving an unknown angle? an unknown side?

For #6, ask students:

- When determining an exact value in trigonometry, which special triangles are you expected to use?
- What are the exact value ratios for 30° angles? 45° angles?

For #7, encourage students to sketch and label the parallelogram. Ask them:

- Which diagonal is the longest? Is it the diagonal opposite the obtuse angle?
- Does this form an obtuse triangle? Explain.
- What trigonometry law would you use to determine the diagonal length? Explain why.

For #8 and 9, encourage students to sketch and label the triangle. Ask them to state which trigonometry law they would use and explain why.

For #13, have students sketch and label a diagram. Have them indicate the maximum width on the diagram and draw a line to show it. (Look for a line that connects the bottom sides of the two equilateral triangles.) Ask students:

- How can you determine the maximum width of Moondog?
- How can you determine the lengths of the bottom sides of the two equilateral triangles?
- How can you find the length of the line you drew to show the maximum width?

For #16, ask students what the sum is of all the interior angles between the line segments that are 7 mm, 23 mm, and 16 mm in length.

Extend

For #23, encourage students to sketch a diagram and indicate the direction and distance the two ships have travelled. Ask students:

- How far does the ship travelling at 11.5 km/h go from 4 p.m. to 6 p.m.?
- How far does the ship travelling at 13 km/h go from 4 p.m. to 6 p.m.?
- What is the angle between N and N38°E (for the ship going 11.5km/h)?
- What is the angle between S and S47°E (for the ship going 13 km/h)?
- What is the angle between the two ships' directions?
- Which trigonometry law would help determine the distance between the ships? Explain.

For #24, ask students what the possible values for the cosine ratio of any angle are.

For #26, ask students:

- How can you determine the distances between points A and B? points A and C? points B and C?
- What is the sum of ∠ACB and ∠ACD?

For #27, ask students for the formula for the area of a triangle.

For #28, you may wish to have students label the vertices A, B, C counterclockwise starting at the top. Ask students:

- How can you determine the measure of ∠CAB? ∠ABC? ∠BCA?
- Can you draw radii from the centre of the circle to each of the three points on the circle? Try it.
- What relationship exists between the two angles opposite the equal sides of isosceles triangles?
- What relationship exists between ∠ACO and ∠CAO? ∠OBC and ∠BCO? ∠BAO and ∠ABO?
- How can you use these relationships to determine ∠ACO, ∠CAO, and ∠AOC?
- If you know ∠AOC, ∠ACO, and side AC, can you determine the length of the radii? Explain.

For #29, ask students:

- How can you determine the distances AB, BC, and AC?
- Can you prove the cosine law using substitution into the cosine law formula? Explain.
- What do you notice about the expressions representing c^2 and $a^2 + b^2 - 2ab \cos C$?

Create Connections

For #33, which is a Mini Lab, have students work in pairs or small groups. Students may need some guidance in their construction and some coaching to answer the related questions for each step of the Mini Lab.

Consider the following suggestions to assist students:

- For step 1, to construct the triangle, suggest using a ruler, protractor, and a compass. Have students draw the longest line segment of 8 cm and label the ends C and A. Use a ruler and compass to measure a distance of 4 cm, and from point C, scribe an arc above AC. At point A, using the ruler and compass, scribe an arc of 6 cm above AC that intersects the first arc at point B. Join points A and B and points C and B. Use the protractor to produce right angles at the endpoints of each line segment. Again, use the ruler and compass to scribe a distance of 4 cm, 6 cm, and 8 cm. Label the endpoint of the 4-cm line from point B as point G, and label the endpoint of the line from point C as point H, forming the square BCHG. Label the endpoint of the 6-cm line from point B as point F, and label the endpoint of the line from point A as point D, forming the square ABFD. Label the endpoint of the 8-cm line from point A as point E, and label the endpoint of the line from point C as point I, forming the square ACIE. This method of construction should help to minimize construction errors.
- For step 2b), ask students:
  - What are the angle measures for any square?
  - What is the measure of ∠FBA? ∠GBC?
  - What is the sum of ∠GBF and ∠ABC?
  - Does this logic work for ∠DAE and ∠BAC? for ∠HCI and ∠BCA? Try it.
- For step 2c), ask students:
  - Given that you know an angle and the measure of the two adjacent sides, what law can you use to find the measure of the sides GF, ED, and HI?
- For step 3, ask students:
  - When you construct an altitude, what is the measure of the angles on the base line?
  - What kind of triangle is formed by one side of the square, part of the base line, and the altitude?
  - What primary trigonometric ratio can you use to determine the height of the altitude?
  - What formula can you use to determine the area of a triangle given the length of the base and the altitude?
- For step 4, allow students to brainstorm possible answers and then share ideas as a class.

