Radical Expressions and Equations

***Pre-Calculus 11*, pages 270–271**

Suggested Timing
20–30 min

Blackline Masters
BLM 5–2 Chapter 5 Prerequisite Skills
BLM U3–1 Unit 3 Project Checklist

Key Terms
rationalize
conjugates
radical equation

What's Ahead

In this chapter, students develop their abilities to simplify radical expressions and perform the four operations (addition, subtraction, multiplication, and division) on these expressions. In order to add and subtract radical expressions, students need to be able to identify like radicals during the simplification process. Then, in order to divide and simplify expressions with a radical in the denominator, students need to be able to rationalize the denominator. They learn this skill prior to learning how to divide radicals. Students then use their ability to perform operations on radical expressions to solve radical equations. Students also determine the restrictions on the values of variables in radical expressions that are real numbers, and they learn to identify extraneous roots in radical equations.

Planning Notes

As a class, talk about the chapter opener. Have students mention situations in which they believe radical expressions and equations can be used or where they have seen them used in everyday life. Direct them to the contour graph showing temperature. You might mention that many formulas, such as the one for calculating the amount of current that an appliance uses, involve radical equations. You might have students use the photo collage for other ideas about how radical equations are used. You may wish to mention that a planet is shown to illustrate space exploration and the skier shown is Jan Hudec of Calgary, AB. The storm damage shown is a result of the tsunami that occurred after the earthquake in Chile in March 2010.

Tell students that in this chapter they will build on their existing knowledge and skills of radical expressions and equations.

As students progress through the chapter, have them record the Key Terms and develop their own definitions. They can refer to the definitions in the student resource. Have students explain their understanding of the Key Terms.

Direct students to the Career Link about meteorologists. Have them discuss what they know about the work that meteorologists do, and how math skills are related to their work. As a class, you might highlight the careers related to radical equations throughout the chapter. Challenge students to consider what other careers might use radical expressions and radical equations.

Unit Project

You might take the opportunity to discuss the Unit 3 project described in the Unit 3 opener. The Project Corner at the end of section 5.1 gives students who are interested in the Milky Way galaxy an opportunity to explore radical expressions and equations. The Project Corner at the end of section 5.2 offers students who are interested in space science some information about topics they might explore for the project.

The Project Corner boxes provide information related to the unit project. These features are not mandatory but they are recommended because they provide some background for the final report for the project assignment.

If you are going to develop a project rubric with the class, you may want to start now. See pages 279 and 280 in this Teacher's Resource for information on working with students to develop a class rubric.

Chapter Summary

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. If they have used Foldables™ before, you may wish to have them select a style they found useful to keep their Chapter 5 notes in. Discuss other methods of summarizing information. For example, many students may have used different types of graphic organizers, such as a mind map, concept map, spider map, Frayer model, and KWL chart. Discuss which one(s) might be useful in this chapter.

Encourage students to use a summary method of their choice. Allowing personal choice in this way will increase student ownership in their work. It may also encourage some students to experiment with different summary techniques.

Give students time to develop the summary method they have chosen. Ask them to include some method of keeping track of what they need to work on. Explain the advantage of doing this.

Meeting Student Needs

- Consider having students complete the questions on **BLM 5–2 Chapter 5 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Have students recall what they know about radical expressions. You might have students create a poster that features the terms *radical*, *radicand*, *index*, *mixed radical*, and *entire radical*. Display the poster in the classroom for reference.
- Some students may benefit from recalling the terminology related to square roots and perfect squares.
- You may wish to post the student learning outcomes for the chapter in the classroom, colour-coding the outcomes by section in the chapter. Ensure that students understand the outcomes as written, and be prepared to rewrite some into language they understand. Provide students with their own copy. They can then refer to the outcomes as they work through the chapter. This will help them to self-assess their progress and to identify areas of weakness.
- Hand out **BLM U3–1 Unit 3 Project Checklist**, which provides a list of all the requirements for the Unit 3 project.

ELL

- Encourage students to create their own math vocabulary dictionary for the Key Terms using written descriptions, diagrams, and examples.

Enrichment

- Students may wish to brainstorm and research other careers that involve radical equations.
- Encourage students to speculate on how the accuracy of radical measurements might be applied to miniaturization in the electronics industry.

Gifted

- Have students explore this statement: Since $\sqrt{4} + \sqrt{4}$ yields an exact answer, so must $\sqrt{2} + \sqrt{2}$ do so. Have them check the results using a calculator and observe any similarities and differences between the results for the expressions.

Career Link

You may wish to have students who are interested in learning more about meteorologists and atmospheric scientists research these careers, including the training and qualifications required and work-related opportunities. Have students present their findings orally. Explain how this career connects to the chapter.

Web Link

For information about a career in meteorology, go to www.mhrprecalc11.ca and follow the links.

Working With Radicals

5.1

***Pre-Calculus 11*, pages 272–281**

Suggested Timing
80–100 min

Materials
- centimetre grid paper
- ruler
- scissors

Blackline Masters
Master 2 Centimetre Grid Paper
BLM 5–3 Chapter 5 Warm-Up
BLM 5–4 Section 5.1 Extra Practice

Mathematical Processes
- ☑ Communication (C)
- ☑ Connections (CN)
- ☑ Mental Math and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☐ Technology (T)
- ☑ Visualization (V)

Specific Outcomes
AN2 Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–3, 5–7, 8a), d), 9a), b), 10a), d), two of 11–14, 24, 25
Typical	#1–7, 9b), d), 10c), d), two of 14–19, 20, 23–25
Extension/Enrichment	#4, 8d), 9c), d), 10c), d), 16–25

Planning Notes

Have students complete the warm-up questions on **BLM 5–3 Chapter 5 Warm-Up** to reinforce prerequisite skills needed for this section. If you have posted the outcomes, refer to the outcomes for this section.

In advance, you might collect several examples of innovative packaging. Share them with the class, after reading the opening paragraph together.

Investigate Radical Addition and Subtraction

The purpose of this investigation is to help students develop a concrete understanding of addition and subtraction of radicals.

You might have students work in pairs but record their own response to all questions. Distribute centimetre grid paper or **Master 2 Centimetre Grid Paper**. For #1, ensure that students use the entire square of 8 cm by 8 cm grid paper. Most students will be able to determine that the base will be a square that is 4 cm by 4 cm. Ask individual students:

- How might you determine the dimensions of the base if you were given a 10 cm by 10 cm square of grid paper? Assume that the height of the box is half the side length of the base.
- What would the dimensions of the base be for the original 8 cm by 8 cm square of grid paper if the height of the box was twice the side length of the base?

As students work to determine the length of the diagonal, circulate and assess the methods they use. For example, they might use the Pythagorean Theorem, the ratio of sides for a 45°–45°–90° triangle, or a ruler. Observe whether students express the radical as an entire radical or a mixed radical. Find out if they understand these terms and can convert between entire radicals and mixed radicals.

For #2, check how students determine the total length of each possible row. Check if students who are working with an entire radical ($\sqrt{32}$) from #1 are incorrectly doubling the radicand. Some students may not know how to verify their answer. Clarify that they need to check their answer using another method. For example, students could sketch the squares on grid paper with the diagonals forming a long hypotenuse, or they could approximate using a calculator. Encourage students who answered the question symbolically to refer to their constructed box and think about how they could verify their answer by applying their original strategy with more than one box present.

For #3, observe whether students are working with radical expressions or attempting to determine decimal approximations. For each of these approaches, explore the possibilities and/or restrictions of answering the questions using the alternative method.

As a class, have students share their responses to Reflect and Respond #4 and 5. Use prompts to help them conclude that when adding and subtracting radicals, only like radicals can be combined.

Meeting Student Needs

- Some students may benefit from recalling the components of a radical using the terms *index*, *coefficient*, and *radicand*. Have them draw their own diagram of a radical and label it using these terms.
- For #1 to 3, you might have students create models of the boxes using cardboard.
- You may wish to explore adding radicals with students by using monomials first, since they are familiar with monomials. Ask students:
  - Assume that one diagonal is $3x$ m long. How long are three diagonals? ($3x + 3x + 3x = 9x$ m)
  - What process is involved in addition of monomials?
  - How can you apply this process to radicals?

Enrichment

- Invite students to research the origin and evolution of the root symbol and present their findings to the class. They may find the related Web Link in this section helpful.

Common Errors

- Some students may not realize how to use the Pythagorean Theorem to determine the diagonal distance.

R_x Coach students to identify the right triangle containing the hypotenuse (diagonal).

- Some students may attempt to add the radicands when adding two radicals.

