Rational Expressions and Equations

Opener

***Pre-Calculus 11*, pages 308–309**

Suggested Timing
45–60 min

Blackline Masters
BLM 6–2 Factoring Polynomials Flowchart
BLM 6–3 Chapter 6 Prerequisite Skills
BLM U3–1 Unit 3 Project Checklist

Key Terms
rational expression
rational equation
non-permissible value

What's Ahead

In section 6.1, students are introduced to rational expressions, how to simplify them, and how to discover non-permissible values. In section 6.2, they learn that rational expressions can be manipulated with mathematical operations in a similar manner to rational numbers. In particular, this section explores multiplying and dividing rational expressions. Adding and subtracting are explored in section 6.3. In section 6.4, students learn how problems can be modelled using rational equations, and then solved.

Planning Notes

Begin the chapter by discussing where algebra is used in the real world. Ask students:
- How does a business decide how much of their product to produce?
- How is a company's profit calculated?
- How does an airline decide what to charge passengers for flights?
- How does NASA decide at what angle, in what direction, and at what speed to send a rocket?

Students should understand that the answers to these questions involve some type of mathematical expression, formula, or relation. Many of these mathematical models will contain expressions involving fractions with polynomials. Explain to students that these are the types of expressions and equations they will explore in this chapter.

For students to have success with this chapter, they must be proficient in factoring polynomials. Consider developing with students a table summarizing factoring methods (a sample completed table is provided below). Also consider distributing **BLM 6–2 Factoring Polynomials Flowchart**, which provides a strategy, in flowchart form, that students can use when factoring polynomials. Encourage students to internalize the strategy so that they can, at some point, use it without referring to the BLM.

To assist students with their chart, ask:
- What is the meaning of the word *factor* when it is used as a noun? as a verb?
- What factoring methods have you studied?
- For each of these methods, what are the characteristics of the polynomial to be factored?
- How do you find the factors? Create or find an example for each method.

Name of Factoring Method	Characteristics	Examples
Greatest Common Factor	each term of the polynomial has a common factor	$a^2 - ab = a(a - b)$ $35xy - 15y + 5 = 5(7xy - 3y + 1)$
Sum and Product	trinomial with even degree where the coefficient of the highest degree term is 1	$x^2 + 2x - 15 = (x + 5)(x - 3)$ $y^4 - y^2 - 56 = (y^2 - 8)(y^2 + 7)$ $x^2 + 12xy + 20y^2 = (x + 2y)(x + 10y)$
Grouping	four-term polynomial where each group of two terms has a common factor	$xy - 8x + 7y - 56 = x(y - 8) + 7(y - 8)$ $= (x + 7)(y - 8)$
Difference of Two Squares	binomial with two perfect square terms written as a difference	$4x^4 - y^2 = (2x^2 + y)(2x^2 - y)$ $81a^2 - 25 = (9a + 5)(9a - 5)$
Perfect Square Trinomial	trinomial with perfect square terms for the first and last terms, and the middle term is $\pm$ twice the product of the square roots of the first and last terms	$169m^2 + 208m + 64 = (13m + 8)^2$ $9x^4 - 12x^2 + 4 = (3x^2 - 2)^2$
General Trinomial	trinomial (must be factorable with rational coefficients and constant) that cannot be factored by one of the previous methods	$6x^2 + 17x + 12 = (2x + 3)(3x + 4)$ $15a^2 + 32ab - 7b^2 = (5a - b)(3a + 7b)$

Revisit mathematical operations with fractions. Begin with the vocabulary of fractions. Then, discuss the concept of reducing to lowest form and multiplying or dividing by 1. Then have students create a table summarizing these operations, along with a method for each (or do this as a class). A sample of a completed table is shown below.

Operation	Method	Example
Addition	**1.** Find a common denominator. **2.** Rewrite each fraction in equivalent form with the LCD. **3.** Add the numerators. **4.** Simplify.	$\frac{7}{12} + 2\frac{2}{9}$ LCD is 36. $\frac{21}{36} + 2\frac{8}{36}$ $\frac{101}{36} = 2\frac{29}{36}$
Subtraction	**1.** Find a common denominator. **2.** Rewrite each fraction in equivalent form with the LCD. **3.** Subtract the numerators. **4.** Simplify.	$1\frac{5}{6} - 2\frac{7}{10}$ LCD is 30. $\frac{55}{30} - \frac{81}{30}$ $-\frac{26}{30} = \frac{-13}{15}$
Multiplication	**1.** Cancel like factors from the numerators and denominators, if possible. **2.** Multiply numerators and denominators. **3.** Simplify.	$4\frac{1}{2}\left(\frac{7}{12}\right)$ $\frac{9}{2}\left(\frac{7}{12}\right)$ $\frac{(3)(\cancel{3})}{2}\left(\frac{7}{(\cancel{3})(4)}\right)$ $\frac{3}{2}\left(\frac{7}{4}\right)$ $\frac{21}{8} = 2\frac{5}{8}$
Division	**1.** Reciprocate the divisor and rewrite as multiplication. **2.** Cancel like factors from numerators and denominators, if possible. **3.** Multiply numerators and denominators. **4.** Simplify.	$\frac{11}{15} \div 2\frac{2}{5}$ $\frac{11}{15}\left(\frac{5}{12}\right)$ $\frac{11}{(3)(\cancel{5})}\left(\frac{\cancel{5}}{12}\right)$ $\frac{11}{36}$

Chapter Summary

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. Discuss methods for keeping a summary, such as a

- Foldable
- concept map
- Frayer model
- mind map
- spider map
- KWL chart

Meeting Student Needs

- Consider having students complete the questions on **BLM 6–3 Chapter 6 Prerequisite Skills** before beginning the chapter.
- You may wish to provide students with **BLM U3–1 Unit 3 Project Checklist**, which lists all the requirements for the Unit 3 project.
- Post or distribute the chapter outcomes and discuss what will be learned. Give links to future learning.
- Place posters that illustrate rational expressions and simplification of rational expressions in your classroom. Invite students to work through each poster with a partner, identifying the operation being illustrated and the steps involved.
- Students could create flash cards containing the key terms for the chapter (or perhaps even the unit).
- Consider searching (or having students search) the Internet for videos depicting students explaining various math processes. This is often done through song or dramatization. You may wish to show one or two of these videos and then encourage students to create their own as they work through the unit. If done well, the video may serve as a summative evaluation for the unit.

Enrichment

Encourage students to do the math of planetary exploration. In particular, have them calculate how long it might take to travel from Earth to various planets. They can use 10% of the speed of light (speed of light is about 300 000 km/s, so about 30 000 km/s) as the velocity because it has been suggested that this velocity might be achievable. Note that the work at the end of the chapter explores time dilation, which has implications for space travel.

Gifted

The chapter begins with a question about the 12 years it took between the time the Cassini-Huygens project was launched until scientists disclosed new information about Saturn's rings. Ask students to speculate why it took 12 years, and list several possibilities.

Career Link

Students who are interested in learning more about careers related to rational expressions and mathematical modelling may wish to research the following:

- medicine
- acoustics
- lighting
- science
- economics
- urban planning
- space travel
- climate change
- engineering
- manufacturing

Students could then report the results of their research to the class and discuss how this career connects to the chapter.

Web Link

For more information on how mathematics is used in various careers, go to www.mhrprecalc11.ca and follow the links.

Rational Expressions

6.1

***Pre-Calculus 11*, pages 310–321**

Suggested Timing
90–120 min

Materials
- algebra tiles
- coloured sheets with domino shapes on them
- scissors

Blackline Masters
BLM 6–4 Chapter 6 Warm-Up
BLM 6–5 Section 6.1 Extra Practice

Mathematical Processes
- ☑ Communication (C)
- ☑ Connections (CN)
- ☑ Mental Math and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☐ Technology (T)
- ☐ Visualization (V)

Specific Outcomes
AN4 Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2a), c), 3, 4a), c), 5–11, 13, 19–22, 29, 30
Typical	#2, 4, 5, 8–22, 23 or 24, 29–32
Extension/Enrichment	#14, 16, 17, 25–32

Planning Notes

Have students complete the warm-up questions on **BLM 6–4 Chapter 6 Warm-Up** to reinforce prerequisite skills needed for this section.

Have a class discussion about how fractional numbers, fractional equations, and fractional expressions might be related. Help students make the connection that the mathematical operations apply similarly to all three. One approach is to use algebra tiles. Students who have not used algebra tiles previously to model division may need an introduction to them. Ask:

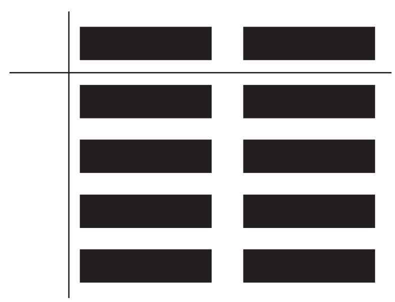
- What does each tile represent?
- How can you model $2x$, $3x + 2$, and $4x - 3$?
- How is division related to the area of a rectangle and the dimensions of the sides?

Begin by demonstrating a division involving a rational expression in the numerator, such as $6x \div 2$.

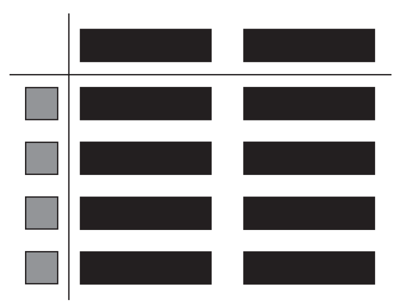
Rearrange six x-tiles into a rectangle, so that the dimension of one side of the rectangle is two 1-tiles.

The dimension of the other side is the quotient, so $6x \div 2 = 3x$.

Use an area model to model division involving rational expressions in both the numerator and denominator, such as $8x \div 2x$. To find the quotient, determine the dimension of the unknown side of a rectangle with an area of $8x$ and one side that has a dimension of $2x$.



The dimension of the other side is 4, so $8x \div 2x = 4$.



Investigate Rational Expressions

Have students work through the Investigate either alone or in pairs. The goal of #1 in the investigation is to help students realize that an algebraic fraction has equivalent forms. In this question, they are determining the simplest form. If students are struggling, ask:

- Can the numerator be represented with algebra tiles in a rectangle where one side is represented by the denominator, $3x$?
- What tiles represent the sides of the rectangle?
- What is the quotient?

To follow-up on the concept presented in #1, ask pairs of students to discuss how they can use the result to create a procedure to find the quotient without using the model. Suggest that they create an example of their own, showing that their procedure will work. They might verify their answer with algebra tiles.

For #2 and 3, students need to understand that division by zero is undefined. Some students may need reminding of the zero property. Have a discussion, asking:

- If $ab = 0$, then what must be true about a and/or b?
- Is there a simpler method than guess and check for finding the values of x for which $x^2 - 4x + 3 = 0$?
- What is the name for an equation in the form $x^2 - 4x + 3 = 0$? Where have you worked with this type of equation previously?

If students are having difficulty with #4, show the following informal proof of why division by zero is undefined:

Start with the assumption that $a = b$.

$a^2 = ab$	Multiply both sides of the equality by a.
$a^2 - b^2 = ab - b^2$	Subtract b^2 from both sides.
$(a - b)(a + b) = b(a - b)$	Factor both sides of the equality.
$\dfrac{(a - b)(a + b)}{(a - b)} = \dfrac{b(a - b)}{(a - b)}$	Divide both sides by $(a - b)$.
$a + b = b$	Cancel $(a - b)$ from the numerator and denominator of each fraction.

Since we started with the assumption that $a = b$, then $b + b = b$ or $2b = b$.

Ask:

- Were there any errors made in the proof?
- What is $(a - b)$ equal to according to the assumption made at the beginning of the proof?
- How does this proof show that division by zero is undefined?

Invite students to explore other examples or proofs that division by zero leads to a contradiction.

Consider having students teach the rule they develop in #5 to other individuals or pairs in the class. Have them explain how their rule works and show examples.