If you are using the Mini Lab for summative assessment, ensure that you present your expectations for the completed work and provide a marking rubric. If you are using the Mini Lab for formative assessment, meet with the class as a whole and ask a couple of groups to lead a discussion of the results.

Project Corner

The Project Corner box introduces and models trilateration, which is another method of determining exact location on a map. Have student pairs explain how they could locate their resource.

Meeting Student Needs

- Encourage students to redraw given diagrams and sketch diagrams to model problems. Invite students to compare diagrams with a classmate.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 2–8 Section 2.4 Extra Practice** to students who would benefit from more practice.

ELL

- Some students may not be familiar with the following terms: *tunnel*, *sea-run brook trout area*, *aircraft tracking station*, *helicopter*, *tuberculosis*, *port*, *angle of depression*, and *trilateration*. Use a combination of descriptions, visuals, and examples to assist in student understanding.
- For #9, some students may not be familiar with sailing race competitions. Ask a student who has knowledge of sailing competitions to explain the different categories. You might refer students to the Did You Know? that follows the question.
- For #14, you may wish to clarify *compass bearings*.
- The language in #18, 19, and 27 may be challenging to some students. Consider not assigning these questions to English language learners, or have them work with a partner who can help them understand the questions.
- For #33, have group members assist students who have difficulty with the language in the procedure.

Enrichment

- Challenge students to develop a contextual cosine law question and provide the solution. Students should correctly apply the cosine law.

Gifted

- Have students consider Halley's Comet which orbits the sun. It reappears every 76 years on average, but the interval varies from 76.0 years to 79.3 years. Ask students to show how the cosine law could be used to track its orbit. Students should show an example of applying the cosine law to solve for a great distance in space. You might ask them to research why the orbits vary. (The orbit of Halley's Comet changes as a result of the gravitational influences of planets and other celestial objects in space whose position changes relative to the comet's path during its elliptical orbit.)
- Challenge students to research the thinking that led to the cosine law. Have them explain the thinking in their own terms. Students may find the Web Link at the end of this section useful. Look for clear and precise mathematical thinking and correct understandings of both the procedural and conceptual thinking.

Common Errors

- Students may not identify the known information and/or what is being asked for.

R_x Ensure that students sketch their own diagram and label all known angles and sides, and identify the angle(s) and side(s) to be determined.

- Students may not select the correct trigonometry law to solve a problem.

R_x Ensure that students record all known values as well as the unknown value(s) to be determined. Then have them select the correct trigonometry law based on this information, substitute the known values, and solve for the unknown value.

Web Link

For information about the cosine law, including examples and practice problems, go to www.mhrprecalc11.ca and follow the links.

For a demonstration of using the cosine law for triangles where two sides and the included angle are given, and a triangle where all three sides are given, go to www.mhrprecalc11.ca and follow the links.

For an explanation of the development of the cosine law, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1a), c), 2a), 3a), b), 4a), c), e), 5, 7 to 9, 12 to 14, 18, and 24. Students who have no problems with these questions can go on to the remaining questions.	• Remind students that they now have several tools to use when solving triangles (e.g., Pythagorean Theorem, primary trigonometric ratios, sine law, cosine law). • For #1 to 3, students apply the cosine law to solve for an unknown angle or side. Encourage students to sketch their own diagram for each question. Have them label the sides of each triangle using appropriate variables to help them use the cosine law. • For #4, students are required to sketch and label their own diagram before solving. They might draw a single △ABC with vertices and sides labelled. This could serve as their reference for each question. • For #5, students analyse the given information and decide which law to use for solving. Allow students to explain their thinking to a classmate. • Encourage students to sketch triangles, if they are not provided, for contextual problems before trying to solve. Some students may find it helpful to refer to the worked examples when attempting to solve these problems. • Remind students to round the answer only at the end of the solution.
Assessment *as* Learning	
Create Connections Have all students complete #30 to 32.	• For #30, encourage students to draw two separate triangles, if necessary, and solve each triangle for the distance the car has travelled. Then, complete the question. • For #31, students compare using the cosine law and the Pythagorean Theorem to solve a right triangle. Encourage them to draw and label a triangle before solving using both methods. Allow students to explain their thinking for part d) to a classmate. • For #32, students complete a summary of methods for solving a triangle. Students may find it beneficial to refer to the summaries they completed in previous sections to help complete this one. The summary is a useful summative tool for students and a useful Assessment *as* Learning tool for you. Have students outline their summary to a classmate, before discussing as a class. Clarify any misunderstandings.