R_x Use examples to show students that adding the radicands is incorrect. You might use a calculator to show that $\sqrt{2} + \sqrt{2} \neq \sqrt{2 + 2}$ Alternatively, you might use a number line to show that the length of $\sqrt{2} + \sqrt{2}$ is not the same as the length of $\sqrt{2 + 2}$.

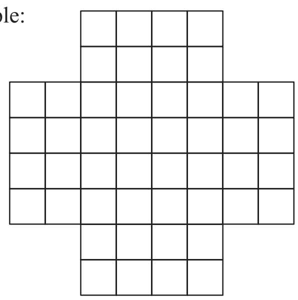
Web Link

For information about the origin of the root symbol, go to www.mhrprecalc11.ca and follow the links.

Answers

Investigate Radical Addition and Subtraction

1. a) Example:



b) $4\sqrt{2}$; Example: Using the Pythagorean Theorem, the diagonal distance is $4\sqrt{2}$ cm.

2. 2 boxes: $8\sqrt{2}$ cm
Example: I determined the sum of two boxes:
$4\sqrt{2} + 4\sqrt{2} = 8\sqrt{2}$
Check: $(2)(4\sqrt{2}) = 8\sqrt{2}$
4 boxes: $16\sqrt{2}$ cm
Example: $4\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} = 16\sqrt{2}$
Check: $(4)(4\sqrt{2}) = 16\sqrt{2}$
6 boxes: $24\sqrt{2}$ cm
Example: $4\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} = 24\sqrt{2}$
Check: $(6)(4\sqrt{2}) = 24\sqrt{2}$

3. a) Yes, the boxes will fit side by side. The boxes will occupy 12 cm of the distance along the shelf. The unoccupied distance will be $(15\sqrt{2} - 12)$ cm.
b) Yes, the boxes will fit corner to corner. The boxes will occupy $12\sqrt{2}$ cm. The unoccupied distance will be $3\sqrt{2}$ cm.

4. $4\sqrt{2} + 4\sqrt{2} + 4\sqrt{2} = 12\sqrt{2}$; $15\sqrt{2} - 12\sqrt{2} = 3\sqrt{2}$

5. Example: $\sqrt{a} + \sqrt{a} = 2\sqrt{a}$; When adding radicals, treat each radical like an object: 1 object + 1 object = 2 objects. Only like objects (radicals) can be added together.

6. Example: $\sqrt{4} + \sqrt{4} + \sqrt{4} + \sqrt{4} = 4\sqrt{4}$; Some students may simplify $4\sqrt{4} = 8$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the investigation. Encourage them to generalize and reach a conclusion about their findings.	• In #5, some students may find it easier to use numerical values before developing a general equation (e.g., $\sqrt{2} + \sqrt{2} = 2\sqrt{2}$). This may help them to draw a conclusion. • Some students might find it easier to respond to #5 if they do #6 first.

Link the Ideas

As a class, discuss the term *like radicals*. You might use parallel examples from algebra to show the difference between unlike and like terms. As in algebra, the coefficients of like terms can vary.

Explain how to determine the restrictions on radicals with variables in the radicand by using examples that involve square roots. For example, you might use the following progression of examples: $\sqrt{2x}$, $\sqrt{x-5}$, and $\sqrt{10-3x}$. Reinforce that if a radical has an even index (such as 2), the radicand must be non-negative. Using an example, show how to determine the values of the variable for which the radical is defined. Explain that the radicand has to be greater than or equal to zero if the root of an even index represents a real number. For $\sqrt{10-3x}$,

$$10 - 3x \geq 0$$
$$-3x \geq -10$$
$$x \leq \frac{10}{3}$$

Explain that if a radical has an odd index, the radicand can be any real number, including negative numbers. Encourage students to produce examples of odd and even radicals, with variables in the radicands. Have them identify restrictions on the variables and then compare their examples with those of a classmate.

Example 1

This example demonstrates how to convert mixed radicals to entire radicals. The questions progress from no variable to one variable to multiple variables.

For part c), ask students to explain why there are no restrictions on the variable. (The radical has an odd index.)

Have students complete the Your Turn questions individually and compare their solutions with those of a classmate. Note that the questions progress in difficulty.

Example 2

This example demonstrates how to convert entire radicals to mixed radicals in simplest form.

Before having students work through the example, ensure that they understand the notes about how to determine if a radical is in simplest form.

For part a), encourage students to use both methods shown for simplifying numerical radicands. Ask them which method they prefer and why. If a radicand is very large and its greatest perfect square factor is not obvious, students may say that they prefer to use prime factorization.

For part c), ask students to identify any restrictions on the variables.

Explain to students that the radical symbol represents only the positive square root and that the sign in front of the symbol is what determines the sign of the answer. Have students add an example to their math journal.

Have students complete the Your Turn questions in pairs using the methods of their choice. Have them compare their solutions with those of another pair who used different methods.

Example 3

Students apply their knowledge of converting mixed radicals (with the same index) into entire radicals, in order to compare and order a set of radicals.

For the expression $4(13)^{\frac{1}{2}}$, students may benefit from recalling how to convert from exponential to radical form.

The Your Turn does not include a term in exponential form. You might ask students to convert $\sqrt{23}$ to exponential form. Have students complete the questions individually and compare their solutions with those of a classmate.

Example 4

This example models how to add and subtract radicals by simplifying radicals and combining like terms. You may wish to remind students of the connection between adding like terms, such as $2x + 4x$, and adding like radicals. Note that the emphasis is for students to work primarily with radicals that contain only numerical radicands and coefficients before moving on to radicals that contain variables.

Have students complete the Your Turn questions individually and compare their solutions with those of a classmate. The green questions beside part c) link back to the Did You Know? on page 275.

Example 5

This example models an application of addition of radicals. Students use trigonometric ratios to generate radical expressions for the base lengths of two triangles. Then, they determine the total length of the bases.

Some students may benefit from having a one-on-one or small-group discussion to help work through the solution.

For the Your Turn, students who decide to solve the problem using special triangles need to be familiar with the ratios of the 45°–45°–90° triangle. Ask them:

- What are the exact trigonometric ratios for 45° angles? $\left(\tan 45° = 1; \sin 45° = \frac{1}{\sqrt{2}}; \cos 45° = \frac{1}{\sqrt{2}}\right)$

Have students compare their solution with those of the rest of the class.

Key Ideas

Emphasize that radicals must be written in simplest form before determining whether two or more radicals are like radicals. This may be a good opportunity to show the parallel between like radicals and like algebraic terms. Use leading questions such as the following:

- What is the only value that can differ between two like radicals? (the coefficient)
- What is true about the indices of two radicals that can be compared? (They are the same.)
- How can you express $7\sqrt{3}$ as the sum of two like radicals? ($3\sqrt{3} + 4\sqrt{3}$)

You might use the Key Ideas to help build the chapter review. Have students record their own summary of the Key Ideas. They can store their section summaries together to create a reference for review purposes. Consider allowing students to use their summary for the chapter test.

Meeting Student Needs

- For Link the Ideas, encourage students to create their own definition and examples of *like radicals*.
- For Example 1, you might begin by having students work through square roots of perfect squares. For example, $\sqrt{36} = 6$ because although $(6)(6) = 36$ and $(-6)(-6) = 36$, the square root symbol $\sqrt{\ }$ denotes only the positive value of the square root. The symbol for the negative square root would be $-\sqrt{\ }$. Sketch a diagram to represent this example. Use the reverse of this concept to assist students to write a mixed radical as an entire radical. For example, (7)(7) or $7^2 = 49$, so $7 = \sqrt{49}$.
- Emphasize that the radicands, when removed, are multiplied by the coefficients. For example, $3\sqrt{18}$ means $(3)\left(\sqrt{18}\right) = (3)\left(\sqrt{(3)(3)(2)}\right)$
$$= (3)(3)\left(\sqrt{2}\right)$$
$$= 9\sqrt{2}.$$
- For Example 1 part c), prompt students to note the root index. For example, $\sqrt[3]{8} = \sqrt[3]{(2)(2)(2)} = 2$.
- For Example 2, have students work in pairs to work through the methods shown. (Students could take turns acting as the instructor for each method.) For part c), you might prompt students to use prime factorization.
- For Example 2, record the following statements on a wall chart:
  A radical is considered in simplest form if all of the following are true:
  – The radicand does not contain a fraction or any factor which may be removed.
  – The denominator does not contain a radical.
  Some students may benefit from visualizing each given radical in the example as a length.
- Students who are interested in learning more about bentwood boxes may be directed to the Web Link near the bottom of this page.
- Allow students to check their work with a calculator. You may need to remind them not to round their calculator values.
- Prior to Example 4, help students make the connection between what they know and what they are learning by answering ten questions that you develop: four questions involving simple addition/subtraction of monomials with like variables, four questions involving simple addition/subtraction of monomials with at least two different variables (e.g., $3x + 2y - 5x + 3y$), and two questions involving simple addition/subtraction of like radicals (e.g., $2\sqrt{3} + 5\sqrt{3}$).