Make sure that #6 to 8 are explored thoroughly and that students have a firm grasp of the concepts. These are important to the work students will do in this section. Some students may have difficulty recognizing that the quotient of two expressions that are additive inverses, or opposites, is -1. Engage students in a discussion about this concept to ensure they understand it. It may help to show them an example, such as $\frac{x - 6}{6 - x}$. Replace x with a value. The result is -1.

Meeting Student Needs

- Before beginning the Investigate, either hand out or work through various rational number questions and operations. Include an example of reducing (simplifying), multiplying, dividing (either by multiplying by the reciprocal or bringing terms to a common denominator and then dividing numerators), adding, and subtracting. Emphasize finding and bringing terms to a common denominator.
- Invite students to use a calculator to attempt various division questions where the denominator is zero. Ask why an error message is returned. What does it mean to divide by zero? Why is this process impossible?

Common Errors

- When simplifying rational expressions, students may divide only one term in the dividend by the divisor. For example,

$$\frac{3x + 6}{3x}$$
$$= \frac{\cancel{3x} + 6}{\cancel{3x}}$$
$$= \frac{1 + 6}{1}$$
$$= 7$$

R_x Ask students to substitute a value for x, such as $x = 4$, and then determine the result.

$$\frac{3x + 6}{3x}$$
$$= \frac{3(4) + 6}{3(4)}$$
$$= \frac{12 + 6}{12}$$
$$= \frac{18}{12}$$

Since $\frac{18}{12} \neq 7$, the simplification for this expression was incorrect.

Ask:
– How can you factor the numerator of the original expression? the denominator?
– Are there any common factors?
– If there are common factors, what is their quotient?
– What is the correct simplification?

Web Link

For some examples of online tutorials on rational expressions, go to www.mhrprecalc11.ca and follow the links.

Answers

Investigate Rational Expressions

1. a), b)

c) $x + 4$

d) No, $\frac{3x^2 + 12x}{3x}$ does not equal $x + 4$ if $x = 0$ since the expressions would be undefined.

2. a) $b \neq 0$ **b)** In each expression you will be dividing by zero.

3. a) $x \neq 0$ **b)** $x \neq 7$ **c)** $x \neq 3, -\frac{1}{2}$

4. a) 0
b) Division by zero is an operation for which you cannot find an answer.

5. Example: The denominator cannot be zero, so factor the denominator, set each factor equal to zero, and solve.

6. division; Example:
$$\frac{2x + 12}{2}$$
$$= \frac{2x}{2} + \frac{12}{2}$$
$$= x + 6$$

7. Example: Both demonstrate division. Neither can have division by zero. With algebraic fractions, variables are present and can take on any value.

8. a) 1
b) -1
c) -1. Either the numerator or denominator could be multiplied by -1 and the sign of each term would change, making the fraction equivalent to 1.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• It is important that students make the correct links in answering #4 and 5. Discuss the responses and ensure that students have the justification for the responses clear in their minds. Have them write their own examples to ensure they can demonstrate their understanding. • For #6, if students are having difficulty, have them simplify simple fractions. Then, have them simplify a rational expression, such as $\frac{4x^2}{2x}$. To ensure that they understand that the common factor must come out of all values in the numerator, they should also simplify an expression such as $\frac{4x^2 + x}{2x}$. • For #8c), have students rewrite the numerator or denominator first so that the terms are in the same order in both. This will be completed by taking out a common factor of -1. Simplifying the common factors will again leave the -1.

Link the Ideas

As a class, discuss the definition of *non-permissible values*. Then, have students work alone or in pairs to answer the following:

- Are there non-permissible values for the expression $\frac{5x - 2}{12}$? Can you make a general statement based on this example?
- Can you find an example in which there is only one non-permissible value? two non-permissible values?
- Write a general rule for determining the number of non-permissible values for single-variable denominators.

Come together again as a class to discuss the answers. Encourage students to place the term, its definition, and examples in their Foldable, graphic organizer, or with their summary notes.

Example 1

This example models how to determine non-permissible values. You may need to discuss rational expressions with more than one variable before beginning Example 1. Prior to working through this example, students have not seen rational expressions with more than one variable, and some students may be confused by them. Remind them of the zero property and discuss why this means that all factors must be set to zero in order to find the non-permissible values. Ask:

- Can the numerator be equal to zero? the denominator?
- Do you find the non-permissible value(s) before or after simplifying the rational expression? Explain.

Example 2

This example illustrates the simplification of rational expressions using factoring and the property of 1. Before discussing Example 2, you may want to help students recall how to obtain equivalent rational numbers. For example, show that $\frac{2}{3} = \frac{2(5)}{3(5)}$, or $\frac{10}{15}$. Ask:

- When the numerator and denominator were multiplied by 5, did the value of the fraction change?
- Multiplying the numerator and denominator by 5 is the same as multiplying the fraction $\frac{2}{3}$ by what value?
- What result, then, do you think would be obtained if both the numerator and denominator where multiplied by -8? 100? y? $(2x + 6)$?
- Can you make a general statement about obtaining equivalent forms of rational numbers?

Emphasize that non-permissible values must be found both before the expression is simplified and after an unsimplified form is found. For example, in the unsimplified expression $\frac{x - 3}{x^2 + 2x - 15}$, there are two non-permissible values: 3 and -5. But, if the simplest form of the expression, $\frac{1}{x + 5}$, is to be changed to another equivalent expression by multiplying by 1, say in the form of $\frac{x - 7}{x - 7}$, then the non-permissible values change to $x \neq -5$ and 7.

Example 3

This example involves an expression in the denominator with two variables. Discuss with students how many solutions there are to degree 1, two-variable equations. Students must recognize that in such cases, all the non-permissible values cannot be listed, but that an equation can be written for the non-permissible values.

Ask students to consider an example where the expression $2x - 9y$ is in the denominator. Ask:

- What is the solution to the equation $2x - 9y = 0$?
- How would you write the equation if x was isolated? $(x = \frac{9y}{2})$
- How would you write the equation if y was isolated? $(y = \frac{2x}{9})$ So, $x \neq \frac{9y}{2}$ and $y \neq \frac{2x}{9}$.

As far as which statement is correct, $x = \frac{9y}{2}$ or $y = \frac{2x}{9}$, it depends on the wording of the question. The first is a response to the question, "What is an expression for the non-permissible values of x?" and the second is a response to the question, "What is an expression for the non-permissible values of y?"

Example 3 also illustrates that the answer can be found by substituting into either the original or simplified rational equation. Have a class discussion after students do the Your Turn. Ask:

- What is the result when substituting the values into the original unsimplified expression?
- What is the result when substituting the values into the simplest expression?
- Why were the results the same?
- Why would you want to simplify an expression before substituting values for the variables?

Key Ideas

Ask students to rewrite the Key Ideas in their own words with examples, and place their work in their summary notes.

Meeting Student Needs

- Provide students with a page containing "Equivalent Fraction Mad Minutes." The sheet could contain two or three sets of 20 questions for students to either reduce to lowest terms or multiply by a form of 1 to produce equivalent fractions. Discuss the process once students have completed the questions. Now link the process used for fractions to working with polynomials.
- Students may wish to practise factoring polynomials using algebra tiles. The visual and tactile representation allows students to understand that they can remove tiles common to the numerator and denominator.
- Students could create "Simplifying Rational Expression" dominoes. Divide students into small groups. Provide each group with sheets of coloured stock paper with blank domino shapes on them. Each group should get a different coloured paper. In their groups, students write a rational expression on one domino and its simplified expression on the end of another domino. When the dominoes are full, they are cut apart. Players fit the dominoes together by matching the simplified expression with its parent expression. Have students play at the end of a period. Collect the dominoes at the end of the class for assessment.

ELL

- Ensure that students understand the vocabulary used in the Link the Ideas section and the examples. Ensure that they understand terms, such as *original*, *beneficial*, etc.
- Suggest students add the following terms to their vocabulary dictionary: *permissible* (and *non-permissible*), *simplify(ing)*, *undefined*, etc. Encourage them to include any term they encounter with which they are not familiar. Suggest that they include a verbal description, diagram, and/or example for each term.

Common Errors

- Simplifying into addition or subtraction is one of the most frequent errors that students make. That is, they cancel part of a factor rather than the complete factor.

R_x Have students consider whether the following simplification is correct:

$$\frac{4x^2 - 25}{4x + 10}$$

$$= \frac{\overset{x}{\cancel{4x^2}} - \overset{5}{\cancel{25}}}{\underset{1}{\cancel{4x}} + \underset{2}{\cancel{10}}}$$

$$= \frac{x - 5}{1 + 2}$$

$$= \frac{x - 5}{3}$$

If they believe it is incorrect, ask them to point out the error and make the correction. If they believe it is correct or are unsure, ask them to replace x with a value in the original unsimplified expression, and then substitute the same value into the simplified expression. The results should be the same, but will not be. Since they are not, ask students to consider where the error is and to make the correction.

Encourage students to factor all polynomials fully before identifying common factors and simplifying.

Answers

Example 1: Your Turn

a) $b \neq 0, c \neq 0$ **b)** $x \neq -2, 3$ **c)** $y \neq -2, 2$

Example 2: Your Turn

a) $\frac{2y + 5}{y} + 5, y \neq -5, 2$ **b)** $\frac{-2}{m + 3}, y \neq -3, 3$

Example 3: Your Turn

a) 4

b) 7.2

c) Example: The rational expression may be easier to evaluate, involving less complex calculations.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Remind students that setting the denominator equal to zero will help find the non-permissible value(s). Remind them that any common factors, inclusive of variables that are factored in the denominator, must also be set to zero and solved. Provide students with an example, such as $x^2 + x$. When factored, this expression becomes $x(x + 1)$. Many students only look to the brackets, and forget to also set the coefficient, x, to 0.
Example 2 Have students do the Your Turn related to Example 2.	• Encourage students to factor the denominator and find the non-permissible values first. Then, have them factor the numerator and simplify common factors. Encourage students to try each method that is presented.
Example 3 Have students do the Your Turn related to Example 3.	• Ask students to identify what they are looking for in part b). • It is important that they have good reasoning for part c). Make a list on the board. Tell students that this list will be a helpful reference, and encourage them to write the list in their notes or in a graphic organizer.

Check Your Understanding

Practise

If students are having difficulty with #1 to 8, have them revisit the examples in the student resource—they are very similar. Make sure that students understand that in #3 to 5, they are being asked to do the same thing: find the non-permissible values. Students should be aware that there are different ways of wording the same request.

Draw students' attention to the fact that #1 and 2 involve the same procedures needed to simplify the expressions in #6 and 8.

Apply

Students will apply what they have learned in this lesson to real-life situations. There are more concrete applications than what was used in the Practise questions. Students may benefit from working with a partner so they can share ideas on how to answer the questions.

For #9, ask students if they can find a value for x so that $x + 5$ and $\frac{x^2 + 2x - 15}{x - 3}$ are not equivalent.

For #10, ask students to form an equivalent expression to $\frac{y}{y - 6}$ by multiplying the numerator and denominator by y. Ask:

- What are the non-permissible values after this is done?
- Do we have a way of knowing what the original unsimplified expression was?

For #11, ask students:

- What is the additive inverse of the integer 8?
- What is the quotient of 8 and its additive inverse?
- What is the additive inverse of $5 - x$?
- What is the quotient of $5 - x$ and its additive inverse?
- Is the quotient of an expression and its additive inverse always -1?

Have a class discussion about #12 to 14. These questions reinforce topics that students may not have absorbed in the examples and the Practise questions. These concepts are

- equivalent expressions must be stated with their non-permissible values
- non-permissible values must be found before simplifying the expression
- the quotient of an expression and its additive inverse, or opposite, is always -1, for all values of the variable for which the expression is defined

To motivate students to do #12, tell them that you will be collecting their three expressions and using selected ones for questions on a quiz, test, or other form of evaluation.

In #13, you have an opportunity to reinforce that each term in the numerator must be divided by the denominator. That is, $\frac{g + 2}{2}$ is in simplest form. The only way division by 2 could be shown is by distributing the divisor of 2 over each term in the dividend: $\frac{g}{2} + \frac{2}{2} = \frac{g}{2} + 1$.

In #15, ask students if there are any other values of n that are unreasonable. Students should be made aware that any value between -4 and 4 has no meaning, because the value of the expression would then be negative.