2 Chapter 2 Review

***Pre-Calculus 11*, pages 126–128**

Suggested Timing

60—90 min

Materials

- centimetre grid paper
- ruler
- protractor

Blackline Masters

Master 2 Centimetre Grid Paper
BLM 2—4 Section 2.1 Extra Practice
BLM 2—5 Section 2.2 Extra Practice
BLM 2—7 Section 2.3 Extra Practice
BLM 2—8 Section 2.4 Extra Practice

Planning Notes

Have students who are not confident identify strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 2–4 Section 2.1 Extra Practice**, **BLM 2–5 Section 2.2 Extra Practice**, **BLM 2–7 Section 2.3 Extra Practice**, or **BLM 2–8 Section 2.4 Extra Practice**.
- Before students begin the questions, invite them to review their summary of the Key Ideas for each section. Invite students to clarify any misunderstandings.
- Have students review prepared flash cards, the concept map of the sine law, and the summary tables.
- If it has not been done already, post all learning outcomes. Invite students to ask questions about any outcomes that they do not understand.
- Display posters at the front of the classroom for easy reference as students work through the review.
- Individualize the chapter review. Have students choose three questions from each section to begin. Correct the questions and analyse errors. Encourage students to request assistance for the questions they are unable to complete successfully. Students can then choose additional practice questions based on their results.

ELL

- Use a combination of visuals, descriptions, and examples to help students understand such terms as *heating lamp*, *angle of elevation*, *cruising altitude*, *deep-sea fishing charter*, and *port*.
- The language in #21 may be challenging to students who are not familiar with golf. Ask a student who has knowledge of golf to explain the terms.

Enrichment

- Challenge students to solve the following problem: In Regina, SK, there is an annual celebration of cultures called Mosaic. Mosaic features different pavilions located throughout the city. Jan decided to visit the Francophone, Caribbean, and Aboriginal Peoples pavilions, which form a triangle when mapped. Starting at the Francophone pavilion, she noticed on the map that the angle toward the Aboriginal Peoples and Caribbean pavilions was 63.5°. She walked 6.3 km to the Caribbean pavilion and noticed that the angle between the Francophone and Aboriginal Peoples pavilions was 51.2°. What distance would she have to travel to go to the Aboriginal Peoples pavilion? What is the distance from the Aboriginal Peoples pavilion back to the Francophone pavilion?

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 2 Review The Chapter 2 Review is an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource.	• Have students revisit any section that they are having difficulty with prior to working on the chapter test. • Encourage students to refer to the summary notes they completed throughout the chapter as well as the concept map about using the sine law to solve oblique triangles.

Chapter 2 Practice Test

2

***Pre-Calculus 11*, pages 129–130**

Suggested Timing
30—45 min

Materials
- centimetre grid paper
- ruler
- protractor

Blackline Masters
Master 2 Centimetre Grid Paper
BLM 2—9 Chapter 2 Test

Planning Notes

Have students indicate which practice test questions they need no help with, a little help with, and a lot of help with. Have students first complete the questions they know they can do, followed by the questions they know something about. Finally, suggest to students that they do their best on the remaining questions. Ensure that students get coaching for questions that they indicated they need help with.

You can assign this practice test as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1—6, 8—10, 12, 14.

Study Guide

Question(s)	Section(s)	Refer to	The student can …
#1, 2, 13	2.1	Link the Ideas Example 2	✓ sketch an angle, from 0° to 360° in standard position and determine its reference angle ✓ determine the quadrant in which an angle in standard position terminates
#3	2.1	Link the Ideas	✓ determine the exact values of sine, cosine, and tangent ratios of a given angle with the reference angle of 30°, 45°, or 60°
#4, 7, 10, 12, 16, 17	2.3	Link the Ideas Example 1 Example 2	✓ use the primary trigonometric ratios to solve problems involving triangles that are not right triangles ✓ sketch diagrams to represent problems involving the sine law ✓ solve problems using the sine law
#5, 8	2.3	Link the Ideas Example 3	✓ solve problems which include the ambiguous case of the sine law
#6	2.2	Link the Ideas Example 3	✓ determine the distance from the origin to a point (x, y) on the terminal arm of an angle ✓ determine the value of $\sin \theta$, $\cos \theta$, or $\tan \theta$ for any point (x, y) on the terminal arm of angle θ
#9, 11	2.4	Example 2	✓ solve problems using the cosine law
#14	2.4	Example 1	✓ sketch diagram and solve problems using the cosine law
#15	2.3 2.4	Link the Ideas Link the Ideas	✓ recognize when to use the sine law to solve a given problem ✓ recognize when to use the cosine law to solve a given problem