Enrichment

- Have students study the bentwood box on page 276 of the student resource. Encourage them to create a scale model of a cube using construction paper for the net of the cube. Have students use their model as a unit cube in which the length of each edge is given a measure of 1 unit. Challenge them to create cubes in which the unit volume increases by a whole number. Have students develop a rule for the relationship between change in volume and edge length.

Web Link

For a description of bentwood boxes including their construction, go to www.mhrprecalc11.ca a and follow the links.

Answers

Example 1: Your Turn

a) $\sqrt{48}$ **b)** $\sqrt{j^7}, j \geq 0$ **c)** $\sqrt[3]{32k^7}$

Example 2: Your Turn

a) $2\sqrt{13}$ **b)** $m\sqrt[4]{m^3}, m \geq 0$ **c)** $3n^3p^2(\sqrt{7n}),\ n \geq 0$

Example 3: Your Turn

$\sqrt{23}, 2\sqrt{6}, 5, 3\sqrt{3}$

Example 4: Your Turn

a) $15\sqrt{7}$ **b)** $\sqrt{6}$ **c)** $-7\sqrt{5x}$

Example 5: Your Turn

$\sqrt{6} + 3\sqrt{2}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• When determining the entire radical, some students may benefit from using an alternative approach, in which the index serves as a guide to determine how many times to multiply the coefficient by itself. For example, you might use $2\sqrt[3]{3}$ to show that the index of 3 means that 2 was cubed (or 2^3). Show how to determine the entire radical: $2\sqrt[3]{3} = \sqrt[3]{(2^3)(3)}$ $= \sqrt[3]{(8)(3)}$ $= \sqrt[3]{24}$
Example 2 Have students do the Your Turn related to Example 2.	• You might show students an alternative approach for converting from entire to mixed radicals. Divide the exponent on each variable in the radicand by the index, and determine the number of times that each exponent divides evenly. This value moves to the exponent of the variable. For example, show students how to simplify $\sqrt[4]{x^7y^{12}}$. $\frac{7}{4} = 1$ and 3 remainder; $\frac{12}{4} = 3$ The solution is $xy^3(\sqrt[4]{x^3})$. • If there is a numerical part in the radicand, have students refer to the methods shown and choose the method they prefer. Alternatively, have students use their math journal and list of perfect squares of numbers from 1 to 20. This may help them to use prime factorization. Students might organize the information using a chart: $(1)(1) = 1$ $\sqrt{1} = 1$ $(2)(2) = 4$ $\sqrt{4} = 2$ $(20)(20) = 400$ $\sqrt{400} = 20$ • For part c), some students may find it easier to use prime factorization.
Example 3 Have students do the Your Turn related to Example 3.	• Some students may need coaching to recall the process for converting powers with rational exponents to radicals. • Recommend to students that they rewrite all mixed radicals as entire radicals. Then, have them estimate the values based on their knowledge of the roots of perfect squares, rather than using a calculator. • Some students will need to use technology for these questions.
Example 4 Have students do the Your Turn related to Example 4.	• It may benefit some students to recall how to add and subtract basic algebraic terms. For example, ask how adding $2x + 3x$ is similar to adding $3\sqrt{2} + 5\sqrt{2}$. • You might have students try adding and subtracting radicals with indices other than 2, so they realize that the process is the same. For example, ask students to solve $2\sqrt[3]{4} + 5\sqrt[4]{2} - 6\sqrt[4]{2} + 5\sqrt[3]{4}$.
Example 5 Have students do the Your Turn related to Example 5.	• Provide a similar question before assigning the Your Turn. • Some students may benefit from recalling the trigonometric ratios. Help them label the adjacent, opposite, and hypotenuse sides on a diagram of a triangle. Have them recall the definitions of sine, cosine, and tangent ratios, and how to use them to determine the unknown sides of a triangle.

Check Your Understanding

Practise

These questions allow students to build basic skills in working with radical expressions, including the operations of addition and subtraction. Students can complete the questions individually or with a partner. After they complete a question, remind students to check the answer. In cases where there is a difference between their answer and the one in the answer key, encourage them to get help from a classmate or from you.

In #1 to 5, students convert between mixed radicals and entire radicals. Beginning with #2d), variables are introduced. Students should realize that when a radicand contains variables, there are restrictions on the variable, in order for the radical to be a real number. They should know that if the index is an even number, the radicand must be non-negative.

In #6, students may use technology to solve these problems by converting expressions to decimal approximations. Although they will need to compare entire radicals in #7 for 6b), you might encourage students to employ a similar method for #6a).

Students identify and combine like terms for #8 to 10. For #9 and 10, tell students they may need to further simplify terms in order to identify all like radical terms.

Apply

These questions provide opportunities for students to apply skills with radical expressions to real-life situations. Use leading questions to help them make the connection to the correct skill or process that is required for a specific question. Allow students to work in pairs to solve at least some of the problems.

Encourage students to sketch a diagram to represent the situation in #12. From the diagram, it may be more apparent that using the Pythagorean Theorem would be helpful.

In #13, students work with a radical that has an index greater than 2.

For #14, encourage students to rewrite the radicand as a product of prime factors in order to be sure that the radical is in simplest form.

Ask students what the term *inscribed* means in the context of #15. (The square fits exactly inside the circle, with its vertices just touching the edge of the circle.)

For #17, encourage students to use grid paper and create an accurate sketch.

Some students may not realize that the backyard in #18 is comprised of two squares and two rectangles. Encourage them to draw a model on grid paper and then determine the exact measurements of the green square and the grey square. Students should then be able to determine the length of one side of the rectangular flower bed and work through the rest of the solution.

Students need to realize that Kristen's partial solution in #19 is correct but incomplete. Her work is mathematically correct but the final solution is not in simplest form.

Extend

These questions require students to extend their knowledge by using new and previously learned skills and processes to solve problems.

For #21, have students sketch their own diagram. Prompt them to determine angles between adjacent line segments.

For #22, prompt students to consider what strategies might be used to solve the problem (such as using proportional sides of similar triangles and applying the Pythagorean Theorem). Help them recall that an inscribed angle that intercepts a semicircle is a right angle.

Create Connections

Consider having students work in pairs or small groups to brainstorm solutions to these problems before developing an individual response.

For #23, you might coach students to convert to mixed radicals before solving. Talk about what determining the exact value means. Help them recall that exact values are radicals or integers (and not decimal approximations).

For #24, students need to realize that all four expressions can be written as mixed radicals with a radicand of 3.

Project Corner

The Project Corner provides students with information about the Milky Way galaxy and the motion of planets. As a class, you might have students use words and a diagram to restate Kepler's third law. Students should understand that there is a mathematical relationship between the time it takes a planet to orbit the Sun and the distance between the planet and the Sun. The time it takes for a planet to orbit the Sun is its *orbital period*. The *semi-major axis* of the ellipse represents the average distance between the planet and the Sun. The semi-major axis is half the major axis, which is the longest distance across the ellipse. (Students might think of the semi-major axis as the longest radius of the ellipse.) Have students brainstorm how to express Kepler's third law using radicals before developing their own response.

Meeting Student Needs

- Encourage students to draw diagrams to model problems.
- Ensure that students spend enough time on the concepts introduced in section 5.1, since they form the basis of success for the rest of the unit.
- Some students may benefit from being able to refer to the perfect squares of 1 to 225. Post these on a classroom wall.
- For #13, use the related Did You Know? to explain astronomical units.
- For #23, some students may benefit from recalling arithmetic sequences.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 5–4 Section 5.1 Extra Practice** to students who would benefit from more practice.

ELL

- Use a combination of visuals, descriptions, and examples to help students understand such terms as *air pressure*, *hurricane*, *wind speed*, *Earth-days*, *tsunami*, *spiral galaxy*, *gravity*, *satellites*, *asteroids*, *comets*, *meteors*, *orbital period*, and *semi-major axes*.

Enrichment

- For #16, students may wish to prove Heron's formula using a right triangle with side lengths of 5 units, 12 units, and 13 units. Ask them to try another right triangle that they know and determine if Heron's formula works for the triangle.

Gifted

- The area of a two-dimensional object can be expressed in squared radicals, and the volume of a three-dimensional object can be expressed in cubic radicals. Challenge students to explore the possibilities of expressing the fourth dimension, time, in radicals.

Common Errors

- Students may forget how to convert from exponential form to radical form.

R_x Help students recall that $b^{\frac{1}{n}} = \sqrt[n]{b}$.

- Some students may express an exact value incorrectly.