For #16, ask students:

- What is the area formula of a circle?
- How does the diameter of the circle you are drawing compare to the side measurement of the square?

- Is there more than one way to show a comparison of the areas? (For example, students may compare the area by writing an expression involving subtraction: $4x^2 - \pi x^2$, or they may use the ratio $\frac{4x^2}{\pi x^2}$.)

For #17, ask students:

- What does the term *integral value* mean?
- How could you use your graphing calculator or a spreadsheet program to help determine what value gives the greatest yield?

In #20, have students use substitution to show that the expressions $\frac{5}{m + 5}$ and $\frac{1}{m + 1}$ are not equivalent.

If students are having difficulty with #21 to 23, refer them to #1 and 2. The questions use the same concepts.

For #24, students will need to recall the formula for the area of a triangle.

For #25b), the concept of moving the negative sign to the numerator is confusing to some students. Suggest that students replace n with a value, such as $n = 7$, in the expressions $\frac{n + 3}{-n}$ and $\frac{n - 3}{n}$. Ask them if the two results are equivalent. After students realize that the two expressions are not equivalent, show them that they must multiply by 1 in the form of $\frac{-1}{-1}$ in order to move the negative to the numerator: $\frac{-1(n + 3)}{-1(-n)} = \frac{-n - 3}{n}$.

Extend

More complicated factoring procedures are involved in #26. Ask students to relate the factoring of $(x + 2)^2 - (x + 2) - 20$ to factoring $a^2 - a - 20$. Guide them through replacing a with $x + 2$:

$$\begin{aligned} a^2 - a - 20 &= (a - 5)(a + 4) \\ (x + 2)^2 - (x + 2) - 20 &= ([x + 2] - 5)([x + 2] + 4) \\ &= (x - 3)(x + 6) \end{aligned}$$

Some students may prefer to expand $(x + 2)^2 - (x + 2) - 20$, resulting in $x^2 + 3x - 18$. This can then be factored, producing the same result.

For #27, students may need to recall the area formulas for parallelograms and triangles.

Students may need some hints for #28c). Guide them through the procedure by making this carpet problem into a simpler problem. Have them imagine having 6 bricks, each of which has a front face with the dimensions 2 cm by 10 cm. Ask:

- If you stacked the 6 bricks to a height of 12 cm, what would be the area of the face of the brick wall? (120 cm^2)
- Take the 6 bricks and set them end to end to form a wall 2 cm high. What would the length of this wall be? (60 cm)
- How is the length of the wall related to the area of the wall face you found?
- How could you apply your findings in this wall question to the carpet question?

Some students may prefer to sketch this situation or use manipulatives to represent it.

Create Connections

Consider using #29, 30, and 32 as summary questions for the lesson.

For #31, consider revisiting the slope formula, negative slopes, and the slopes of vertical and horizontal lines with students. For #31c), you may wish to have students place values for p in a table in a graphing calculator or in spreadsheet program.

Meeting Student Needs

- Provide **BLM 6–5 Section 6.1 Extra Practice** to students who would benefit from more practice.
- Pair students to work through the questions. For questions #1 to 6, each student in the pair should complete a different part and then explain the answer to their partner. If they cannot come to a consensus on the correct answer, they can then consult with you or other student pairs.
- There are 17 questions in the Apply section. Divide students into groups of three. Within the groups, each student should complete five questions. Once finished, students explain their responses to the other two members of their group. This discussion should help each student understand how the concepts developed. Students can then work on the two remaining questions as a group or individually.
- Students may wish to challenge each other to another game of Rational Expression Dominoes.

ELL

- Ensure that students understand the vocabulary in the questions. Ensure that they understand terms, such as *permitted*, *shortcuts*, *preliminary*, *pesticide*, *yield*, *tube*, etc. Encourage them to include in their dictionary a verbal description, diagram, and/or example for each term.

Enrichment

Challenge students to create a script that explains to younger students, through examples, why they should learn fraction operations with real numbers so they will better understand rational number operations.

Gifted

Ask students to brainstorm and investigate why excluded values are worth identifying. What results, mathematically, if excluded values are not investigated? Why should mathematicians care?

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2a), c), 3, 4a), c), 5 to 11, 13, and 19 to 22. Students who have no problems with these questions can go on to the remaining questions.	• It is important that students can confidently solve #1 and 2. If they are having difficulty, provide extra practice before allowing them to move on. • In #3 and 4, students demonstrate that they understand how to find the non-permissible value if the denominator is already factored. It is important that they are confident in these two questions. Note that #3d) has no non-permissible value. Ask students why that is. • For questions #5 to 8, students must be able to factor and recognize that any variable could have a unique non-permissible value. Coach students through a question such as #8a), which lacks an operational sign. This will ensure that students recognize a non-permissible value for each variable. • Operations have already been performed in #9 to 11, 13, and 20 to 23. It is important that students can identify what has been done before answering. Coach students through any of these questions where they have difficulty identifying what process has been performed. • For #10, you may wish to add an additional identical factor to both the numerator and the denominator so that you can show students the previous step. Then, physically show the simplifying of the common factors, resulting in the original question. That should be sufficient to prompt students forward.
Assessment *as* Learning	
Create Connections Have all students complete #29 and 30. Students who have no difficulty with these two questions should be encouraged to try #31 and 32.	• You could have students work in pairs for these questions, or trade their answers for peer review. Note that #29 is similar to #14. Students may wish to refer back or work backward. Ask students, "If an expression has non-permissible values of 1 and −1, what would the factors be in the denominator?" Start students off by writing $x = 1$ and $x = -1$. • For #30, tell students that the first rational expression is the reduced version of the second one. Ask them, "Using factoring, how could you factor the second expression, and then cancel common factors to end up with the first expression?"

Multiplying and Dividing Rational Expressions

6.2

***Pre-Calculus 11*, pages 322–330**

Suggested Timing
90–120 min

Materials
- paper for folding
- coloured pencils
- various manipulatives (e.g., fraction strips)
- grid paper

Blackline Masters
BLM 6–4 Chapter 6 Warm-Up
BLM 6–6 Section 6.2 Extra Practice

Mathematical Processes
- [x] Communication (C)
- [x] Connections (CN)
- [x] Mental Math and Estimation (ME)
- [x] Problem Solving (PS)
- [x] Reasoning (R)
- [ ] Technology (T)
- [x] Visualization (V)

Specific Outcomes
AN5 Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–8, 10–12, 14, 21–23
Typical	#2, 4–6, 8–13, two of 14–18, 21–23
Extension/Enrichment	#11, 15, 16, 23

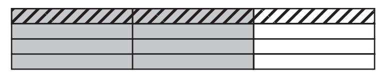
Planning Notes

Have students complete the warm-up questions on **BLM 6–4 Chapter 6 Warm-Up** to reinforce prerequisite skills needed for this section.

Some students may benefit from using manipulatives or diagrams to multiply and divide rational numbers. You may wish to provide these students with a rectangular piece of paper and coloured pencils (alternatively, students could use a single colour or lead pencil and use shading, as shown in the following diagrams). Tell students that they can use paper folding to multiply two fractions, $\frac{2}{3}$ and $\frac{1}{4}$, using the following steps:

1. Fold the paper vertically into three equal sections.
2. Colour (or shade) two of the three sections.

3. Fold the paper horizontally into four equal sections.
4. Colour one of the four rows a second colour (or use a second method of shading).



The result is that two of the sections have been coloured or shaded twice. Ask:
- How many sections are there in total?
- How many sections are a combination of the first and second colour (or the two shading patterns)?
- What does the number of mixed-colour (mixed shading) sections over the total number of sections represent?

Investigate Multiplying and Dividing Rational Expressions

Consider having students work with a partner and record their procedures and explanations. Invite students to switch partners and compare their findings with those of other students. After students have completed the investigation, reconvene as a class to discuss the results. Ask:
- How does *simplifying* rational expressions compare to *multiplying* rational expressions? How are they the same? different?
- How does multiplying rational *numbers* compare to multiplying rational *expressions*? How are they the same? different?
- How does dividing rational *numbers* compare to dividing rational *expressions*? How are they the same? different?
- What role does finding non-permissible values play in multiplying and dividing rational expressions?
- Why is it important to identify all non-permissible values before you start simplifying factors?

Meeting Student Needs

- Before beginning the Investigate, revisit the list of outcomes you gave to students. Discuss the outcome(s) already studied and ask students to rate their progress to date. Remind them to continually monitor their own progress, and to use interventions to ensure understanding. Then, introduce students to the outcomes that will be learned during this section.

- If students are creating videos depicting the chapter concepts, check their progress. Simplifying Rational Expressions should be complete.
- Have a variety of manipulatives available, such as fraction strips.
- Monitor students while they work through the Investigate. Invite students to share with the class the process they used.

ELL

- Ensure that students understand the vocabulary in the opener and Investigate, such as *environmental practices*, *Elder*, *cultural significance*, etc.
- Suggest students add the term *numerically* to their vocabulary dictionary. Encourage them to include any term they encounter with which they are not familiar. Suggest that they include a verbal description, diagram, and/or example for each term.

Common Errors

- Students may simplify parts of an expression, rather than simplifying complete factors. For example,

$$\frac{3x^2 + 7x}{4x} = \frac{3x^2 + 7\cancel{x}}{4\cancel{x}} = \frac{3x^2 + 7}{4}$$

Rx Have students substitute a value for x, such as $x = 2$, into the original expression and into the incorrectly simplified expression: $\frac{3x^2 + 7x}{4x} = \frac{26}{8}$ and $\frac{3x^2 + 7}{4} = \frac{19}{4}$. Since the substitutions do not produce the same result, the simplified expression is incorrect. Discuss with students a simple three-step procedure for multiplying fractions:

1. factor
2. simplify
3. multiply

- When dividing rational expressions, students forget to reciprocate the divisor, or they reciprocate both the divisor and the dividend.

Rx A simple, real-world scenario may help students remember this concept. Ask:
– If you had half of a pizza and divided it equally into three parts to share with two friends, what part of the whole pizza would each person receive?
– Can you draw a diagram to represent the situation?
– What portion of the full pizza does each piece represent? $\left(\frac{1}{6}\right)$
– Can you write a mathematical statement that represents the picture you drew? $\left(\frac{1}{2} \div 3\right)$
– How would you obtain $\frac{1}{6}$ from the division statement $\left(\frac{1}{2} \div 3\right)$?

Help students develop, understand, and internalize a simple four-step procedure for dividing rational expressions:

1. reciprocate the divisor
2. factor
3. simplify
4. multiply

Answers

Investigate Multiplying and Dividing Rational Expressions

1. The product is $\frac{3}{8}$ which is determined by multiplying the numerators and multiplying the denominators.

2. Multiplying the numerators gives $(x + 3)(x + 1) = x^2 + 4x + 3$. The product of the denominators is 8. The product of the two rational expressions is $\frac{x^2 + 4x + 3}{8}$.

3. To divide by a rational number, multiply by its reciprocal.

$$\frac{2}{3} \div \frac{1}{6} = \frac{2}{3}\left(\frac{6}{1}\right) = \frac{12}{3} = 4$$

4.
$$\frac{x-3}{x^2-9} \div \frac{x}{x+3} = \left(\frac{x-3}{x^2-9}\right)\left(\frac{x+3}{x}\right) = \frac{\overset{1}{\cancel{(x-3)}}\overset{1}{\cancel{(x+3)}}}{\underset{1}{\cancel{(x-3)}}\underset{1}{\cancel{(x+3)}}(x)} = \frac{1}{x}$$

Multiplying by the reciprocal allows the expression to be simplified if the numerators and denominators share like factors.

5. $x \neq -3, 0, 3$. Set the denominators in the original expression and the final expression equal to zero. Then, solve each equation for x to determine the non-permissible values.

6. The rules for operations with numerical fractions can be used for operations with rational expressions as long as the necessary factoring is done.

Answers

7. Factor all portions of all numerators and denominators. Determine all non-permissible values. Divide any factors that reduce to 1. Simplify.