Assessment	Supporting Learning
Assessment *as* Learning	
Chapter 2 Self-Assessment Have students use their responses on the practice test and work they completed earlier in the chapter to identify skills or concepts they may need to reinforce.	• Students may wish to review their chapter summary notes and concept map before they begin the practice test. Students can use these to identify any areas of weakness. • Before the chapter test, coach students in areas in which they are having difficulties.
Assessment *of* Learning	
Chapter 2 Test After students complete the practice test, you may wish to use **BLM 2–9 Chapter 2 Test** as a summative assessment.	• Consider allowing students to use their summary notes and concept map to complete the practice test.

CHAPTER 2

Unit 1 Project

***Pre-Calculus 11*, page 131**

Suggested Timing

60–90 min

Blackline Masters

BLM U1–1 Unit 1 Project Checklist
BLM U1–2 Chapter 2 Task Map

Mathematical Processes

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

General Outcome

Develop trigonometric reasoning.

Specific Outcomes

T1 Demonstrate an understanding of angles in standard position [0° to 360°].
T2 Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.
T3 Solve problems, using the cosine law and sine law, including the ambiguous case.

Planning Notes

Take time to review the Project Corners in Chapter 2. Help students recall the skills involved in creating and/or interpreting a map.

For the Chapter 2 Task, students create or use the exploration map provided on **BLM U1–2 Chapter 2 Task Map** to determine the total distance of the route to the discovery of their chosen natural resource. They also determine all measures of a triangular region that could be developed.

During their project work, students may wish to talk about their ideas with each other. Peer coaching will assist many students in improving their skills.

Assessment	Supporting Learning
Assessment *for* Learning	
Unit 1 Project This unit project gives students an opportunity to apply and demonstrate their knowledge of • the sine law • the cosine law	• You may wish to have students use the part of **BLM U1–1 Unit 1 Project Checklist** that provides a list of the required components for the Chapter 2 part of the Unit 1 project. • Discussing the Project Corner boxes at the end of some sections of Chapter 2 will assist students in describing their route of discovery of the resource and the planned area of the resource.

UNIT 1

Unit 1 Project Wrap-Up

***Pre-Calculus 11*, page 132**

Suggested Timing

60–90 min plus individual time

Blackline Masters

Master 1 Project Rubric
BLM U1–1 Unit 1 Project Checklist
BLM U1–3 Unit 1 Project Rubric

Mathematical Processes

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

General Outcome

Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes

RF9 Analyze arithmetic sequences and series to solve problems.
RF10 Analyze geometric sequences and series to solve problems.

General Outcome

Develop trigonometric reasoning.

Specific Outcomes

T1 Demonstrate an understanding of angles in standard position [0° to 360°].
T2 Solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.
T3 Solve problems, using the cosine law and sine law, including the ambiguous case.

Planning Notes

At the beginning of the unit, students were challenged to consider one of Canada's natural resources that they would be interested in learning more about by collecting and presenting data. Project Corner boxes throughout Chapters 1 and 2 provided sample data on various natural resources for students to consider and work with. The boxes in Chapter 2 also included information about triangulation and trilateration that provided opportunities for students to develop the skills needed for the project wrap-up.

Students are now asked to develop the final report, which involves the following:

- using the data collected to show what they have learned about arithmetic sequences or series, geometric sequences or series, and/or infinite geometric series
- using what they have learned about sine and cosine laws in developing a fictitious account of a recent discovery of their resource, including a map of the area showing relevant distances
- using both sets of skills in a proposal to develop the new resource area over a number of years

Make sure students are aware of the project information provided on page 3, the chapter tasks on pages 71 and 131, and the Unit 1 Project Wrap-Up on page 132 of the student resource. If students did not complete each part of the chapter task at the end of Chapters 1 and 2, have them do so now, and then use this information to prepare their presentation.

Have students use **BLM U1–1 Unit 1 Project Checklist** to make sure that all parts of their project have been completed. As a class, brainstorm different ways students can do their presentations. You may wish to limit the time each student has to present.

Students do best if they know exactly how they will be evaluated. One way to increase student motivation is to work with the class to create a rubric for the project. You may wish to use the **Master 1 Project Rubric** template and review the general holistic points within the 1–5 scoring levels. Explain to students how they might achieve each of these levels in the Unit 1 project.