R_x Remind students that exact values require radical or integer values.

Web Link

For information about Kepler's third law, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 3, 5 to 7, 8a), d), 9a), b), 10a), d), and two of 11 to 14. Students who have no problems with these questions can go on to the remaining questions.	• Provide additional coaching with Examples 1 and 2 to students who need support with #1, with Example 2 for #2, with Example 3 for #6, and with Example 4 for #8 to 10. These sets of questions are particularly important because they form the basis of work in later sections. Have students who need coaching try some of the unassigned questions for additional practice. Use their work as assessment *for* learning. • There are multiple approaches to #1 to 10. Have students identify the method they prefer to use. Suggest that they use more than one method to answer a question, when possible. • For #11, 13, and 14, students may be more successful if you coach them to substitute values into the given equations. • For #12, encourage students to sketch and label their own diagram.
Assessment *as* Learning	
Create Connections Have all students complete #24 and 25.	• Some students may benefit from working with a partner to plan their responses. • For #24, remind students that the easiest way to compare radicals is to convert them to entire radicals or rewrite them using a common radicand. Ask them which approach would be easier to use in this question. • For #25, have students consider when the ± sign can and cannot be used to determine the root of a number.

5.2 Multiplying and Dividing Radical Expressions

***Pre-Calculus 11*, pages 282–293**

Suggested Timing

90–110 min

Materials

- regular hexagon template or compass and ruler
- ruler
- grid paper

Blackline Masters

Master 2 Centimetre Grid Paper
BLM 5–3 Chapter 5 Warm-Up
BLM 5–5 Section 5.2 Extra Practice

Mathematical Processes

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Specific Outcomes

AN2 Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–3, 4a), b), 5a), c), 6, 7, 8a)–c), 9–11, 13, 14, 28, 29, 33
Typical	#1–3, 4c), d), 5b), d), 6, 7, 9–13, 15, 17, 28–30, 33
Extension/Enrichment	#3, 5, 11–13, 17, 21–28, 30–33

Planning Notes

Have students complete the warm-up questions on **BLM 5–3 Chapter 5 Warm-Up** to reinforce prerequisite skills needed for this section.

As a class, read the opening paragraph and invite students to share what they know about the hexagonal cloud feature on Saturn.

Investigate Radical Multiplication and Division

In this investigation, students develop a concrete understanding of multiplication and division of radicals.

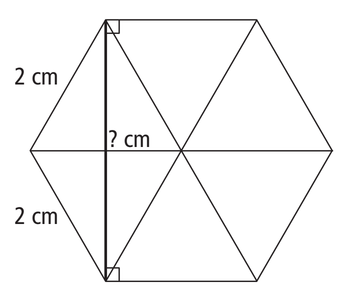
You might have students work in pairs but record their own response to all questions.

For Part A, provide students with a template for a regular hexagon or have them construct the hexagon using a compass and ruler. If students are constructing their own hexagon, you might model how to do so:

- Mark a point and draw a horizontal line through it.
- Using a compass, construct a circle with this point as the centre.
- Construct another circle the same size, using the point of intersection of the line and the circle as the centre.
- Draw diameters from the centre of the circle to each point of intersection of the two circles.
- Construct a hexagon by joining the points of intersection of the circle and the diameters.

As you circulate, observe how students approach the problem. Many students may realize that the height of one equilateral triangle is half the distance across the hexagon. Observe whether they use special angles or trigonometry to determine the height. Either of these methods would be sufficient to verify their answer for #3. If students verify their answer using other methods, such as considering the larger right triangle where the height of the hexagon is one leg of the triangle, invite them to present their solutions to the class. For #3, you might show the sketch that follows and ask students:

- How might you determine the shortest distance between parallel sides of the hexagon using special triangles? the cosine law?



As a class, have students share their response to Reflect and Respond #5. Have them note the different methods that were used to verify radical division.

For #7 in Part B, students can use special ratios for the large isosceles triangle or the three small isosceles triangles. Or, students may decide to use the Pythagorean Theorem with the large isosceles triangle.

As a class, have students share their response to Reflect and Respond #9 to 11. For #11, students will likely indicate that they would use radicals that are perfect squares as examples.

Meeting Student Needs

- Ensure students have a sketch of a 30°–60°–90° triangle and a 45°–45°–90° triangle with the sides clearly labelled to use as a reference.
- Some students may benefit from recalling the cosine law from Chapter 2. (The cosine law can be used to find the length of an unknown side of any triangle when the lengths of two sides are known, as well as the measure of the angle between them.)

Gifted

- Invite students who have an interest in space exploration to research the cloud feature on Saturn. Have them present their findings in a format of their choice to the class. Students may find the related Web Link in this section helpful.

Web Link

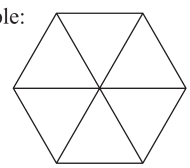
For instructions and a demonstration about how to construct a regular hexagon using a compass, go to www.mhrprecalc11.ca and follow the links.

For an image and information about the hexagonal cloud pattern on Saturn, go to www.mhrprecalc11.ca and follow the links.

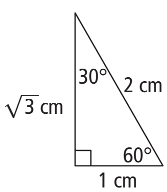
Answers

Investigate Radical Multiplication and Division

1. Example:



2. $2\sqrt{3}$ or $\sqrt{12}$; Example: The angle measures are 30°, 60°, and 90°.



3. Example: $\tan 60° = \frac{x}{1}$, $x \approx 1.732$, and $\sqrt{3} \approx 1.732$

4. 12 500 km

5. Example:

$$\sqrt{12\,500^2 - 6250^2} = \frac{\sqrt{468\,750\,000}}{2}$$
$$2\sqrt{117\,187\,500} = \sqrt{468\,750\,000}$$
$$\sqrt{468\,750\,000} = \sqrt{468\,750\,000}$$

6. small rectangle: $\sqrt{3}$ m by $\sqrt{3}$ m; area = 3 m^2;
large rectangle: $\sqrt{3}$ m by $2\sqrt{3}$ m; area = 6 m^2

7. $3\sqrt{6}$ m, $\sqrt{54}$ m

8. Example:

$$\sqrt{3\sqrt{3}^2 + 3\sqrt{3}^2} = \sqrt{54}$$
$$\sqrt{\sqrt{27}^2 + \sqrt{27}^2} = \sqrt{54}$$
$$\sqrt{27 + 27} = \sqrt{54}$$
$$\sqrt{54} = \sqrt{54}$$

9. $\sin 60° = \frac{\sqrt{3}}{2}$; $\cos 60° = \frac{1}{2}$; $\tan 60° = \sqrt{3}$;
$\sin 30° = \frac{1}{2}$; $\cos 30° = \frac{\sqrt{3}}{2}$; $\tan 30° = \frac{1}{\sqrt{3}}$;
$\sin 45° = \frac{1}{\sqrt{2}}$; $\cos 45° = \frac{1}{\sqrt{2}}$; $\tan 45° = 1$.

10. Example: When multiplying radicals, multiply the coefficients and then multiply the radicands. When dividing radicals, divide the coefficients and then divide the radicands.

$$(3\sqrt{4})(2\sqrt{9}) = 6\sqrt{36}$$
$$= 36$$
$$\text{Check: } (3\sqrt{4})(2\sqrt{9}) = (6)(6)$$
$$= 36$$

11. Example: I would use perfect squares because the product and quotient are whole number values. Using familiar values makes it easier to verify the solutions.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the investigation. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students complete #10 and 11 before 9, and talk about the responses as a class. For #11, students may choose perfect squares. If they do, their examples may help students come to understanding more easily than the values in #9. • As a class, have volunteers post their answers for #9 on the board. Have students record these in their math journal or graphic organizer for review purposes.

Link the Ideas

As a class, work through Link the Ideas. Point out to students that generally, they will be less likely to make simplification errors if they simplify radicals before multiplying. This is modelled in the second method used to solve $(2\sqrt{7})(4\sqrt{75})$ on page 284 in the student resource. You might encourage students to adopt this method by asking them:

- Which method requires factoring large radicands? (the first method)
- Which method will make it easier to avoid simplification errors? (the second method)

Reinforce that radicands can only be multiplied and divided if they have the same index.

Encourage students to produce and record their own examples of multiplication and division. Then, have them explain their solutions to a classmate.

Example 1

In this example, students multiply radicals, including binomial terms.

It may be helpful to walk students through the similarities between multiplying radical expressions and multiplying algebraic expressions. For example, the steps involved in multiplying $(3x - 4)(2x + 1)$ include multiplying using the distributive property and combining like terms, which is a similar process to combining like radicals, as shown in part c).

Have students work through the solutions before assigning the Your Turn questions. Have students attempt the questions individually and compare their solutions with those of a classmate.