8. Change the division question to multiplication, keeping in mind that you need to check for non-permissible values. Then, use the process for multiplying rational expressions.

$$\frac{x+5}{x^2-25} \div \frac{x}{x-5}$$
$$= \left(\frac{x+5}{x^2-25}\right)\left(\frac{x-5}{x}\right)$$
$$= \left(\frac{\overset{1}{\cancel{x+5}}}{\underset{1}{(\cancel{x+5})}\underset{1}{(\cancel{x-5})}}\right)\left(\frac{\overset{1}{\cancel{x-5}}}{x}\right)$$
$$= \frac{1}{x}$$

From original expressions, $x \neq -5, 5$, and from the inverted divisor $x \neq 0$.

9. Fractions and rational expressions only have meaning when the values for the variables result in meaningful numerical values in both the numerator and denominator. A denominator cannot take on a value of zero, so non-permissible values must be identified.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings	• If students have difficulty explaining how the process works, redo the division shown in #3, with the factors of 6 showing. Simplifying the 3s should prompt students to think about how to answer #4 and 7.

Link the Ideas

The Link the Ideas section presents an excellent example to reinforce that multiplication of rational expressions follows the same procedure as multiplying rational numbers, but with the added necessity of determining non-permissible values for the variables.

Before working through the examples, remind students to follow the correct order of operations when there is more than one operation per question. If multiplication and division are found in the same question, they must be completed as they appear from left to right.

Example 1

Ask students:

- What do you think would happen if you multiplied the numerators and the denominators before factoring? Suggest they try it.
- How does this example relate to the rational number example in the Link the Ideas section?
- What would the answer be if all the factors were simplified? Explain.
- At what step do you determine the non-permissible values? Why?

Remind students of the three-step procedure for multiplying fractions: factor, simplify, and then multiply.

Example 2

Some students may not be familiar with the method of bringing rational numbers to a common denominator and then dividing the numerators. Give students a couple of examples, such as the two below, and ask them to determine the procedure and explain why it works.

Examples:

$$\frac{7}{8} \div \frac{2}{3}$$
$$= \frac{21}{24} \div \frac{16}{24}$$
$$= \frac{21}{16}$$

$$3\frac{1}{4} \div 2\frac{5}{8}$$
$$= \frac{13}{4} \div \frac{21}{8}$$
$$= \frac{26}{8} \div \frac{21}{8}$$
$$= \frac{26}{21}$$

Reinforce the comparison between dividing rational numbers and dividing rational expressions as given in the student resource ahead of Example 2.

Discuss with students which method they find preferable — using a common denominator or multiplying by the reciprocal. Show them what would be involved if the common denominator method was used in Example 2:

$$\frac{(x+2)(x-2)(x+5)}{x(x-4)(x+5)} \div \frac{x(x+3)(x-2)}{x(x+5)(x-4)}$$

$$= \frac{(x+2)\cancel{(x-2)}(x+5)}{x(x+3)\cancel{(x-2)}}$$

$$= \frac{(x+2)(x+5)}{x(x+3)}$$

Complex rational numbers and rational expressions can be confusing for students. Remind them that a fraction bar is another symbol for division. Suggest that they use boldface to emphasize the fraction bar that separates the two rational expressions in a complex fraction.

You may want to discuss the use of brackets or the raised dot to indicate multiplication. Some students may expect the multiplication symbol, $\times$. Explain that this symbol is avoided because it can be confused with the variable x.

Discuss the need to find the non-permissible values for both the numerator and the denominator of the divisor.

Develop a simple four-step procedure for dividing rational expressions:

1. reciprocate the divisor
2. factor
3. simplify
4. multiply

Example 3

Begin by asking students to recall the order of operations. Then, for the example, discuss with students the choice of finding the quotient first and then multiplying, versus rewriting the entire expression as one multiplication expression, as shown in the example.

Some students may need to be reminded that the quotient of an expression and its additive inverse, or opposite, is -1. For example, $\frac{2m-3}{3-2m} = -1$ and $\frac{(2m-3)}{-1(-3+2m)} = \frac{1}{-1}$ or -1.

Discuss which polynomials in a division question must be examined to find all of the non-permissible values.

Key Ideas

Ask students to rewrite the Key Ideas in their own words, including examples, and place their work in their summary notes. (Note that the third bullet in the Key Ideas is an excellent summary of finding the non-permissible values in a division statement.)

Meeting Student Needs

- Divide students into groups and have them build more Rational Expression Dominoes. For the dominoes in this section, have students focus on multiplying and dividing. The end of one domino will contain a question, and the end of another domino will contain the answer to that question. Each group should create 12 to 15 dominoes.
- Create a series of questions similar to $\frac{3+8}{8+5} = \frac{3+\cancel{8}}{\cancel{8}+5} = \frac{3}{5}$. Ask students to determine if the solution is correct and, if not, to identify the error. Students will develop the understanding that numbers cannot be simplified when added or subtracted. You might also divide students into groups to create their own, similar examples. They will put their question and a proposed solution on the front of a paper; this proposed solution may be correct or it may contain an error. On the back of the paper, students will state whether the proposed solution contains an error and, if so, what the correct solution is. Groups could then exchange questions to work through.

Common Errors

- There are three common errors that some students make when finding non-permissible values:
  1. They forget to identify the non-permissible values before simplifying and multiplying.
  2. They forget to identify the non-permissible values for the numerator of the divisor in a division statement.
  3. They forget to identify the non-permissible values for the equivalent forms of a rational expression. The non-permissible values must be determined in each case before the expression is simplified.

R_x Remind students to always consider their results in terms of both the original and simplified expression. They should ask themselves, "Does substitution result in any term equalling zero? If so, what are the implications for the expression?"

Answers

Example 1: Your Turn

a) $\frac{dh}{d-2}$, $r \neq 0$, $d \neq 2$

b) $\frac{y-3}{r+1}$, $r \neq 0, -1, 1$, $y \neq -3$

Example 2: Your Turn

$\frac{c}{c+7}$, $c \neq -7, -1, 0, 7$

Example 3: Your Turn

$\frac{1}{2}$, $x \neq -4, 3, -\frac{4}{3}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Remind students that it is important to state the non-permissible values before they begin simplifying factors. Doing so helps ensure that no non-permissible values are missed.
Example 2 Have students do the Your Turn related to Example 2.	• Encourage students to write the complex rational expression in the form of a division statement. • Remind students that it is important to state the non-permissible values for the existing divisions' denominators before simplifying or inverting the second fraction. • Again, encourage students to find the non-permissible value of the reciprocal fraction before simplifying.
Example 3 Have students do the Your Turn related to Example 3.	• Remind students that it is important to state the non-permissible values for the existing questions' denominators before finding the reciprocal or simplifying common factors. • Again, encourage students to find the non-permissible value of the reciprocal fraction before simplifying. Remind them to only find the reciprocal for the rational expression that occurs immediately following the division sign, and not the reciprocal for all terms.

Check Your Understanding

Before beginning the Practise questions, you may want to revisit factoring polynomials of the type $ax^2 + bx + c$. For example,

$2z^2 - 13z + 20$
$2w^2 - w - 6$
$3x^2 - 5x - 2$
$8y^2 - 2y - 3$
$3y^2 - 3y - 2$

All the polynomials of this form in this lesson are factorable.

When students are simplifying a rational expression that involves factoring polynomials of the form $ax^2 + bx + c$, you can give them the hint to look at the factors in either the numerators or denominators of the expression they are simplifying: often one of the factors is a factor of the $ax^2 + bx + c$ polynomial.

Some students may try using the quadratic formula and working backward to find the factors. For example, in #8d), if students used the quadratic formula to solve $8y^2 - 2y - 3 = 0$, they would obtain the solutions $\frac{3}{4}$ and $-\frac{1}{2}$. They could then work backward to form the factors from $y = \frac{3}{4}$ and $y = -\frac{1}{2}$: $(4y - 3)$ and $(2y + 1)$.

Practise

Students need to find products in #1, 2, 5, and 7. They should be familiar with the term *product* and understand it is the answer to a multiplication statement.

Students are asked to find the reciprocal of a rational expression in #3, which they must be able to do when dividing rational expressions.

Students will divide rational expressions in #4, 6, 8, and 9. Students should be familiar with the word *quotient* and understand that it is the answer to a division statement.

You may want to mention that all questions except #3 and 7 ask for the non-permissible values.

The need to find the non-permissible values of the numerator in the divisor of a division statement is reinforced in #9.

Apply

If students are having difficulty with #12, ask them how the two division statements are similar and different. Then, provide students with a rational number example, such as the following:

$$\frac{5}{6} \div \frac{1}{2} \qquad \frac{1}{2} \div \frac{5}{6}$$

$$= \frac{5}{6}\left(\frac{2}{1}\right) \qquad = \frac{1}{2}\left(\frac{6}{5}\right)$$

$$= \frac{10}{6} \text{ or } \frac{5}{3} \qquad = \frac{6}{10} \text{ or } \frac{3}{5}$$

Ask them what summary statement they can make. ($\frac{5}{3}$ and $\frac{3}{5}$ are reciprocals)

You could also ask students to prove that the quotients will always by reciprocals by asking them to simplify the quotients $\frac{A}{B} \div \frac{C}{D}$ and $\frac{C}{D} \div \frac{A}{B}$.

Students have used similar procedures to the one used in #13 in their science courses.

Students work with a common error in #14, reciprocating the dividend instead of the divisor. Ask them to find an example involving rational numbers to show that reciprocating the dividend will result in an incorrect answer.

Students work with the areas of rectangles and triangles in #15 and 16. Some students may need to be reminded of the formulas for these areas.

The use of rational expressions in an equation is introduced in #18. Remind students to follow the usual procedures for solving an equation: isolate the variable V_1 by performing the same operations to both sides of the equation.

Extend

Students may recognize the connection with Chapter 5, particularly with rationalizing binomial denominators, when working on #19. Similarly, #20 connects to work they did in Chapter 2. Students may need to be reminded that the acceleration due to gravity is a constant, so they will use the value 9.8 m/s^2 in parts a) and b).

Create Connections

For #21, ask students how multiplying and dividing rational expressions are like multiplying and dividing fractions? How are the two different?

Students will probably recognize a connection to their previous work in coordinate geometry when working on #22. Students may need to be reminded about the slope formula and the relationship between the slopes of two perpendicular lines.

Students' knowledge of trigonometry will also connect to #23. Ask students if they can find other identities involving the basic trigonometric ratios.

Project Corner

Space travel is a topic that many students find interesting. The Project Corner presents several examples of areas related to space travel in which rational expressions could be applied. Students could research and locate some of the expressions used. Encourage the class to brainstorm additional topics that might involve the use of rational expressions.

Meeting Student Needs

- Provide **BLM 6–6 Section 6.2 Extra Practice** to students who would benefit from more practice.
- Students could write a paragraph explaining the Key Ideas in this section. Include a list of dos and don'ts to remember.
- Invite students to create a multiplication or division question with a final answer of $\frac{(x+5)}{(x+3)}$. To complete this task, the problem solving approach would be to work backward.
- Ensure students continue to work on presentations if they are doing one. They may choose to rap, dramatize, videotape a lesson, interview each other, create a computer slide presentation, or some other final product.

ELL

- Ensure that students understand the vocabulary in the questions and Project Corner, such as *plywood*, *windmill*, *fog dissipation*, *canister*, *dry ice*, *dilation*, *phenomenon*, *gravitational mass*, *planetary*, *compelling*, etc. Suggest that they include a verbal description, diagram, and/or example for each term.

Enrichment

The Project Corner discusses time dilation, in which time slows as the speed of light or an immense mass is approached. The formula that predicts the effect of this is a rational expression. Ask students to consider and discuss how time dilation might affect space travel.

Gifted

Multiplying has been compared to fast adding, meaning that $4(30) = 120$ is easier to find than adding 4 thirty times. Encourage students to explore the similarities and differences between the process of multiplying rational expressions and the equivalent processes of adding with numbers.

Common Errors

- Students may confuse the procedures for adding rational expressions with those for multiplying rational expressions. That is, they may find a common denominator for the two factors they are multiplying. This may not result in an error, but it is unnecessary.

