Absolute Value and Reciprocal Functions

Opener

***Pre-Calculus 11*, pages 356–357**

Suggested Timing

30–40 min

Materials

- current exchange rates for a variety of countries

Blackline Masters

BLM 7–2 Chapter 7 Prerequisite Skills
BLM U3–1 Unit 3 Project Checklist

Key Terms

absolute value
absolute value function
piecewise function
invariant point
absolute value equation
reciprocal function
asymptote

What's Ahead

In this chapter, students learn about absolute values, determine the absolute value of numerical expressions, compare and order absolute values, and solve problems involving absolute value. They graph and determine the intercepts, domain, and range of absolute value functions of the form $y = |ax + b|$ and $y = |ax^2 + bx + c|$. Next, students learn how to write linear and quadratic absolute value functions in piecewise notation. They learn how to solve absolute value equations graphically and algebraically. Students determine the extraneous solutions for absolute value equations and explain why the equation $|f(x)| = b$ for $b < 0$ has no solution. Next, they apply what they have learned to solve absolute value problems. Finally, students learn about reciprocal functions by comparing the graphs of a function and its reciprocal. They graph and analyse the graphs of the reciprocal of given linear and quadratic functions.

Planning Notes

As a class, talk about the chapter opener and ask students for some other applications of absolute value in everyday life. You might coach students to think about possibilities, such as elevators ascending and descending, changes in sea level, time before and time after an event, and changes in values of stocks on the stock market.

Use the example of exchange rates to introduce the concept of reciprocal relationships. Ask students to share personal experiences:

- Why is it important to know how to convert Canadian currency to other currencies?
- Where have you travelled? What currencies have you encountered?
- How did you know the value of your Canadian dollars?

You might provide students with current exchange rates and ask them to convert between several currencies. Ask them to explain how the relationship between the two currencies is reciprocal.

Have students use the graphic of the balloon at different depths to explain how it shows a reciprocal relationship.

Direct students to the photo collage showing currency, a bridge, a barometer, and a thermometer. Ask them to speculate on how these items are related to absolute value and/or reciprocal functions. Note that the bridge shown is Broadway Bridge in Saskatoon, SK.

Tell students that in this chapter, they will build on their existing knowledge and skills with linear and quadratic functions.

As students progress through the chapter, have them record the Key Terms and develop their own definitions and examples. They can refer to the definitions in the student resource.

Direct students to the Career Link about commercial divers. Have them discuss what they know about the work that divers do, and how math skills are related to their work. Ask students what theoretical and practical training divers must undergo.

Unit Project

You might take the opportunity to discuss the Unit 3 project described in the Unit 3 opener. The Project Corner at the end of section 7.1 gives students who are interested in space tourism an opportunity to explore how absolute value might be involved in designing space vehicles.

The Project Corner box provides information related to the unit project. This feature is not mandatory but is recommended because it provides some background for the final report for the Unit 3 project assignment.

If you are going to develop a project rubric with the class, you may want to start now. See pages 279–280 in this Teacher's Resource for information on working with students to develop a class rubric.

Chapter Summary

Discuss with students the benefits of keeping a summary of what they are learning in the chapter. If they have used Foldables™ before, you may wish to have them select a style they found useful to keep their notes in for Chapter 7. Discuss other methods of summarizing information. For example, many students may have used different types of graphic organizers, such as a mind map, concept map, spider map, Frayer model, and KWL chart. Discuss which one(s) might be useful in this chapter.

Encourage students to use a summary method of their choice. Allowing personal choice in this way will increase students' ownership in their work. It may also encourage some students to experiment with different summary techniques.

Give students time to develop the summary method they have chosen. Ask them to include some method of keeping track of what they need to work on. Explain the advantage of doing this.

Ensure that students summarize the key concepts in their own words to help them recall the content when they review the chapter. Encourage students to note any concepts that they are struggling with. This will help them identify which concepts need extra review and practice.