Example 2

In this example, students apply radical multiplication.

As students work through the solution, encourage them to sketch their own diagrams.

For part b), two methods are shown to determine the height of the triangle. Have students try both methods before asking them which method they prefer and why. Encourage them to think about whether one method is more efficient than the other.

Have students complete the Your Turn questions with a partner and compare their solution with that of another pair who used a different method to determine the height of the triangle. Have them discuss which method is more efficient.

Example 3

This example models how to divide radicals and rationalize expressions that contain a radical in the denominator.

Before working through Example 3, as a class walk through the teaching notes on pages 286 and 287 that explain how to rationalize the denominator. Using an example of a monomial square-root denominator, show how to multiply both the numerator and the denominator by the radical term from the denominator. (Check that students understand that multiplying both the numerator and the denominator by $\sqrt{3}$ is equivalent to multiplying by 1.) Then, collect like terms. You might wish to discuss some reasons for rationalizing the denominator. For example, it is easier to manipulate and perform mathematical operations on expressions that have rational denominators rather than on expressions that contain radicals in the denominator.

Using an example of a binomial square-root denominator, show multiplying both the numerator and the denominator by the conjugate of the denominator. Explain that the conjugate is determined by rewriting the binomial, using the opposite operation for one term (usually the second term). Ensure that students understand the connection between the factors of a difference of squares and conjugates. Encourage them to produce and record their own example before moving on.

Walk through the solutions as a class. For part c), you might have students compare the initial expression and the final expression. Ask them:

- Which is easier to work with, $\frac{11}{\sqrt{5} + 7}$ or $\frac{7 - \sqrt{5}}{4}$?

  Why do you think so? (It is easier to apply mathematical operations to the final expression since it has a rational denominator.)

Have students complete the Your Turn questions individually before comparing their solutions with those of a classmate. You might have students verify one solution using decimal approximations.

Key Ideas

As a class, have students summarize the rules for multiplying and dividing radicals. Walk through the example showing the multiplication of two binomials containing radicals. Alternatively, you might provide a question similar to the one given and a solution that contains a common error, and ask students to identify the error.

You might provide an example of an expression with a radical binomial in the denominator and ask students to list the steps for simplifying the expression.

Have students record their own summary of the Key Ideas. Have them store the summaries of the Key Ideas for each chapter section together for review purposes.

Meeting Student Needs

- For Link the Ideas, you may decide to walk through the examples on the board together as a class. If so, use a different colour for the coefficients than for the radicands. Illustrate how the radicands can be simplified during the multiplication process. Break the radicands into prime numbers rather than working with larger numbers. For example, $\left(3\sqrt{30}\right)\left(4\sqrt{20}\right) = (3)(4)\left(\sqrt{(3)(2)(5)(2)(2)(5)}\right)$. Then, continue the process of simplifying. Encourage students to record their own summary of multiplying and dividing radicals.
- Some students will benefit from working with smaller numbers.
- Reinforce that any radical multiplied by itself becomes an integer. This process is very important when rationalizing a denominator. Provide sample questions for students to practise this skill, such as $(\sqrt{3})(\sqrt{3}) = 3$ and $(\sqrt{5})(\sqrt{5}) = 5$.
- For Example 1, parallel teach using polynomials. Review multiplying monomials and then multiply radicals. Review the distributive property using a monomial and a binomial and then demonstrate the same process using radicals. Follow the same process when multiplying two binomials. Linking new learning with previous knowledge will help students be successful.
- For Example 2 part b), Method 2, you might have students do a comparison. If the sides of a 30°–60°–90° triangle are 1, $\sqrt{3}$, and 2, respectively, then the base of the triangle in Example 2 is half the hypotenuse, $2\sqrt{2}$. The side opposite the 60° angle in the diagram is the product of $\sqrt{3}$ and the length of the base in the diagram $(2\sqrt{2})(\sqrt{3}) = 2\sqrt{6}$.
- Explain to the class that frieze patterns are patterns that repeat in one direction. Discuss how the image in Example 2 resembles a frieze pattern and how the traditional beadwork patterns of the Plains First Nations people, such as Blackfoot, Lakota, Nakota, Dakota, etc, can also be used as models for frieze patterns.
- Encourage students to record their own explanation and examples for the terms *rationalize* and *conjugates*.
- When modelling how to divide, you may wish to highlight the steps using different colours. You might circle the coefficients to draw attention to the simplified coefficient and circle the simplified radicand. For example,

  $$\frac{4\sqrt[3]{6}}{2\sqrt[3]{3}} = 2\sqrt[3]{\frac{6}{3}}$$
  $$= 2\sqrt[3]{2}$$

- Emphasize that when rationalizing the denominator or multiplying by the conjugate, the purpose is to remove the square root sign from the denominator.
- For Example 3, you might illustrate simple division first. For example, $\frac{\sqrt{6}}{\sqrt{2}} = \sqrt{3}$ or $4\frac{\sqrt{12}}{\sqrt{6}} = 4\sqrt{2}$.

  Colour-coding or circling the radicands may be helpful for visual learners.
- For Example 3, invite students to work in groups to create a poster that features an example of simplifying an expression with a radical binomial in the denominator. Have them record the steps for simplifying the expression beside the example.
- Use an example, such as $\frac{4}{\sqrt{2x} + 3}$, to help students recall how to identify the values of x for which the radical is a real number.
- Pair students to walk through the Key Ideas. Have them develop an example to illustrate each Key Idea and take turns to explain the process of solving an example to their partner.

Enrichment

- Challenge students to research frieze patterns, choose a pattern, and develop a related problem and its solution. Invite them to post their work in the classroom.

Common Errors

- When rationalizing monomial denominators that are mixed radicals, students may multiply the numerator and denominator by the mixed radical instead of by the radical portion only.

Rx Using an example, show students that when the coefficient of the radical is included, the coefficient is factored out from both the numerator and the denominator in the simplification process.

- Some students may multiply an expression with a radical binomial in the denominator by the exact same expression instead of by the conjugate.

Rx Using an example, show how a radical term remains in the denominator if they do that.

Web Link

For an explanation and examples of rationalizing the denominator and conjugates, go to www.mhrprecalc11.ca and follow the links.

Answers

Example 1: Your Turn

a) $15\sqrt{2}$ **b)** $6\sqrt[3]{33} - 8\sqrt[3]{22}$ **c)** $-37\sqrt{7} - 11\sqrt{14}$
d) $-8c^2\sqrt{22} + 6\sqrt{33c}$, $c \geq 0$

Example 2: Your Turn

$A = \sqrt{290}$ m^2

Example 3: Your Turn

a) $2\sqrt{17}$ **b)** $\frac{-7\sqrt[3]{3p^2}}{6p}, p \neq 0$ **c)** $\frac{6\sqrt{5} + 8}{29}$ **d)** $\frac{12\sqrt{x} - 6}{4x - 1}, x \geq 0$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Reinforce that to multiply radicals, you multiply coefficients by coefficients and radicands by radicands. • When using the distributive property, remind students to be aware of the correct sign in cases involving a negative coefficient and/or an expression inside brackets that requires subtraction. • Some students may not have learned to use the distributive property when multiplying binomials. Help reactivate students' knowledge of other methods (e.g., FOIL, vertical, box) and encourage them to use the method they prefer.
Example 2 Have students do the Your Turn related to Example 2.	• Provide a similar question before assigning the Your Turn. • Have students sketch and label the triangle to help visualize the problem. • Provide coaching for students who need support to square mixed radicals, and explain the connection to the Pythagorean Theorem. • Have students refer to their math journal for the table that lists the exact sine, cosine, and tangent values for 30°, 45°, 60°, and 90° angles. Remind them to use exact values and not decimal approximations.
Example 3 Have students do the Your Turn related to Example 3.	• Provide similar questions before assigning the Your Turn. • Reinforce that the product of a pair of conjugates is a difference of squares. Help students understand by multiplying $(x - 2)(x + 2)$, and then explaining the link to $a^2 - b^2$. Use a numerical example to show students that multiplying conjugates always results in $a^2 - b^2$. • If needed, coach students by asking them to verbalize what they will multiply the numerator and denominator by for each question before recording their answer. • Remind students that in the case of monomial radicals in the denominator, they should multiply only by the radical and not the coefficient. If students have difficulty identifying this, have them include the coefficient and express the fraction in lowest terms in the answer.

Check Your Understanding

Practise

These questions allow students to build basic skills in multiplying and dividing radical expressions. Students can complete the questions individually or with a partner. After they complete a question, remind students to check the answer. In cases where there is a difference between their answer and the one in the answer key, encourage them to get help from a classmate or from you.

For #1, students multiply monomial terms before moving on to binomial terms in #2.