Ask questions such as:

- What are the big ideas in the unit? (for example, understanding arithmetic and geometric sequences and series, understanding the sine and cosine laws)
- Which of the big ideas are involved in the project?
- What part of the project could you complete or get partially correct to indicate that you have a basic understanding of what was learned in the two chapters? Should you get a pass mark if you can show that you understand arithmetic and geometric sequences but not series and can model the sine law but not the cosine law? The answer is yes. Help students to understand that completed projects and partial understandings of concepts, in which most of the major concepts are started correctly, should be sufficient for a level 3 score.

- What would be on a level 1 project? What might you start on correctly? What could be considered a significant start?
- What would be expected for a level 5 project? What should it include? Try to help students realize that a level 5 project may have a minor error or omission that does not affect the final result.
- Knowing the expectations of levels 1, 3, and 5 projects, what would you expect for a level 4? Help students to understand that this is still an honours level and therefore the work should be reflective of this. However, even an honours project may have a minor error or omission. Explain the difference between a major conceptual error and a minor miscalculation or omission. Understanding this point will help clarify for students the expectations and differences between a pass and above-average result, and may encourage some students to work toward the highest level. Repeat the process for a level 2 project.

BLM U1–3 Unit 1 Project Rubric models a completed rubric for this project. Note that this is one idea for completing a rubric. Your class rubric may have more detail. Use what your students have developed or this example to ensure that students understand the criteria for an acceptable level, as well as what would warrant either an unacceptable or an honours grading.

Assessment	Supporting Learning
Assessment *of* Learning	
Unit 1 Project This unit project gives students an opportunity to apply and demonstrate their knowledge of the following: • arithmetic sequences and series • geometric sequences and series • infinite geometric series • sine and cosine laws Work with students to develop assessment criteria for this project. **Master 1 Project Rubric** provides a holistic descriptor that will assist you in assessing students work on the Unit 1 project.	• You may wish to have students use **BLM U1–1 Unit 1 Project Checklist**, which provides a list of the required components for the Unit 1 project. • Reviewing the Project Corner boxes at the end of some sections of Chapters 1 and 2 will assist students in developing appropriate data presentations. • Make sure students recognize what is expected for the minimum requirements for an acceptable project as well as the difference between level 5 and level 4. • Clarify the expectations and the scoring with students using **Master 1 Project Rubric** or the rubric you developed as a class. It is recommended to review the scoring rubric at the beginning of the project, as well as intermittently throughout the project to refresh student understanding of the project assessment.

Cumulative Review and Test

UNIT 1

***Pre-Calculus 11*, pages 133–137**

Suggested Timing
60—90 min

Materials
- 0.5 centimetre grid paper
- centimetre grid paper
- ruler
- protractor

Blackline Masters
Master 2 Centimetre Grid Paper
Master 3 0.5 Centimetre Grid Paper
BLM U1—4 Unit 1 Test

Meeting Student Needs
- Encourage students to draw and label diagrams, when appropriate.

ELL
- For terms that students are unfamiliar with, coach them, or have another student coach them, using a combination of visuals, descriptions, and examples.

Enrichment
- Have students develop new questions that are similar to the questions in the review and test and in the chapters, or completely original. Students can then exchange questions and answer them for further practice.

Planning Notes

Have students work independently to complete the review and then compare their solutions with those of a classmate. Alternatively, you may wish to assign the cumulative review to reinforce the concepts, skills, and processes learned so far; if students encounter difficulties, provide an opportunity for them to identify strategies with other students. Encourage them to refer to their notes, and then to the specific section in the student resource. Once they have determined a suitable strategy, have students add it to their notes. Consider having students make a list of the questions that they found difficult. They can then use the list to help them prepare for the unit test.

Assessment	Supporting Learning
Assessment *for* Learning	
Cumulative Review, Chapters 1 and 2 The cumulative review provides an opportunity for students to assess themselves by completing selected questions pertaining to each chapter and checking their answers against the answers in the back of the student resource.	• Have students review their notes and tests from each chapter to identify items that they had problems with, and do the questions related to those items. Have students do at least one question that tests skills from each chapter. • Have students revisit any chapter section they are having difficulty with.
Assessment *of* Learning	
Unit 1 Test After students complete the cumulative review, you may wish to use the unit test on pages 136 and 137 as a summative assessment.	• Consider allowing students to refer to their summary notes and concept map of the sine law. • You may wish to have students complete **BLM U1–4 Unit 1 Test**, which provides a sample unit test. You may wish to use it as written or adapt it to meet the needs of your students.