Meeting Student Needs

- You may need to clarify the term *absolute value* for some students. Explain that absolute value is the distance of a number from zero. Ask students to think of their own example.
- Some students may not be familiar with concepts related to foreign currency. You may need to explain that each country has its own currency, and that the value of a country's currency varies in comparison to another country's currency. Explain the meaning of *exchange rate* and how it is determined. You might have students choose a foreign currency and research its current exchange rate in Canadian dollars. Alternatively, have the current currency rate of several countries available and ask students to choose a holiday in one country. Tell them they have C$1000 to spend. Ask them to convert C$1000 to the foreign currency. Have them make the conversion using a calculator instead of the Internet.
- You may wish to provide the formulas related to Boyle's law $\left(P = \frac{C}{V} \text{ or } V = \frac{C}{P}\right)$. Ask students:
  - What do these formulas have in common? (They are inversely proportional.)
  - What formulas do you know that appear to be similar to the formulas for Boyles's law?
  - Do these other formulas represent reciprocal functions? Explain.
  - Where else have you worked with reciprocals?
- Consider having students complete the questions on **BLM 7–2 Chapter 7 Prerequisite Skills** to activate the prerequisite skills for this chapter.
- Some students may benefit from reactivating their knowledge of setting up and solving proportions.
- Some students may benefit from reactivating their knowledge and skills with linear and quadratic functions.
- You might reinforce the Key Terms after they are introduced by posting sheets of paper on a classroom wall, each labelled with one term. Have student pairs move around the room and contribute to a definition and an example for each Key Term. Once all contributions have been made, have students review the entries. As a class, debrief each sheet. Display the sheets throughout the chapter.
- You may wish to post the student learning outcomes for the chapter in the classroom, colour-coding the outcomes by section in the chapter. Ensure that students understand the outcomes as written, and be prepared to rewrite some into language they understand. Provide students with their own copy. They can then refer to the outcomes as they work through the chapter. This will help them to self-assess their progress and to identify areas of weakness.
- Hand out to students **BLM U3–1 Unit 3 Project Checklist**, which provides a list of all the requirements for the Unit 3 project.

ELL

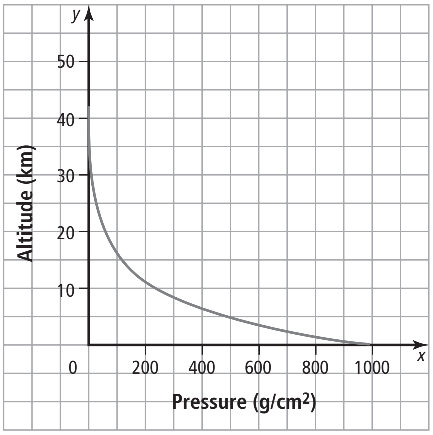
- Encourage students to create their own math vocabulary dictionary for the Key Terms using written descriptions, diagrams, and examples.

Enrichment

- Invite students who are interested to explore Boyle's law. They may find the Web Link at the end of this section interesting.
- Suppose an engineer is monitoring temperature fluctuations in a processing plant. She has to decide whether averaging the temperature differences from the desired setting is as informative as determining the absolute values of the temperature differences. Ask students to describe what differences, if any, the engineer would find using both methods.

Gifted

- Challenge students to represent Boyle's law with a mathematical equation that describes the relationship between the pressure and the volume of a gas. They may find the related Web Link in this section helpful.
- Challenge students to use the following chart to determine the air pressure at various altitudes. Ask them to suggest why the relationship between altitude and pressure exists, and what the mathematical relationship might be.



Career Link

You may wish to have students who are interested in learning more about commercial diving research the career, including the training and qualifications required, and work-related opportunities. Have students present their findings orally and explain how commercial diving connects to the chapter.

Web Link

For information and an applet that illustrates Boyle's law, go to www.mhrprecalc11.ca and follow the links.

For information about a career as a commercial diver, go to www.mhrprecalc11.ca and follow the links.

7.1 Absolute Value

***Pre-Calculus 11*, pages 358–367**

Suggested Timing

60–90 min

Materials

- grid paper
- ruler

Blackline Masters

Master 2 Centimetre Grid Paper
BLM 7–3 Chapter 7 Warm-Up
BLM 7–4 Section 7.1 Extra Practice

Mathematical Processes

- ☑ Communication (C)
- ☑ Connections (CN)
- ☑ Mental Math and Estimation (ME)
- ☑ Problem Solving (PS)
- ☑ Reasoning (R)
- ☐ Technology (T)
- ☑ Visualization (V)

Specific Outcomes

AN1 Demonstrate an understanding of the absolute value of real numbers.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2, 4–7, two of 8–12, 21, 23
Typical	#1, 3–7, three of 8–14, 20–24
Extension/Enrichment	#6, 10, 13, 15–19, 21, 23–25

Planning Notes

Have students complete the warm-up questions on **BLM 7–3 Chapter 7 Warm-Up** to reinforce prerequisite skills needed for this section. If you have posted the outcomes, refer to the outcomes for this section.