Students must use the distributive property to simplify the products in #2 and 3.

For #4 and 5, students multiply two binomials containing radicals. Variables are not introduced until #5. For #5c), ensure that students multiply two factors of the binomial. These questions provide good practice for rationalizing binomial denominators successfully in later questions.

Students rationalize monomial denominators in #6 to 8.

For #9, students determine conjugates of binomials prior to rationalizing binomial denominators in #10 and 11.

Apply

Allow students to work in pairs to solve the problems. Tell students to sketch diagrams whenever possible.

In #13, students analyse a common error when distributing, in which Malcolm multiplies both the coefficient and the radicand by a whole number.

Coach students who struggle with #16 to realize that the isosceles triangle can be constructed from two identical right triangles, and that the hypotenuse can be determined using the Pythagorean Theorem. Encourage students to sketch their own diagram using grid paper or **Master 2 Centimetre Grid Paper**.

For #17, students need to multiply the two fractions and then rationalize the denominator of the product. Some students may fail to realize that the denominators do not form a conjugate pair.

Extend

Students need to realize that the diagonal of the cube in #22 is the same length as the diameter of the sphere.

For #23, help students realize that the coordinates of the midpoint of a line segment are the mean of the coordinates of the endpoints.

For #24, students must rewrite expressions with negative exponents as reciprocals with positive exponents.

Some students may need to be reminded of the quadratic formula for #25.

Create Connections

Consider having students work in pairs or small groups to brainstorm solutions to these problems before developing an individual response. Remind students that they will need to use previous math knowledge to solve the problems.

For #28 and 29, students connect their learning to concepts learned earlier, including simplifying algebraic expressions and factoring.

In #30 and 31, students connect their ability to work with radical expressions with quadratic relations from earlier in this course. To help students determine whether the equations given in #31 are solutions, prompt them about the methods they might use to solve a quadratic equation (such as using factoring). Alternatively, students may choose to substitute each value of m and evaluate the quadratic equation.

For #33, which is a Mini Lab, have students work in pairs. If the Mini Lab is used for summative assessment, ensure that you present your expectations for the completed work and provide a marking rubric for the assignment. If the Mini Lab is used for formative assessment, meet with the class as a whole and ask a couple of student pairs to lead a discussion of the results.

Project Corner

The Project Corner provides students with information about escape velocity and space exploration. As a class, you might have students brainstorm possible applications of radicals in space related to the concepts described in the student resource before recording some possibilities to research. They may find the related Web Links at the end of this section helpful.

Meeting Student Needs

- Encourage students to draw diagrams to model problems whenever possible.
- Use the Did You Know? related to #15 to elaborate on the pendulum in the HSBC building.
- Have students review the rules for simplifying, adding, subtracting, multiplying, rationalizing the denominator, dividing, and multiplying by the conjugate prior to beginning the Check Your Understanding questions. This may help them to clarify the concepts.

- Have students work in pairs or small groups. Assign each pair or group one question in the Apply section. Have them solve the problem and then prepare a short presentation to the class including any challenges in the problem, the solution, and the strategies used to determine the solution.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 5–5 Section 5.2 Extra Practice** to students who would benefit from more practice.

ELL

- Use a combination of visuals, descriptions, and examples to help students understand such terms as *golden rectangle*, *pendulum*, *period*, *skidoo race*, and *snowboarder*.
- Use the visual related to #30 to help explain what a snowboard cross event involves.
- The language in the Project Corner may be challenging for some students. Consider having English language learners work with a partner who can help them understand the following terms: *escape velocity*, *celestial objects*, *artificial gravity*, and *emulation*.

Enrichment

- For #14, invite students to research the proportions of the golden rectangle. They may find the related Web Link in this section interesting.
- As an alternative to the Mini Lab about patterns of numbers involving radicals, challenge students to research an application of radicals, such as designs of star blankets. Have students present their findings to the class.
- Have students explore the following statement found on page 284 in the student resource: In general, $\left(m\sqrt[k]{a}\right)\left(n\sqrt[k]{b}\right) = mn\sqrt[k]{ab}$, where k is a natural number, and m, n, a, and b are real numbers. If k is even, then $a, b \geq 0$. Ask them to explain the restrictions placed on a and b.

Gifted

- For #15, have students research pendulums. Challenge them to develop a problem and a solution involving the motion of a pendulum, and exchange their problem with that of a classmate. Students may find the related Web Link in this section interesting.
- Have students reflect on the following statement: Mathematicians, such as Descartes, began to adopt the use of imaginary numbers, where $i^2 = -1$. Challenge students to explore the use of such numbers in the real world.

Common Errors

- Students may not determine exact values when asked to do so.

R_x Remind students to express exact values as radicals, rational numbers in fraction form, or integers.

Web Link

For applets that show the golden ratio and the spiral centre of the golden rectangle, go to www.mhrprecalc11.ca and follow the links.

For an applet that shows the motion of a pendulum, go to www.mhrprecalc11.ca and follow the links.

For information about escape velocity, go to www.mhrprecalc11.ca and follow the links.

For information about artificial gravity, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 3, 4a), b), 5a), c), 6, 7, 8a) to c), 9 to 11, 13, and 14. Students who have no problems with these questions can go on to the remaining questions.	• Encourage students to sketch diagrams, even for questions that provide diagrams. • Some students may benefit from working with a partner. • Remind students that for #1 to 5, it may be easier to simplify the radicals, where applicable, before multiplying. • Prompt students to verbalize that when multiplying radicals, they multiply coefficients by coefficients and radicands by radicands. If students are confused, have them sketch a diagram of a mixed radical and label the coefficient and the radicand. • For #8a) and c), ask students to identify what is different about these questions. Ask them to explain how part c) can use two radical signs. • For #9 and 10, remind students that only the middle sign changes when determining the conjugate. Ask students to identify the conjugate of several examples you provide before they answer the questions. For example, $2 + \sqrt{5}$, $2 - \sqrt{71}$, $\sqrt{6} - 3$, and $-5 - 3\sqrt{11}$. This should assist them with #14.
Assessment *as* Learning	
Create Connections Have all students complete #28, 29, and 33. Students who have no difficulty with these questions can attempt #30 to 32.	• Some students may benefit from working with a partner to plan their responses. • For #28 and 29, encourage students to use examples to develop their explanation. Consider collecting students' responses to these questions and checking for weaknesses in their thinking. Provide coaching to students, as needed. Have students place their checked responses into their graphic organizer for review purposes.

5.3 Radical Equations

***Pre-Calculus 11*, pages 294–303**

Suggested Timing
100–120 min

Materials
- three metre sticks (or one metre stick and markings for 100 cm on wall and floor) per group
- grid paper
- graphing calculator or computer with graphing software

Blackline Masters
Master 2 Centimetre Grid Paper
BLM 5–3 Chapter 5 Warm-Up
BLM 5–6 Section 5.3 Extra Practice

Mathematical Processes
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Specific Outcomes
AN3 Solve problems that involve radical equations (limited to square roots).

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–6, 8a), b), 9a), d), 10, 12–14, 24
Typical	#1, 2, 4, 7, 8c), d), 10–12, two of 14–17 or 19 and 20, 24–26
Extension/Enrichment	#6, 10, 11, 18–22, 24–27

Planning Notes

Have students complete the warm-up questions on **BLM 5–3 Chapter 5 Warm-Up** to reinforce prerequisite skills needed for this section.

As a class, have students respond to the opening questions about the length of a ramp and the angle of elevation, and how the relationship between these measurements determines safety standards. Referring to the Did You Know? about Shaw Millennium Park, have students mention skate parks they know and any safety standards in place.

Investigate Radical Equations

In this investigation, students develop an equation describing the relationship between vertical and horizontal distances.

Begin by asking students what they think the relationship is between the changes in vertical distance compared to horizontal distance. Intuitively, they may expect the change to correspond directly (e.g., the horizontal distance will change by 5 cm if the vertical distance changes by 5 cm).

Have students work in small groups. Provide three metre sticks per group and grid paper or **Master 2 Centimetre Grid Paper**. If you do not have enough metre sticks for each group to have three metre sticks, have students mark 100 cm on paper taped to the wall and the floor. Each group will need one metre stick to gradually slide down and away from the wall.

Clarify for students that they need to write the function for the vertical distance, v, that the top of the ruler moves down the wall as a function of the change in the horizontal distance, h.

Have students respond to the Reflect and Respond questions individually and then compare their response to #9 with that of a classmate. As a class, have students share their response to #7 to 9. Direct their attention to the definition of a *radical equation*.

Meeting Student Needs

- Try to ensure that there are sufficient metre sticks so that all students can be actively involved in the investigation.
- Encourage students to create their own definition and an example of a *radical equation*.
- Discuss the learning outcomes. Inform students that all rules for operations on radicals should be applied when solving radical equations.