R$_x$ Show students the following worked solution:

$$\left(\frac{x+1}{9x}\right)\left(\frac{3x^2}{x^2-1}\right)$$

$$= \left(\frac{(x+1)(x+1)(x-1)}{9x(x+1)(x-1)}\right)\left(\frac{3x^2(9x)}{9x(x+1)(x-1)}\right)$$

$$= \left(\frac{\overset{1}{\cancel{(x+1)}}\overset{1}{\cancel{(x+1)}}\overset{1}{\cancel{(x-1)}}}{\underset{3}{\cancel{9x}}\underset{1}{\cancel{(x+1)}}\underset{1}{\cancel{(x-1)}}}\right)\left(\frac{\overset{x}{\cancel{3x^2}}\overset{1}{\cancel{(9x)}}}{\underset{1}{\cancel{9x}}\underset{1}{\cancel{(x+1)}}(x-1)}\right)$$

$$= \frac{x}{3(x-1)}$$

The answer is correct, but students did not have to find the common denominator. Instead, they could have simply factored the terms, simplified, and then multiplied. Ask:

- Is the answer correct?
- What steps are unnecessary?
- What is a simpler why of obtaining the product?

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 8, 10, and 11 to 14. Students who have no problems with these questions can go on to the remaining questions.	• For #1 and 2, remind students to factor the denominator first and determine the non-permissible values before cancelling factors. The same is true for #4 and 8, where students should identify the non-permissible values of the original equation and of the reciprocal expressions. • For #5 to 9, you may need to remind students of the meaning of *quotient* and *product*, since they frequently confuse the two. • Students may have difficulty identifying what operation is asked for in #10. Ask them, "If you had $100 and wanted to know how many $5 bills this amount represents, how would you complete the task?" You may wish to have them complete #15 before #10; the two questions are similar. • #14 requires students to apply what they have learned about simplifying rational expressions and completing both operations.
Assessment *as* Learning	
Create Connections Have all students complete #21 to 23.	• You may wish to have students work in pairs for these questions. • You may need to revisit the slope formula with students before beginning #22. It may also be useful to discuss the importance of brackets and why we use them.

6.3 Adding and Subtracting Rational Expressions

***Pre-Calculus 11*, pages 331–340**

Suggested Timing

90–120 min

Materials

- fraction strips
- pattern blocks
- grid paper

Blackline Masters

BLM 6–4 Chapter 6 Warm-Up

BLM 6–7 Section 6.3 Extra Practice

Mathematical Processes

- ☑ Communication (C)
- ☑ Connections (CN)
- ☑ Mental Math and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☐ Technology (T)
- ☐ Visualization (V)

Specific Outcomes

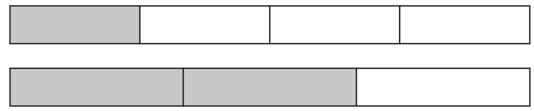
AN5 Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–4, 5a)–c), 6a)–c), 7a), b), 8, 9, 10a), b), 12, 24, 28
Typical	#2–4, 5b), d), f), 7c), d), 8, 9, 10a), b), 11–13, 15b), one of 14, 16, or 17, 18, 25, 27, 28
Extension/Enrichment	#11b), 15d), 18–24, 26–28

Planning Notes

Have students complete the warm-up questions on **BLM 6–4 Chapter 6 Warm-Up** to reinforce prerequisite skills needed for this section.

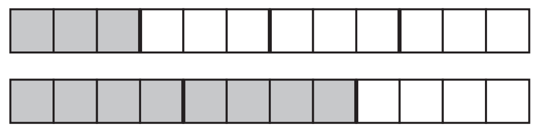
Students may be unfamiliar with using diagrams or manipulatives for adding and subtracting fractions as in the Investigate. Give each student two strips of paper the length of a notebook page and 2 to 3 cm wide. Explain that they will use these strips to find the sum $\frac{1}{4} + \frac{2}{3}$. Ask students to fold one of the strips into fourths and shade one of the resulting sections. Then, fold the second strip into thirds, shading two of the sections.



To add the two, the coloured parts of each strip must contain parts that are the same size. Ask:

- How would you fold the strips so that both strips have parts that are the same size?
- What part of the whole is each section of the strip?
- How many coloured sections are there?

  So, $\frac{1}{4} + \frac{2}{3} = \frac{11}{12}$



Ask students to try another simple addition of fractions question using diagrams or manipulatives. Then, ask them to move beyond the concrete to the symbolic.If you created a table of fraction operations at the beginning of the chapter (see Chapter Opener notes), refer students to the first two rows of the table:

Fraction Operations		
Operation	**Method**	**Example**
Addition	**1.** Find a common denominator. **2.** Rewrite each fraction in equivalent form with the LCD. **3.** Add the numerators. **4.** Simplify.	$\frac{7}{12} + 2\frac{2}{9}$ LCD is 36. $\frac{21}{36} + 2\frac{8}{36}$ $2\frac{29}{36}$
Subtraction	**1.** Find a common denominator. **2.** Rewrite each fraction in equivalent form with the LCD. **3.** Subtract the numerators. **4.** Simplify.	$1\frac{5}{6} - 2\frac{7}{10}$ LCD is 30. $\frac{55}{30} - \frac{81}{30}$ $-\frac{26}{30}$ $\frac{-13}{15}$

Investigate Adding and Subtracting Rational Expressions

Discuss #2 and 3 in the investigation. Ask:

- How are the procedures for adding and subtracting rational numbers the same as adding and subtracting rational expressions?

- How are the procedures for adding and subtracting rational numbers different from adding and subtracting rational expressions?

You may need to revisit how to find least common denominators for rational numbers. This procedure can then be extended to finding least common denominators for rational expressions.

Ask students to create and complete a table like the one below.

Example	Situation	Method
$\frac{1}{8} + \frac{3}{4}$		
$\frac{2}{5} + \frac{4}{7}$		
$\frac{5}{12} + \frac{11}{18}$		

This exercise will help students discover that there are basically three situations that can occur when finding the LCM of two or more numbers. Ask:

- What is the least common denominator for each addition question?
- How did you find the LCD for each question?
- Can you list other addition examples that fit each situation?

Here is a sample completed table:

Example	Situation	Method
$\frac{1}{8} + \frac{3}{4}$	The common factor is one of the denominators.	Use the larger denominator as the LCD. The LCD is 8.
$\frac{2}{5} + \frac{4}{7}$	The denominators have no common factors.	Multiply the denominators to find the LCD. The LCD is 35.
$\frac{5}{12} + \frac{11}{18}$	The denominators have a common factor.	The LCD must contain the greatest number of any factor that appears in the denominator of either fraction. If a factor appears once in either or both denominators, include it only once. If a factor appears twice in any denominator, include it twice. $12 = (2)(2)(3)$ $18 = (2)(3)(3)$ LCD $= (2)(2)(3)(3)$, or 36

Next, extend student knowledge of finding the LCD of rational numbers to finding the LCD to rational expressions. How might they be the same? How might they be different?

Meeting Student Needs

- Before beginning the Investigate, revisit the list of outcomes you gave students. Discuss the outcome(s) already studied and ask students to rate their progress to date. Remind students to continually monitor progress and use interventions to ensure understanding. Introduce the outcomes that will be learned during this section.
- If students are creating videos, check on their progress. Multiplying and Dividing Rational Expressions should be complete.
- Have students brainstorm the steps involved in adding and subtracting rational numbers. Post the list, with examples, on the board.
- Students may wish to use pattern blocks or fraction strips for the first question of the Investigate.
- Students could work through the Investigate in pairs or groups of four, with each student doing one or two questions. Alternatively, they could work together on the questions within their groups, discussing the process as they go.

Common Errors

- Some students add or subtract the numerators without first writing the fractions in equivalent form, with a common denominator.

R_x Show students the following examples and ask what was done incorrectly in each case.

Example 1: $\frac{1}{4} + \frac{2}{3} = \frac{3}{7}$

Example 2:

$$\frac{1}{4} + \frac{2}{3}$$
$$= \frac{1}{12} + \frac{2}{12}$$
$$= \frac{3}{12}$$

Then, ask students to do the question using the correct procedure.

- Students sometimes cancel numerator to numerator or denominator to denominator.

R_x Present the following example. Ask if the addition was done correctly and, if not, where the error was made. Then, ask students to use the correct procedure. What is the correct answer?

$$\frac{7y(y-1)}{8} + \frac{5y^2}{8}$$
$$= \frac{7\cancel{y}^{1}(y-1)}{8} + \frac{5y^{\cancel{2}\,1}}{8}$$
$$= \frac{35y(y-1)}{8}$$

- Students often cancel numerators and denominators before adding.

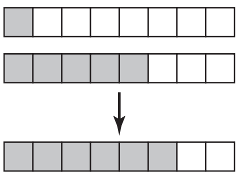
R$_x$ Present the following example. Ask if the addition was done correctly and where the error was made. Then, ask students to use the correct procedure. What is the correct answer?

$$\frac{2x}{4x(2x-3)} + \frac{8}{4x(2x-3)}$$
$$= \frac{2\cancel{x}^{1}}{4\cancel{x}_{1}(2x-3)} + \frac{\cancel{8}^{2}}{\cancel{4}_{1}x(2x-3)}$$
$$= \frac{2+2}{4x(2x-3)}$$
$$= \frac{\cancel{4}^{1}}{\cancel{4}_{1}x(2x-3)}$$
$$= \frac{1}{x(2x-3)}$$

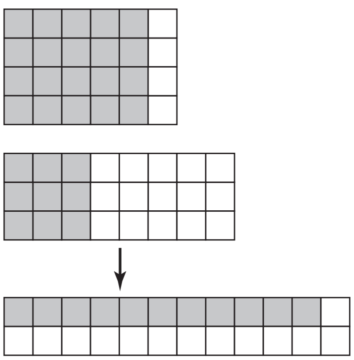
Answers

Investigate Adding and Subtracting Rational Expressions

1. a) $\frac{1}{8} + \frac{5}{8} = \frac{6}{8}$, or $\frac{3}{4}$



b) $\frac{5}{6} - \frac{3}{8} = \frac{20}{24} - \frac{9}{24}$



2. a) $x \neq 0$; $\frac{12x-1}{x}$

b) $1, x \neq 3$

c) $x \neq -2, 2$; $\frac{2(x+8)}{(x-2)(x+2)}$

3. Example: For $x = 4$, part a) is $\frac{47}{4}$, and part c) is 2, which both check by substituting into the original and the simplified expression.

4. Example: In both parts, the first step was to determine non-permissible values. In part b), there already was a common denominator, so it was just a matter of subtracting the numerators. Part c) needed a common denominator before the subtraction operation could be performed.

5. Example: Factor and reduce the original fractions. Find the non-permissible values. Then, find the LCD and make equivalent fractions. Add or subtract the numerators, leaving the answer over the common denominator. Factor and reduce, if possible.

6. Example: Once you learn one, you know how to do another.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Both of these questions provide students with an opportunity for assessment *as* learning. Some students may find it easier to describe and explain by referencing an example. Have them generate their own example that could be used for this purpose.

Link the Ideas

Encourage students to write the summary in their own words and include examples of their own in their Foldable, graphic organizer, or with their summary notes.

Create a table similar to the one in the Planning Notes, but in this case for finding the LCD for rational expressions. Have students create this table individually, in pairs, or in small groups, or create the table as a class. Here is a sample completed table.

Example	Situation	Method
$\frac{5}{x+3} + \frac{1}{4x+12}$	The common factor is one of the denominators.	Use the larger denominator as the LCD. The LCD is $4x + 12$.
$\frac{2}{5x} + \frac{4}{x-1}$	The denominators have no common factors.	Multiply the denominators to find the LCD. The LCD is $5x(x - 1)$, or $5x^2 - 5x$.
$\frac{7}{x^2-9} + \frac{5}{x^2+5x+6}$	The denominators have a common factor.	The LCD must contain the greatest number of any factor that appears in the denominator of either fraction. If a factor appears once in either or both denominators, include it only once. If a factor appears twice in any denominator, include it twice. $x^2 - 9 = (x + 3)(x - 3)$ $x^2 + 5x + 6 = (x + 3)(x + 2)$ LCD is $(x + 3)(x - 3)(x + 2)$.