In advance, you might have students research the temperature extremes for Nunavut or Canada. (They may find the related Web Links at the end of this section useful.) As a class, have them use the data to generate some additional examples of weather extremes. Then, ask students:

- Why might communities in southern Canada sometimes have colder temperatures in winter than communities in the Arctic?
- Does the distance between a community and the equator affect weather? Explain.
- Does sea level affect the weather in a community? Explain.

Ask students why values are subtracted in order to determine temperature range. Alternatively, you might illustrate the temperature range using a number line. Ask if the differences could be determined by adding values.

For the Did You Know? about temperature change, ask students to write a subtraction statement that shows the total temperature change of 41 °C.

Investigate Absolute Value

The purpose of this investigation is to help students develop a concrete understanding of absolute value.

Have students work individually to complete the investigation. Provide them with grid paper or **Master 2 Centimetre Grid Paper**.

As a class, have students discuss their responses to Reflect and Respond #8 and 9. For #9, ask students if a negative distance exists, and ask them to explain why. (Distance is a numerical value that represents how far apart two objects are. Distance is greater than zero, so it is always positive.)

Meeting Student Needs

- Some students may benefit from reactivating their skills in adding and subtracting integers.
- Challenge students to research the coldest and hottest temperature recorded for the community in which they live, and then determine the total temperature difference.
- Some students may prefer to use masking tape and make a number line on a tiled floor. Have them mark zero on the masking tape. Have each tile represent a unit on the number line. Alternatively some students may prefer making the number line using coloured blocks or linking cubes (ensuring that adjacent blocks are different colours).
- Have students consider the driving distance from school to a local landmark. Ask them:
  - How many kilometres away is the landmark?
  - Is the distance the same when you start at school and walk to the landmark or start at the landmark and walk to school? Explain.

Common Errors

- Students may ignore the absolute value symbol (e.g., $|-3| = -3$).

Rx Reinforce that the absolute value is the distance of a number from zero, and that distance is always positive.

- Students may apply the absolute value to each value inside the absolute value symbol. For example,

$$\begin{aligned}&|12 \times (-3) + 5|\\&= 12 \times (3) + 5\\&= 36 + 5\\&= 41\end{aligned}$$

Rx Remind students to take the absolute value after they evaluate the expression inside the absolute value symbol. Tell them to take the absolute value of the one value remaining.

Web Link

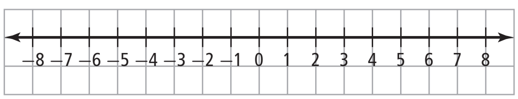
For information about the temperature extremes in Nunavut, go to www.mhrprecalc11.ca and follow the links.

For information about extreme weather in Canada, go to www.mhrprecalc11.ca and follow the links.

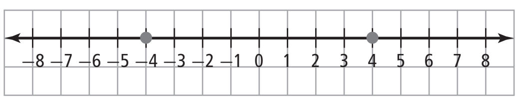
Answers

Investigate Absolute Value

1.



2.



−4 is 4 units from 0; 4 is 4 units from 0

3. a) Example: Plot 7 and 2; 5 units between the points

b) Look for two ways. Example:
- By counting the distance between the points on the number line, there are 5 units.
- By subtracting, $7 - 2 = 5$ *or* $2 - 7 = -5$.

4. Example: Plot −3 and −7; 4 units between the points. By counting the distance between the points, there are 4 units. By subtracting, $-3 - (-7) = 4$ *or* $-7 - (-3) = -4$.

5. Example: Plot −5 and 4; 9 units between the points. By counting the distance between the points, there are 9 units. By subtracting, $4 - (-5) = 9$ *or* $-5 - (4) = -9$.

6. Example: The values are positive or negative depending on the order of the subtraction. The values are always positive when counting the distance between two points.

7. Example: The values could be positive or negative.

8. Example: 1 and 6, −7 and −2, and −2 and 3. I checked to make sure that each pair of points is 5 units apart on a number line. Subtracting the first value from the second value in each set of numbers always yields 5.

9. Example: The distance from each point to zero is always 3 units. When you move a distance of 3 units to the left or to the right, or forward or backward, in each case you have moved a distance from zero. Consider a triangle. The distance from vertex A to vertex B represents how far apart the points are. Since distance is always greater than zero, distance is always positive.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Some students may simply count the units initially. Ensure they make the link between the distances they counted and the fact that the absolute value is positive. Coach them to realize that they counted in positive (not negative) numbers.

Link the Ideas

As a class, discuss the term *absolute value*. Have students record their own definition and examples of the absolute value of a positive and a negative number.

Example 1

This example allows students to practise determining the absolute value of a number.

Have students evaluate each expression before checking the given solution.

Have students complete the Your Turn questions individually.