Enrichment

- Encourage students to research the ratio of height to length of a ramp for wheelchair accessibility. What concepts did the designers include when creating the criteria for building ramps?

Answers

Investigate Radical Equations

3. Example: Distances are approximate.

Horizontal Distance From Wall, *h* (cm)	Vertical Distance Down Wall, *v* (cm)
0	0
10	0.5
20	2
30	4.6
40	8.4
50	13.4
60	20
70	28.6
80	40
90	54.4
100	100

4. The data are nonlinear. Example:
- I plotted the data to determine if they are linear.

5. The vertical side is $100 - v$ cm, the horizontal side is $200v - v^2$ cm, and the hypotenuse is 100 cm.

6. $v(h) = 100 - \sqrt{10\,000 - h^2}$; $v(10) \approx 0.501$ cm; $v(50) \approx 13.397$ cm

7. Example: $h \approx 66.14$ cm; using a metre stick, $h = 64$ cm

8. 10 cm

9. Example: Radical equations have variables in the radicands. Linear equations have no exponents on the variables, and quadratic equations have an exponent of 2 on the variable.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the investigation. Encourage them to generalize and reach a conclusion about their findings.	• Students may find it helpful to talk about the response to #6 before you assign #7 to 9. Listening to other learners explain their thinking about how to write the equation may help students who struggle with developing equations. • Ask students to identify as many different ways of expressing the equation as possible (e.g., using a chart or technology). • Allow students to work with a partner so they can assist each other with any technology-related issues. • Post students' answer to #9 on the board. Have students compare their answer with the ones posted. Ask which type of equation best represents the data generated in the investigation.

Link the Ideas

At this point, students should be aware of the restrictions on variables in radicals within the real-number system that have an even index (the radicand must be greater than or equal to zero). Remind students to set the radicand to greater than or equal to zero and then isolate the variable.

Example 1

In this example, students determine the restrictions on a variable in part a) before solving a radical equation with one radical term in part b).

Have students work through the solution. In part b), begin by isolating the radical expression. After solving for the variable, reinforce the importance of checking that the value is a solution to the original equation. Encourage students to adopt this practice whenever they solve equations.

Have students attempt the Your Turn question individually and compare their solution with that of a classmate.

Example 2

In this example, students are exposed to a radical equation with an extraneous root.

Direct students to observe that when squaring both sides of the equation $(n + 7)^2 = \left(\sqrt{5 - n}\right)^2$, it is critical to square both sides, not only the individual terms of the equation. Reinforce the importance of always checking possible solutions, since one root in this example is extraneous.

Have students attempt the Your Turn question individually and compare their solution with that of a classmate.

Example 3

In this example, students solve a radical equation with two radical expressions.

Reinforce the strategy of isolating a radical expression before squaring both sides. In this case, neither of the two generated solutions is extraneous.

Have students attempt the Your Turn question individually and compare their solution with that of a classmate.

Example 4

In this example, students solve a problem involving a commonly used scientific equation for kinetic energy.

Although Method 2 may be more commonly used to solve the problem, students should be able to solve for a variable by rearranging the radical equation, as modelled in Method 1.

The Your Turn question requires students to solve for a variable (the diagonal of a cube). Have students work with a partner and compare their solution with that of another pair.

Key Ideas

As a class, review the proper order of operations to solve a radical equation. Also, check students' process for determining restricted values.

Have students record their own summary of the Key Ideas. Have them store their summaries for each chapter section together for review purposes.

Meeting Student Needs

- Some students may benefit from an explanation of how to solve an equation, such as $5 + (2x + 1) = 12$. Demonstrate that the process of isolating the variable is similar for solving a radical equation.
- For Link the Ideas, talk about the restriction on the radicand. Ask students why the radicand must be a value greater than or equal to zero and in which situations. Consider modelling a solution along with the steps for solving a radical equation. For example,

$$\begin{aligned} 2\sqrt{x+1} + 6 &= 12 && \text{Isolate the variable.} \\ 2\sqrt{x+1} &= 6 \\ \sqrt{x+1} &= 3 \\ \left(\sqrt{x+1}\right)^2 &= 3^2 && \text{Square both sides.} \\ x + 1 &= 9 && \text{Solve for the variable.} \\ x &= 8 \end{aligned}$$

- Direct students to the Did You Know? on page 296 of the student resource to help recall reversing the direction of the inequality symbol when multiplying or dividing both sides of an inequality by a negative number.
- For Example 2, it may be helpful for students to recall the strategies for factoring trinomials.
- For Example 2, students need to understand that substituting the values for the variables in the original equation is the most effective method to check for extraneous roots.
- Encourage students to create their own definition and an example of an *extraneous root*.
- For Example 3, you may wish to model the solution (along with the steps) for a radical equation with two radicals. If so, show isolating the radical twice (isolate one radical, square both sides, isolate the remaining radical, and then square both sides).
- For Example 4, give students the option of trying either method.

Common Errors

- In Example 3, some students may incorrectly square the binomial expression. For $\sqrt{x + 2} = 3 - \sqrt{x + 7}$, they may simply square each term and obtain $x + 2 = 9 - x + 7$.

R_x Remind students to use the distributive property and multiply the binomial. For $\sqrt{x + 2} = 3 - \sqrt{x + 7}$, show $x + 2 = 16 - 6\sqrt{x + 7} + x$ before going on to solve the equation.

Answers

Example 1: Your Turn

$y \geq 0$; $y = 60$

Example 2: Your Turn

$m \geq -\frac{3}{2}$; $m = 11$

Example 3: Your Turn

$j = 79 - 10\sqrt{57}$
$j \approx 3.5$

Example 4: Your Turn

$d = \sqrt{3A}$; When $d = 23.3$ cm, $A \approx 181$ cm^2.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Remind students to isolate the radical expression. • Students may forget to determine the restrictions when solving the equation. Encourage them to do this first before solving. Have students compare their restrictions with those of a classmate before solving the equation.
Example 2 Have students do the Your Turn related to Example 2.	• It is important that students differentiate between squaring a binomial and squaring a binomial radicand. Before assigning the Your Turn, explain why it is necessary to expand a binomial expression, such as $(n + 7)^2$, but not binomial radicands, such as $\left(\sqrt{5 - n}\right)^2$. Understanding this is key to solving radical equations. • Remind students that *expanded form* means using the distributive property to obtain $n^2 + 14n + 49$, and not simply $n^2 + 7^2$. • Remind students to state the restrictions before solving, and to verify the solution.
Example 3 Have students do the Your Turn related to Example 3.	• Remind students to isolate one radical to each side of the equal sign. Have them identify which radical is a monomial term and which radical belongs to a binomial term. It is important that students recognize the binomial side of the equation, since it requires binomial multiplication for simplification. Using an example such as $\sqrt{x + 2} = 3 - \sqrt{x + 7}$, model how to solve the equation by squaring both sides and using the distributive property for the binomial expression.
Example 4 Have students do the Your Turn related to Example 4.	• Provide a similar question before assigning the Your Turn. • Encourage students to sketch their own diagram. • Have students identify which method is easier to use. • Remind students of the importance of verifying their answer and checking if the answer is reasonable in the given context.

Check Your Understanding

Practise

These questions allow students to build basic skills in solving radical equations. Students can complete the questions individually or with a partner.

In #1, students practise squaring a radical expression, as a first step to solve radical equations.

Students are expected to describe the steps in solving a specific radical equation in #2.

In #3 to 6, students solve radical equations involving a single radical expression containing variables that are first degree.

In #7 and 8, the variables are second degree, which may require factoring or using the quadratic formula. Observe whether students are using the proper process for isolating the radical expression and then solving for the variable.

Students solve equations that contain two radical expressions in #9 and 10.

Apply

Allow students to work in pairs to solve the problems. Tell students to sketch diagrams whenever possible.

For #11, students may benefit from coaching to understand why the equation $\sqrt{3y - 1} - 2 = 5$ has a real solution. Cover up the radical part of the equation and ask:

- What number value makes $? - 2 = 5$ true? (7; Since 7 is positive, and a radical expression must be non-negative, then a real solution exists for this radical equation.)

In #12, a common error is shown. Instead of squaring the expression $x - 3$, the student has incorrectly distributed the squaring to each term. If necessary, re-teach students this concept with numerical examples. For example, $(5 - 3)^2 \neq 5^2 - 3^2$.

Left Side		Right Side
$= (5 - 3)^2$		$= 5^2 - 3^2$
$= (2)^2$		$= 25 - 9$
$= 4$		$= 16$
Left Side	$\neq$	Right Side

Students apply radical equations in #13 and 14. Although the coefficient of each radical term is a decimal, the process required to solve the radical equation does not change.