When discussing Case 1 in the Link the Ideas, elicit from students the fact that the numerator may be simplified by collecting like terms. After that has been done, they should check whether the numerator and denominator have any common factors that can be divided to express the single rational expression in simplest form.

Example 1

When the subtrahend contains a polynomial of more than one term, students often make the error of subtracting the first term, but do not use the subtraction operation on the second term. For example, $(3x - 1) - (4x - 10) = -x - 11$. Students may find it helpful to rewrite the subtraction statement as an addition statement:

$$\begin{aligned}&(3x - 1) - (4x - 10)\\ &= (3x - 1) + (-4x + 10)\\ &= -x + 9\end{aligned}$$

Example 2

Students may need help understanding how the LCD is found in part a). If you have not guided them through examples similar to the table in the Link the Ideas notes, now would be a good time to do so.

A common error that occurs in questions similar to part c) is that students try to cancel the $\frac{1}{x}$ term from the numerator and denominator. You may need to walk students through the procedure shown. In the first line, remind students that $1 = \frac{x}{x}$ and $x = \frac{x}{1}$. Encourage students who are having difficulty to rewrite the original expression using these forms of 1 and x:

$$\frac{\frac{x}{x} + \frac{1}{x}}{\frac{x}{1} - \frac{1}{x}}$$

Consider showing students the alternative way of simplifying the expression, which is multiplying the numerator and the denominator by the LCD, x. Some students may prefer this method.

$$\begin{aligned}&\frac{1 + \frac{1}{x}}{x - \frac{1}{x}}\\ &= \frac{x\left(1 + \frac{1}{x}\right)}{x\left(x - \frac{1}{x}\right)} && \text{Multiply by 1 in the form of } \tfrac{x}{x}.\\ &= \frac{x + \frac{x}{x}}{x^2 - \frac{x}{x}} && \text{Expand.}\\ &= \frac{x + 1}{x^2 - 1} && \text{Simplify.}\\ &= \frac{(x + 1)}{(x + 1)(x - 1)} && \text{Factor.}\\ &= \frac{\overset{1}{\cancel{(x + 1)}}}{\underset{1}{\cancel{(x + 1)}}(x - 1)} && \text{Simplify.}\\ &= \frac{1}{(x - 1)}\end{aligned}$$

Take the time to have students demonstrate to you that they can complete the Your Turn questions correctly.

Key Ideas

Ask students to rewrite the Key Ideas in their own words with examples, and place their work in their summary notes.

Meeting Student Needs

- You may wish to provide a few quick examples of polynomials that involve a leading negative coefficient, such as $-(x - 3)$, $-(2y + 3)$ and $-3(x - 4)$. You could also explore the concept of factoring the common factor, for example, $12x + 4 = 4(3x + 1)$.
- In the solution for Example 2, part b), some students may wish to use FOIL to expand the numerator prior to combining all terms over the common denominator.
- Have students create Rational Expression Dominoes using adding and subtracting rational expression questions. They will put the question on the end of one domino, and the answer on the end of another.

Common Errors

- Students sometimes forget to check whether the sum or difference they obtain can be simplified.

R$_x$ Remind students to look for common factors between and within terms. For example, the expression $\frac{2}{4x + 12}$ could be simplified to $\frac{1}{2x + 6}$ by factoring the common 2. The denominator can be simplified further to $\frac{1}{2(x + 3)}$.

Answers

Example 1: Your Turn

a) $\frac{-1}{n}$; $n \neq 0$

b) 3; $m \neq \frac{3}{4}$

c) $\frac{2x^2 - 7x - 5}{(x - 3)(x + 1)}$; $x \neq -1$ and 3

Example 2: Your Turn

a) $\frac{3p + 1}{(p - 1)(p + 1)}$; $p \neq -1, 1$

b) $\frac{4x - 5}{(x - 2)(x + 3)(x + 1)}$; $x \neq -3, -1, 2$

c) $\frac{2}{y + 2}$; $y \neq -2, 0, 2$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Remind students to determine the non-permissible values before adding and/or subtracting, and cancelling.
Example 2 Have students do the Your Turn related to Example 2.	• Remind students that factoring the denominator will help them see the factors that are already common to each term (if there are any). A good rule for them to remember when determining the LCD is that they can *gain* factors, but they can never *lose* factors. Provide them with an example, such as $\frac{2x}{(x + 3)^2} + \frac{x}{x + 3}$, which requires two factors of $(x + 3)$ in the denominator. • For part c), prompt students to write the question as a division statement, with each expression having a common denominator. You may wish to ask, "Why can you not find the reciprocal immediately and then simply cancel the *y*'s as common factors?"

Check Your Understanding

Practise

If students are having difficulty with #1 and 2, which involve adding and subtracting rational expressions with common denominators, suggest they factor the denominator out of each expression as a common factor, and then simplify.

Students have to find more than one common denominator in #4. They then have to find the LCD. Ask students, "Why do you think it is preferable to work with the LCD rather than another denominator?"

In #6, students face a similar problem as in #5. Both questions involve adding and subtracting rational expressions with unlike denominators. However, in this case some of the polynomials are binomials and trinomials.

Students must first simplify the expressions before they find the LCD in #7. There are no similar examples in the student resource, so students may need some coaching. Ask, "Why would you want to simplify the expression before finding the LCD?"

Apply

The common error illustrated in #8 is that of incorrectly subtracting the numerators when the subtrahend has more than one term. Review this concept with students, since some may not have understood it previously and others may need to reactivate this knowledge.

The simplification of complex fractions is illustrated in #10. This is similar to Example 2, part c). Ask students to try both methods of simplification that are discussed in the example:

- Write the numerator and denominator as a single fraction, and then rewrite the division as multiplication.
- Multiply the numerator and denominator by the LCD of the fractions to simplify the complex fraction to a regular rational expression. (See notes for Example 2.)

In #11, some students may be intimidated by expressions that contain only variables. Suggest that they assign a monomial to each capital variable. For example, let $A = x$, $B = 2$, $C = 3y$, and $D = 5$. Substituting these terms and simplifying is another way of showing that both expressions are equivalent.

In #12, watch for the common errors of not squaring the denominators of the fractions and incorrectly squaring a binomial. Ask:

- Is the statement $\left(\frac{x}{2}\right)^2 = \frac{x^2}{2}$ true or false? Explain.
- Is $(x - 1)^2 = x^2 + 1$ an example of the correct way to square a binomial? If no, what is the correct method?

Some students may find #13 confusing. Suggest that they give m a value, such as $m = 2$. This can be used to get a numerical value for the number of weeks. Then, students can write the difference as a rational expression and compare it to their numerical value.

Real-life situations in which rational expressions are added and subtracted are presented in #14 and 16.

You may want to discuss BEDMAS before students work through #15. Ask students:

- In each of the parts what operation would you perform first?
- What is the order of operations?

To motivate students to do #17, you could make this a hand-in assignment. You might even tell students that you will be selecting some student-written problems to be included in an upcoming quiz, test, or other form of evaluation.

You could use #19 as a lead-in to a discussion about the distributive property. In effect, the polynomial in the denominator is being distributed over each term in the numerator. Present students with the example $\frac{3x - 7}{x}$. Ask:

- Dividing by x is the same as multiplying by what expression?
- How can you distribute this expression over the numerator, $3x - 7$?
- What is the simplified expression?

$$\begin{aligned}&\tfrac{1}{x}(3x - 7)\\ &= \frac{3x}{x} - \frac{7}{x}\\ &= 3 - \frac{7}{x}\end{aligned}$$

Be careful that students do not make the following common error when considering this example:

$$\begin{aligned}&\frac{3x - 7}{x}\\ &= \frac{3x}{x} - 7\\ &= 3 - 7\\ &= -4\end{aligned}$$

You might even present this solution and ask students what error was made.

Use 19b) for a class discussion. It will serve as a reminder that rational expressions can have many equivalent forms. Ask students:

- Can you find another expression equivalent to $\frac{3x - 7}{x}$?
- Can you find two rational expressions that could be added or subtracted so the sum/difference is the equivalent expression you wrote?

If students are having difficulty with #20, have them revisit #11. Both questions involve substituting values for the variables into the expression before simplification and after simplification.

Extend

For #22, you may need to revisit the slope formula with students. You might also suggest that students check their prediction by entering different values of p on a graphing calculator or spreadsheet program.

To make #23 less intimidating, suggest that students remove the brackets and group together the fractions with common denominators. You might also suggest that they replace the variables with values before simplification.

Create Connections

Students could make an excellent journal entry with #24.

For #25, students can compare performing operations on rational numbers to performing operations on rational expressions.

For #26, make sure students understand that this method only works with numerators of 1. Have them try examples with numerators other than 1. Ask:

- Does this process work for the expression $\frac{1}{5} + \frac{1}{6}$?
- Does it work for the expression $\frac{2}{3} + \frac{7}{10}$?
- Show algebraically that $\frac{1}{a} + \frac{1}{b} = \frac{a + b}{(a)(b)}$.

It would be helpful for students to do #26 before they do 27. In #27a), they can use the relationship that was shown in 26.

The exploration in the Mini Lab, #28, recalls a method that students may have discovered in earlier grades to add and subtract fractions. It is a cross product procedure, where the common denominator is not necessarily the LCD, but is the product of the two denominators. The numerators of the equivalent fractions are found by multiplying the numerator of one fraction with the denominator of the other fraction. For example,

$$\frac{5}{3} + \frac{7}{6}$$
$$= \frac{30 + 21}{18}$$
$$= \frac{51}{18}$$

This method is simple to prove algebraically:

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}$$

In the Mini Lab, students explore using this cross product procedure in reverse.

Meeting Student Needs

- Provide **BLM 6–7 Section 6.3 Extra Practice** to students who would benefit from more practice.
- Allow students time to add to their video presentation.
- Students should try one part of each Practise question. If successful, allow them to choose questions in the Apply section. Those who experience difficulty should complete more parts of the questions in the Practise section before moving on.

ELL

- Ensure that students understand the vocabulary in the questions, such as *ohms*, *resistors*, *resistance*, *convex*, *focal length*, etc.
- Encourage students to include any term in their dictionary that they encounter and with which they are not familiar. Suggest that they include a verbal description, diagram, and/or example for each term.

Enrichment

The formula for resistors arranged in parallel is given in #20. Ask students to compare the formulas for electrical resistance when resistors are in series, versus when they are in parallel. What might explain the similarities and differences between the two formulas?

Gifted

The logic calculators use to perform operations is introduced in #11. Extend this discussion of calculators and computers to discuss binary numbers. Share with students that the number 55 in decimal is equivalent to the binary number 110111:

1	1	0	1	1	1
32	16	8	4	2	1

$(1 \times 32) + (1 \times 16) + (0 \times 8) + (1 \times 4) + (1 \times 2) + (1 \times 1)$, or $32 + 16 + 0 + 4 + 2 + 1 = 55$

Challenge students to explore why the binary number system is better suited to an electronic device. (Each digit can be represented by an electrical current being off (0) or on (1). Computer memory is made up of *bytes*, which in turn are made up of eight aligned *bits*. Each bit represents a digit in a binary number. By turning the current on or off for each bit, one byte can represent a binary number of eight digits, or 0 through 255.)

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 4, 5a) to c), 6a) to c), 7a), b), 8, 9, 10a), b), and 12. Students who have no problems with these questions can go on to the remaining questions.	• For #1 and 2, students are provided with like denominators. If they are having difficulty, ask "How would you add $\frac{1}{3}$ and $\frac{2}{3}$? Would you add the 3s?" (a common mistake) You may wish to ask them if the non-permissible value for each term will be different or the same. This should prompt them to realize that unlike denominators have different non-permissible values. This is something that you can reiterate when students get to #6 and 7. • For #3 and 4, students have minimal work to find the LCD. The denominators are not equal, but their values should be quickly determined. Note that #4 asks for two denominators. If this confuses students, have them find the LCD only. • You may need to guide students through #10a). Have them write the complex rational as a division statement. Prompt them to find the LCD and make changes for each of the two rational expressions separately. Remind students that they should be finding the non-permissible values for both the original and the reciprocal before they cancel factors. • Encourage students to draw a diagram for #12 and label the legs of the triangle. Have them verbalize the Pythagorean Theorem, prompting them if necessary. Remind them that squaring a value is like multiplying it by itself. You may need to show them an example, such as $\left(\frac{2x}{5}\right)^2 = \frac{4x^2}{25}$.
Assessment *as* Learning	
Create Connections Have all students complete #24 and 28. Students who have no difficulty with these two questions should be encouraged to try #25 to 27.	• You may wish to have students work in pairs for these questions. • Students explain the similarities and differences of adding and subtracting rational expressions from rational numbers in #24, which makes this a good assessment *as* learning question. Students should keep this in their notes for the chapter review. Encourage students to write their own example. • For #28, have students work with a partner to find the missing values for *A* and *B*. Have pairs compare their answers with those of another group.

6.4 Rational Equations

***Pre-Calculus 11*, pages 341–351**

Suggested Timing

120–150 min

Blackline Masters

BLM 6–4 Chapter 6 Warm-Up

BLM 6–8 Section 6.4 Extra Practice

Mathematical Processes

- ☑ Communication (C)
- ☑ Connections (CN)
- ☑ Mental Math and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☐ Technology (T)
- ☐ Visualization (V)

Specific Outcomes

AN6 Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–7, two of 8–11, 14, 19, 24, 25, 27
Typical	#1–7, one of 9–11, one of 12–14, one of 15–19, 24–27
Extension/Enrichment	#3, 7, 20–27

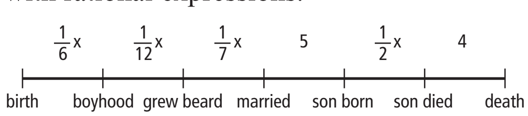
Planning Notes

Have students complete the warm-up questions on **BLM 6–4 Chapter 6 Warm-Up** to reinforce prerequisite skills needed for this section.

Investigate Rational Equations

For steps 1 and 2, ask students:

- If we assume Diophantus's age is a whole number, what does that tell you about his age if the fractions $\frac{1}{6}$, $\frac{1}{7}$, and $\frac{1}{12}$ represent different parts of his life?
- What would a time line of his life look like? Label it with rational expressions.



- Could the riddle be solved by working backward?
- Is there another method to find Diophantus's age other than writing an equation?
- Is there more than one equation that could be used to find his age?

Meeting Student Needs

- Before beginning the Investigate, revisit the list of outcomes you gave students. Discuss the outcome(s) already studied and ask students to rate their progress to date. Remind students to continually monitor progress and use interventions to ensure understanding. Then, introduce the outcomes that will be learned during this section.
- If students are creating videos, check on their progress. Adding and Subtracting Rational Expressions should be complete.
- Have students work in small groups to go through the Investigate. Students can compare equations that they have developed to represent Diophantus's life, and determine which equation is correct.
- Students could research the work completed by Diophantus and make a short presentation to the class outlining his major contributions.

Common Errors

- Students often get the procedures for adding and subtracting rational expressions confused with those for solving equations that involve rational expressions. For example, in the equation $\frac{2}{x} - \frac{3}{2x} = \frac{1}{3}$ they find a common denominator for the left side of the equation, and then are unsure how to proceed.

$$\frac{2}{x} - \frac{3}{2x} = \frac{1}{3}$$
$$\frac{4}{2x} - \frac{3}{2x} = \frac{1}{3}$$
$$\frac{1}{2x} = \frac{1}{3}$$

R_x Guide students to the point shown above and then guide them through two alternative methods.

Method 1: Suggest that the next step could be to cross-multiply:

$$3 = 2x$$
$$\frac{3}{2} = x$$

Method 2: They could find a common denominator for the fractions $\frac{1}{2x}$ and $\frac{1}{3}$:

$$\frac{1}{2x} = \frac{1}{3}$$
$$\frac{3}{6x} = \frac{2x}{6x}$$

Since the denominators are the same, the two numerators could be set equal to form an equation that will give the answer.

$3 = 2x$

$\frac{3}{2} = x$

Web Link

For more information on mathematical modelling, go to www.mhrprecalc11.ca and follow the links.

Answers

Investigate Rational Equations

1. a) boyhood: $\frac{1}{6}x$; beard: $\frac{1}{12}x$; married: $\frac{1}{7}x$; son born: $+ 5$; son's age $= \frac{1}{2}x$; died 4 years later: $+ 4$

b) add 5

c) $x = \frac{1}{6}x + \frac{1}{12}x + \frac{1}{7}x + 5 + \frac{1}{2}x + 4$

2. a) 84; $84x = 14x + 7x + 12x + 420 + 42x + 336$

b) $x = 84$

c) He was 84 years old.

3. Example: Multiply each term on both sides by the LCD within the equation. Solve the equation.

4. Example: When adding and subtracting rational expressions, a common denominator must be determined so that equivalent sizes can be operated on. Similarly, in an equation, equivalent portions of a variable must be determined so operations can be done.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings	• Students may have difficulty creating their equation in #1. Have them work as partners and then share their equation with other pairs to solve. • Suggest that students work in groups of three. Each group member creates a riddle of their own, similar to the Diophantus riddle. They pass their riddle to a member of their group, who creates an equation for solving the riddle. The third member solves the equation and gives it back to the original creator to check the answer. If it is incorrect, the group members revisit the riddle and equation to identify where the error was introduced and to correct it.

Link the Ideas

Encourage students to write the summary in their own words, and to include examples of their own in their Foldable, graphic organizer, or with their summary notes.

Students may wish to use this alternative method to solve the equation given in the Link the Ideas:

$$\frac{x}{4} - \frac{7}{x} = 3$$

$$\frac{x^2}{4x} - \frac{28}{4x} = \frac{12x}{4x}$$

Rewrite the equation with all fractions having a common denominator of $4x$.

$$x \neq 0$$

$$x^2 - 28 = 12x$$

Since the denominators are the same, form an equation with the numerators.

$$x^2 - 12x - 28 = 0$$

Solve the quadratic equation as shown in Chapter 4. That is, factor and then set each factor equal to zero.

Example 1

Ask:

- How could you solve this equation using an alternative method?
- Which method do you prefer? Explain.

Here is an alternative method for solving this equation:

$$\frac{2}{z^2 - 4} + \frac{10}{6z + 12} = \frac{1}{z - 2}$$

Factor each denominator to find the non-permissible values and the LCD:

$$\frac{2}{(z - 2)(z + 2)} + \frac{10}{6(z + 2)} = \frac{1}{z - 2}$$

The non-permissible values are ± 2 and the LCD is $6(z - 2)(z + 2)$.

Rewrite each fraction in equivalent form with the LCD as the denominator:

$$\frac{12}{6(z - 2)(z + 2)} + \frac{10(z - 2)}{6(z + 2)(z - 2)} = \frac{6(z + 2)}{6(z - 2)(z + 2)}$$

Since the denominators are all the same, form an equation with the numerators:

$$\begin{aligned} 12 + 10(z - 2) &= 6(z + 2) \\ 12 + 10z - 20 &= 6z + 12 \\ 10z - 8 &= 6z + 12 \\ 4z &= 20 \\ z &= 5 \end{aligned}$$

Suggest students use this alternative method to solve the Your Turn.

Example 2

Ask students:

- What does the word *extraneous* mean?
- What does *finding the root* mean?
- Can you solve the equation using the alternative method we explored in Example 1?
- How does Example 2 show that it is important to always check for non-permissible values?

Example 3

This example is similar to the one students did in the Investigate Rational Equations section.

Example 4

Before going over this example, ask students:

- What is the quadratic formula?
- When do you need to use the quadratic formula?

After students have worked through the example, challenge them to find another way to solve the problem. Some students may come up with the following method:

In this table, y represents the time for the trip to Flin Flon.

	Distance (mi)	Rate (mph)	Time (h)
Trip to Flin Flon	70	$\frac{70}{y}$	y
Return to Flin Flon	70	$\frac{70}{\frac{17}{2} - y}$	$\frac{17}{2} - y$
Total			$\frac{17}{2}$

Since the rate on the return from Flin Flon was 6 mph slower, the equation is

$$\frac{70}{y} - 6 = \frac{70}{\frac{17}{2} - y}$$

$$\frac{70}{y} - 6 = \frac{140}{17 - 2y}$$

$$y(17 - 2y)\left(\frac{70}{y} - 6\right) = \left(\frac{140}{17 - 2y}\right)(y)(17 - 2y)$$

$$70(17 - 2y) - 6y(17 - 2y) = 140y$$

$$12y^2 - 382y + 1190 = 0$$

$$6y^2 - 191y + 595 = 0$$

Use the quadratic formula.

$$y = \frac{170}{6} \text{ or } \frac{7}{2}$$

Since the total time was $8\frac{1}{2}$ h, the $\frac{170}{6}$ answer is not possible. So, the time to travel to Flin Flon was $\frac{7}{2}$ h or $3\frac{1}{2}$ h. To determine the speed, substitute $y = 3\frac{1}{2}$ into the expression $\frac{70}{y}$:

$$\frac{70}{\frac{7}{2}} = 20$$

The speed travelling to Flin Flon was 20 mph.

Key Ideas

Ask students to rewrite the Key Ideas in their own words with examples, and place their work in their summary notes.

Meeting Student Needs

- Have students complete equations containing rational coefficients prior to beginning this section. Emphasize removing the fraction(s) by multiplying by the least common multiple.
- Either create a poster containing the steps involved in solving a rational equation as a class, or have students create posters in small groups. Display the poster(s) in the classroom, along with an example. Colour-code each step of the process, and highlight the process with the appropriate colour within the example.
- For Example 2, ask students to write the step-by-step process illustrated in the solution. Remind them to be cautious when working with $-(k + 2)(k + 1)$. Ask them to explain why this part of the solution process may need particular attention.
- Encourage students to work through Examples 3 and 4 in pairs or groups of three. In Example 4, ask, "Why was the quadratic equation used?"

- Students should discuss how to determine whether a solution is valid. As a class, discuss why it is important to consider solutions within the context of the question before declaring a solution valid.

ELL

- Ensure that students understand the vocabulary in the examples, such as *trappers*, *inappropriate*, *context*, etc.

Answers

Example 1: Your Turn

$y \neq 3, 6; y = 12$

Example 2: Your Turn

$x \neq -2, 3; x = \frac{5}{3}$

Example 3: Your Turn

Working together they can finish in $\frac{12}{7}$ h.

Example 4: Your Turn

The train's average speed was 80 km/h.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Students should now be familiar with finding the non-permissible values. Remind them that a solution cannot be a non-permissible value.
Example 2 Have students do the Your Turn related to Example 2.	• Remind students that they must check that the solutions to rational equations are not non-permissible values. If one should appear, it cannot be included and is called an *extraneous root*.
Example 3 Have students do the Your Turn related to Example 3.	• Encourage students to use the table method demonstrated in the example for their problem. Have them verbalize the similarities between their question and the example (minutes are parallel to hours). This should help them pattern a table and write their equation.
Example 4 Have students do the Your Turn related to Example 4.	• Encourage students to use the table method demonstrated in the example for their problem. Have them verbalize the similarities between their question and the example. The formula distance = rate(time) is one that they have seen before and should be easily applied to the question. Remind students that the presence of a quadratic means that they may need to factor or apply the quadratic formula to solve the problem. It is also important to remind students that the value they obtain needs to be evaluated for its appropriateness to the context.

Check Your Understanding

Practise

Questions #1 to 4 are similar to Examples 1 and 2 in the student resource. Remind students to check for solutions that are also non-permissible values. These extraneous solutions must be eliminated.

Apply

Remind students to not only compare their solutions to non-permissible values, but also to check if their solutions are reasonable for the situation. For example, in #5c) there are two solutions, but the solution $-\frac{3}{7}$ is not practical as it would result in a side measurement that has a negative value.

Questions #6 and 7 require the use of the quadratic formula.

Students might approach #8 and 9 either as a two-variable system of equations or as a single-variable equation:

Method 1: Two-Variable System
Let x represent the first number and let y represent the second number.