Some students may determine the square root in #16 using a guess and check strategy. However, they are unlikely to determine the extraneous root in this way. Use the opportunity to reinforce that determining the algebraic solution (instead of using systematic trial and error) helps ensure that all possible solutions are identified.

In #17, students may benefit from coaching to translate the word problem into a radical equation. Help them decode twice the product of the maximum height, h, and the acceleration due to gravity.

Extend

For #19, students apply their skills with radical equations to solve for a literal coefficient.

For #20a), students may benefit from referring back to problems they solved earlier in this section, in order to help them identify the types of equations that result in extraneous solutions. For part b), they need to make the connection that two solutions will involve solving a quadratic as part of a radical equation.

Students create a radical equation that contains two radical expressions in #21. You may need to remind them that the greater expression for time is on the moon since its gravity is smaller.

Prompt students to realize that they will need to use the quadratic formula or technology to solve #22.

For #23b), encourage students to complete the square. Some students may need to review the process for solving a quadratic equation.

Create Connections

Consider having students work in pairs or small groups to brainstorm solutions to these problems before developing an individual response. Remind students that they will need to use previous math knowledge to solve the problems.

Students may benefit from coaching to answer #25. Ask them:

- What happens to a negative number when you square it?

You may wish to have students solve for x in $x + 2 = 5$ and then square both sides and solve for x. After comparing the two solutions, students will likely have a better understanding of how an extraneous root can be produced.

For #27, which is a Mini Lab, you might have students work in pairs but record an individual response, before comparing their results with those of another student pair. If the Mini Lab is used for summative assessment, ensure that you present your expectations for the completed work and provide a marking rubric for the assignment. If the Mini Lab is used for formative assessment, meet with the class as a whole and ask volunteers to lead a discussion of the results.

Meeting Student Needs

- Encourage students to draw diagrams to model problems, whenever possible.
- For #2, allow students to provide either an oral or a written response. Consider allowing students who can orally explain the process for deriving the solution to dictate the steps to a scribe. Have the scribe follow the instructions exactly as written. Encourage the first student to make modifications as needed to arrive at the correct solution.
- The level of the questions in #9 is more challenging. Ensure students are well prepared for the required algebra.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 5–6 Section 5.3 Extra Practice** to students who would benefit from more practice.

ELL

- Use a combination of descriptions, visuals, and examples to help students understand such terms as *collision investigators*, *skid mark*, *beam*, *acceleration due to gravity*, *horizon*, *spacecraft*, and *continued radical*.

Enrichment

- Encourage students to create their own written strategy for solving radical equations. Tell them to think of the big picture in terms of creating an algorithm for students who may be unsure about what to do first and when to consider extraneous roots. Set a space limitation for the strategy, such as one side of an index card. Encourage students to be as concise and clear as possible.

- Invite students to research a context presented in one Apply level question. Challenge them to develop a problem involving radical equations that is related to the context they chose, and provide a solution.

Gifted

- Challenge students to explore the relationship between the historical development of radical equations, the development of inequalities, and the development of imaginary numbers. Ask them to report on how radical equations, inequalities, and imaginary numbers are connected.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 6, 8a), b), 9a), d), 10, and 12 to 14. Students who have no problems with these questions can go on to the remaining questions.	• Encourage students to sketch diagrams, even for questions that provide diagrams. • For #1, ask students which terms are monomials or binomials for squaring. • For #3 and 4, coach students to explain the difference between the two questions. Remind them to isolate the radical. • Once students have completed #8a), ask them whether multiple answers are appropriate and why. Have them note the difference between #8a) and b). (Part a) requires squaring a binomial; part b) does not.) Reinforcing this point should help them with #10 and 12. • Have students identify the method they prefer to use for solving #14b). Challenge them to solve the problem again using a different strategy.
Assessment *as* Learning	
Create Connections Have all students complete #24. Encourage students who have no problem with this question to attempt #25 to 27.	• Some students may benefit from working with a partner to plan their responses. • For #24, encourage students to use examples to develop their explanation. Consider collecting students' responses to these questions and checking for weaknesses in their thinking. Provide coaching as needed. Have students store the revised response in their graphic organizer.

5

Chapter 5 Review

***Pre-Calculus 11*, pages 304–305**

Suggested Timing
60—90 min

Blackline Masters
BLM 5–4 Section 5.1 Extra Practice
BLM 5–5 Section 5.2 Extra Practice
BLM 5–6 Section 5.3 Extra Practice

Planning Notes

Have students who are not confident identify strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test.

Meeting Student Needs

- Encourage students to sketch a diagram, whenever possible.
- Students who require more practice on a particular topic may refer to **BLM 5–4 Section 5.1 Extra Practice**, **BLM 5–5 Section 5.2 Extra Practice**, and **BLM 5–6 Section 5.3 Extra Practice**.
- Before students begin the questions, invite them to review their summary of the Key Ideas for each section. Have students clarify any misunderstandings.
- For #13 and 14, have students recall how to rationalize the denominator. Ask them how to rationalize a binomial denominator.
- If it has not been done already, post all learning outcomes. Invite students to ask questions about any outcomes that they do not understand.
- Consider setting up stations in the classroom, each one corresponding to a section in the chapter. Based on the assessment of their own progress, invite students to go to the station for the section with which they had the most difficulty. Encourage them to work in pairs to answer the questions. Give students an opportunity to move to the other two stations during the class.
- Individualize the chapter review. Have students choose three questions from each section to begin. Correct the questions and analyse errors. Encourage students to request assistance with the questions they are unable to complete successfully. Students can then choose additional practice questions based on their results.

ELL

- For #21, use the Did You Know? about the Vakta to clarify the meaning of *coast guard* and *Coast Guard cutter*.

Enrichment

- In this chapter, students learned about how radical equations are used to model a variety of relationships that have applications in engineering and science. Challenge them to create a timeline that traces the development of our capacity to solve such problems mathematically. The timeline should include the names of the relevant mathematicians and their contributions to mathematical thinking.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 5 Review The Chapter 5 Review is an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource.	• Have students revisit any section that they are having difficulty with prior to working on the chapter test. • Encourage students to refer to the summary notes they completed throughout the chapter.

Chapter 5 Practice Test

5

***Pre-Calculus 11*, pages 306–307**

Suggested Timing
45—60 min

Blackline Masters
BLM 5–7 Chapter 5 Test

Planning Notes

Have students start the practice test by writing the question numbers in their notebook. Have them indicate which practice test questions they need no help with, a little help with, and a lot of help with. Have students first complete the questions they know they can do, followed by the questions they know something about. Finally, suggest to students that they do their best on the remaining questions. Ensure that students get coaching for questions that they indicated they need help with.

You can assign this practice test as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1–11, 13–15.

Study Guide

Question(s)	Section(s)	Refer to	The student can …
#1, 11	5.1	Example 1	✓ convert between mixed radicals and entire radicals
#2	5.1	Link the Ideas Example 1 Example 2	✓ identify restrictions on the values for a variable in a radical expression
#3	5.1	Example 4	✓ simplify radical expressions using addition and subtraction
#4	5.2	Example 1	✓ perform multiple operations on radical expressions
#5	5.3	Example 2	✓ determine the roots of a radical equation algebraically
#6, 8, 13, 15	5.2	Example 3	✓ perform multiple operations on radical expressions ✓ rationalize the denominator
#7	5.1	Example 3	✓ compare and order radical expressions
#9	5.3	Example 1 Example 2 Example 3	✓ solve equations involving square roots ✓ determine the roots of a radical equation algebraically ✓ identify restrictions on the values for the variable in a radical equation
#10	5.3	Example 2 Example 3	✓ determine the roots of a radical equation algebraically ✓ identify restrictions on the values for the variable in a radical equation
#12	5.1	Example 5	✓ solve problems involving radical expressions
#14, 17	5.3	Example 4	✓ solve problems involving radical expressions
#16, 18, 19	5.3	Example 4	✓ model and solve problems with radical equations

Assessment	Supporting Learning
Assessment *as* Learning	
Chapter 5 Self-Assessment Have students use their responses on the practice test and work they completed earlier in the chapter to identify skills or concepts they may need to reinforce.	• Students may wish to review their chapter summary notes before they begin the practice test. Students can use these to identify any areas of weakness. • Have students work through the practice test individually. Invite them to refer to any related information posted on the classroom wall or their own notes. • Before the chapter test, coach students in areas in which they are having difficulties.
Assessment *of* Learning	
Chapter 5 Test After students complete the practice test, you may wish to use **BLM 5–7 Chapter 5 Test** as a summative assessment.	• Consider allowing students to use their summary notes to complete the practice test.