$x + y = 25$

$\frac{1}{x} + \frac{1}{y} = \frac{1}{4}$

Method 2: Single-Variable Equation
Let x represent the first number and let $25 - x$ represent the second number.

$\frac{1}{x} + \frac{1}{25 - x} = \frac{1}{4}$

The simplest method to solve the system of equations is substitution, which will result in the student solving an equation similar to the equation written with one variable.

Students may need some coaching for #10. Ask:

- If n represents the number of students going on the trip originally, how would you represent the total each student paid before the six students dropped out?
- How would you represent the total paid by each student after the six students dropped out?
- How can you use the two expressions you just wrote to represent the difference between each student's amount before the six students dropped out and after?

In #11, students may have trouble representing two consecutive integers using the same variable. Ask:

- How could you use what you learned in #9 to help you write an equation for this question?
- If you need to use the quadratic formula, what value do you have to eliminate?

For #14, draw students' attention to the Did You Know? feature. This will serve as a hint to write the expression for the rate with the current and the rate against it.

If students are having difficulty with #12 and 13, direct them to Example 3 and the Your Turn that follows it. Before students attempt #13, ask:

- What part of the pool is filled by hose A in 2 h?
- How would you represent what part of the pool is filled by hose B in 2 h if x represents the time it takes hose B to fill the pool by itself?

From these coaching questions, students should realize that (the part of the pool filled by pipe A) + (the part of the pool filled by pipe B) = (full pool), or $\frac{2}{3} + \frac{2}{x} = 1$.

For #17, tell students to be aware of dimensional consistency since the time is given in both minutes and hours. This question can be set up in two ways:

1) Let the variable x represent the rate travelling east of Swift Current.

	Distance (km)	Rate (km/h)	Time (h)
West of SC	275	$x - 10$	$\frac{275}{x-10}$
East of SC	300	x	$\frac{300}{x}$
Time Difference			$\frac{1}{2}$

For this approach, the equation is $\frac{275}{x-10} - \frac{300}{x} = \frac{1}{2}$.
The solution to this equation will be the speed travelling east of Swift Current. Adding 10 to the solution will give the speed travelling west of Swift Current.

2) Let the variable y represent the time travelling east of Swift Current.

	Distance (km)	Rate (km/h)	Time (h)
West of SC	275	$\frac{275}{y+\frac{1}{2}}$	$y + \frac{1}{2}$
East of SC	300	$\frac{300}{y}$	y
Time Difference		10	

For this approach, the equation is $\frac{300}{y} - \frac{275}{y+\frac{1}{2}} = 10$.
The solution to this equation will be the time travelling east of Swift Current. To find the rate, substitute the solution into the expressions: $\frac{275}{y+\frac{1}{2}}$ for the rate west of Swift Current, and $\frac{300}{y}$ for the rate east of Swift Current.

Students may recognize that #18 is similar to #14. The difference is that in #14 the rate of the current is known, and the rate of paddling in still water is the unknown. In #18, the unknown rate is the rate of the current.

When completing the table for #19, suggest that students let the variable represent the number of pages per day. If they let the variable represent the number of days, a more complicated quadratic equation results. Either way, they will need to use the quadratic formula. The following is a suggestion for completing the table:

	Reading Rate in Pages Per Day	Number of Pages Read	Number of Days
First Half	x	259	$\frac{259}{x}$
Second Half	$x + 12$	259	$\frac{259}{x+12}$

The equation is $\frac{259}{x} + \frac{259}{x+12} = 21$

For #20, students may need prompting to enter the salt solution percents as decimals in the formula.

Extend

If students replace b with $\frac{1}{a}$ in #21, a complex-complex fraction results. This would make the question very confusing to solve. Instead, students should substitute for $\frac{1}{a}$ in the second rational equation and solve for b. Remind them that they need to include both values of b when solving for a.

For #22, students may think they have to eliminate the negative value for x when solving their equation. However, the positive and negative values are both needed to find the two pairs of numbers.

Students are required to rearrange the variables in formulas in #23. Successfully doing so will help students work with formulas in their chemistry and physics courses.

Create Connections

Students could make an excellent journal entry for #24. They could place their response in their Foldable, graphic organizer, or with their summary notes.

In #25a), it is not clear whether the 490 pages represents how many pages each printer prints in 14 min or the total combined pages that the two printers print in 14 min. This needs to be clarified for students. Also, parts b) and c) are excellent discussion questions that could be extended to a survey or research project.

Students should be able to relate to the scenario in #26. They often want to know the scores that they need to strive for to obtain a certain average.

Meeting Student Needs

- Provide **BLM 6–8 Section 6.4 Extra Practice** to students who would benefit from more practice.
- Allow students time to complete their presentations. A small summary of the entire unit should be included.
- Once students have completed #7, encourage them to sketch the rectangle. "Frame" various objects in the classroom, and discuss.

ELL

- Ensure that students understand the vocabulary in the questions, such as *consecutive*, *kayak(ers)*, *downstream* and *upstream*, *solution* (i.e., a mixture of liquids), *harmonic*, etc.
- Encourage students to include any term in their dictionary that they encounter and with which they are not familiar. Suggest that they include a verbal description, diagram, and/or example for each term.

Gifted

The formula $\Delta t' = \Delta t\sqrt{1 - \frac{v^2}{c^2}}$ is used to calculate how time is affected by the speed, v, of the object. In this formula, Δt represents the change in time for a moving object and $\Delta t'$ represents the change in time for a stationary object. Einstein's theory of relativity predicts that as an object approaches the speed of light, its time is significantly reduced compared to that of a stationary object. Have students calculate the change in time for an object travelling at 150 000 km/s given that c (the speed of light) is approximately 300 000 km/s. Assume the change in time for the stationary object is 3 h.

Common Errors

- Students may confuse the procedures they learned in section 6.4 with the procedures for adding and subtracting rational expressions they learned in section 6.3. For example, they may offer the following solution:

  $\frac{2x + 1}{x} + \frac{9x}{x - 1}$

  LCD is $x(x - 1)$

  $$x(x - 1)\left(\frac{2x + 1}{x}\right) + x(x - 1)\left(\frac{9x}{x - 1}\right)$$

  $$= \overset{1}{\cancel{x}}(x - 1)\left(\frac{2x + 1}{\underset{1}{\cancel{x}}}\right) + x\overset{1}{\cancel{(x - 1)}}\left(\frac{9x}{\underset{1}{\cancel{x - 1}}}\right)$$

  $= (x - 1)(2x + 1) + x(9x)$

  $= 2x^2 - x - 1 + 9x^2$

  $= 11x^2 - x - 1$

R_x Ask students:
- Is this an expression or an equation?
- Are the steps performed correctly? If not, where are the errors?
- What is the correct answer?

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 7, two of 8 to 11, 14, and 19. Students who have no problems with these questions can go on to the remaining questions.	• For #1, students are asked to simply eliminate the fractions. Remind them that each term must be multiplied by the LCD and not just the first term on each side of the equal sign (a common mistake). Ask students to identify the non-permissible value for this question, even though this is not asked for. • For #2, students are asked to solve completely. Students should only start #2 when they have confidently demonstrated that they can do #1. • For #4, ask students, "Does it matter what the solution to a rational equation is? Is anything acceptable?" Have them verbalize their thinking, and clarify any misunderstandings before they start this question. • For #14 and 19, allow students to demonstrate that they can take the information from a question and organize it in a table in order to write an equation. You may wish to have students partner with a classmate to complete these two questions.
Assessment *as* Learning	
Create Connections Have all students complete #24, 25, and 27. Students who have no difficulty with these questions should be encouraged to try #22 and 23.	• You may wish to have students work in pairs for these questions. • Students are asked to differentiate between expressions and equations in #24a). For students having difficulty remembering the difference between these, tell them to look for the letters *equa* in the word *equation*. Have them associate this with *equal*. • For #25, suggest that students use a table. This question deals with a real-world application that students can relate to. For part c), have students brainstorm ideas and place them on the board. This may help students think of ways to conserve paper and ink. • For #27, have students complete the work for part a) independently, and then compare their response with partners. Have them work in pairs for parts b) and c). Some students may need to be coached to help them determine that the quadratic formula is needed for exact answers.

Chapter 6 Review

6

Pre-Calculus 11, pages 352–354

Suggested Timing
90–120 min

Blackline Masters
BLM 6–5 Section 6.1 Extra Practice
BLM 6–6 Section 6.2 Extra Practice
BLM 6–7 Section 6.3 Extra Practice
BLM 6–8 Section 6.4 Extra Practice

Planning Notes

Have students who are not confident with the concepts discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions with which they need no help, a little help, and a lot of help. They can use this list to help them prepare for the chapter test.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 6–5 Section 6.1 Extra Practice**, **BLM 6–6 Section 6.2 Extra Practice**, **BLM 6–7 Section 6.3 Extra Practice**, and **BLM 6–8 Section 6.4 Extra Practice**.
- Have students put together the entire set of dominoes they created. They could then play a game in groups of four. Allow 20 to 30 min, and then determine which person, pair, or group has successfully completed the most dominoes.
- Students who completed a video should show their performances or presentations to the class. Students may self-assess and peer-assess the product(s).
- Students should complete one or two questions from each section to determine their progress. Once they determine their strengths and weaknesses, more time should be spent on the sections with which they require assistance.

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 6 Review The Chapter 6 Review is an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource.	• You may wish to have students work in partners. • Prior to working on the chapter test, have students revisit any section with which they are having difficulty.

6

Chapter 6 Practice Test

***Pre-Calculus 11*, page 355**

Suggested Timing

60–75 min

Blackline Masters

BLM 6–9 Chapter 6 Test

Planning Notes

Have students start the test by writing the question numbers in their notebook. Have them indicate which questions they need no help with, a little help with, and a lot of help with. Have students first complete the questions they know they can do, followed by the questions they know something about. Finally, suggest to students that they do their best on the remaining questions.

This test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1–9.

Study Guide

Question(s)	Section(s)	Refer to	The student can …
#1, 6	6.1	Link the Ideas Example 1	✓ determine the non-permissible values for a rational expression
#2	6.1	Link the Ideas Example 2	✓ simplify a rational expression
#3, 8	6.3	Link the Ideas Example 2	✓ determine, in simplified form, the sum or difference of rational expressions in which the denominators are not the same and which may or may not contain common factors ✓ simplify an expression that involves two or more operations on rational expressions
#4	6.2	Link the Ideas Example 2	✓ determine, in simplified form, the product or quotient of rational expressions
#5, 7	6.4	Link the Ideas Example 1	✓ determine the solution to a rational equation algebraically, and explain the process used to solve the equation
#9	6.4	Example 3	✓ model a situation using a rational equation
#10	6.3 6.4	Link the Ideas Example 1 Example 2 Link the Ideas Example 1	✓ determine, in simplified form, the sum or difference of rational expressions with the same denominator ✓ determine, in simplified form, the sum or difference of rational expressions in which the denominators are not the same and which may or may not contain common factors ✓ determine the solution to a rational equation algebraically, and explain the process used to solve the equation
#11	6.4	Link the Ideas Example 1	✓ determine the non-permissible values for the variable in a rational equation ✓ determine the solution to a rational equation algebraically, and explain the process used to solve the equation
#12, 13	6.4	Example 3	✓ determine the non-permissible values for the variable in a rational equation ✓ solve problems by modelling a situation using a rational equation

Assessment	Supporting Learning
Assessment *as* Learning	
Chapter 6 Self-Assessment Have students use their responses on the test and work they completed earlier in the chapter to identify skills or concepts they may need to reinforce.	• Students may wish to review the Create Connections comparisons they have created throughout the chapter before they begin the test. These can also be used to identify any areas of weakness. • Before the chapter test, coach students in areas in which they are having difficulties. • If students are unsure what questions they should try, they should be able to complete: #1 to 9.
Assessment *of* Learning	
Chapter 6 Test After students complete the test, you may wish to use **BLM 6–9 Chapter 6 Test** as a summative assessment.	• Have students revisit any notes and graphic organizers that they have produced through the chapter. Consider allowing students to use their chapter summary to complete the practice test.