Example 2

This example demonstrates comparing and ordering absolute values. Have students work through the example and record the final solution using the original notation in the question, as follows:

$-3.4, -0.01, |-0.1|, \left|-\frac{12}{5}\right|, \left|-2\frac{1}{2}\right|, |4.75|, 5, |-6.5|$

Have students complete the Your Turn questions individually and compare their solution with that of a classmate.

Example 3

In this example, students evaluate absolute value expressions. Direct students to the note that appears before Example 3. Reinforce that when working with absolute values expressions, the order of operations applies. Tell students to follow the order of operations for the expression within the absolute value symbol, and to take the absolute value only after there is one numerical value remaining. You might have students develop and record their own example of evaluating an expression inside an absolute value symbol. Then, have them explain their solution to a classmate.

Have students work through the example on their own before looking at the given solution.

Have students complete the Your Turn question individually and compare their solution with that of a classmate.

Example 4

In this example, students apply absolute value to solve a problem involving change in stock value.

Have students work through the given solution. Ask individual students why the initial value is not simply subtracted from the final value (i.e., $|\$13.85 - \$13.55| = \$0.30$). (The amount \$0.30 represents net change, not total change.) Direct students to the Did You Know? to clarify *net change of a stock*.

Assign the Your Turn to students to complete individually. As a class, have students compare their solutions.

Key Ideas

As a class, talk about the Key Ideas and check for understanding. Ask students:

- How is the absolute value symbol similar to the use of brackets in the order of operations? How is it different?

Use the Key Ideas to help build the chapter review. Have students record their own summary of the Key Ideas. They can store their summaries for each section together to create a reference for review purposes. Consider allowing students to use their summary for the chapter test.

Meeting Student Needs

- For Link the Ideas, encourage students to create their own definition and example of *absolute value*.
- For Example 2, ensure that students record the solution using the original values.
- For Example 4, you might have students go to a set of stairs in the school. Tell them the floor represents zero. Ask a volunteer to walk up or down one or more steps and stop (stepping up represents positive units and stepping down represents negative units). Have students record the value that the movement represents. Have the volunteer move up or down a different number of steps. Again, have students record the value. Repeat the activity a few more times. When done, have students determine the total change in steps.
- To help clarify understanding about determining the absolute value of a numerical expression, give students a deck of cards. Tell them red cards represent negative values and black cards represent positive values (similar to values entered in a bank account). Have students choose several cards and determine the absolute value of the sum of the cards.

  Alternatively, have students form groups of four. Have each student choose to be positive or negative. You will need multi-sided die of four different colours. Tell students that each colour represents an operation. For example, assume that red = addition, blue = subtraction, green = multiplication, and yellow = division. Give each group any four die. Have students each roll their own dice and then determine the absolute value of the combined roll. For example, student A is positive and has a green dice and students B, C, and D are negative. Student B has a green dice, student C has a blue dice, and student D has a red dice. If student A rolls an 8, student B a 4, student C a 12, and student D a 2, there is more than one possible numerical expression. The numerical expression could be $|8(-4)(-12) - (-2)|$, which represents [(A)(B)(C) − (D)], or it could be $|(-4)(-12) - (-2) + 8|$, which represents [(B)(C) − (D) + (A)]. Have students roll the die again and repeat the activity.
- Some students may wish to create a poem or rap featuring the Key Ideas in this section.

ELL

- For Example 4, use descriptions and examples to help students understand such terms as *stock market*, *stock* and *bond values*, *fluctuate*, *volatile*, and *per share*.

Answers

Example 1: Your Turn

a) 9 is 9 units away from 0 on the real-number line, so $|9| = 9$.

b) -12 is 12 units away from 0 on the real-number line, so $|-12| = 12$.

Example 2: Your Turn

$-2, -1.25, |-0.5|, 1.05, \left|-\frac{13}{4}\right|, \left|-3\frac{1}{3}\right|, |3.5|, |-5.75|$

Example 3: Your Turn

a) 1 **b)** 4 **c)** 11

Example 4: Your Turn

$$|1 - 6| + |6 - 2| + |2 - 4| + |4 - 1|$$
$$= |-5| + |4| + |-2| + |3|$$
$$= 5 + 4 + 2 + 3$$
$$= 14$$

Or, 1st to 6th means 5 floor changes, down to 2nd adds 4 changes, up to 4th adds 2 changes, down to 1st adds 3 changes, for a total of 14 floor changes.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Have students refer to the Link the Ideas for the absolute value of a negative number.
Example 2 Have students do the Your Turn related to Example 2.	• Remind students to change only the values within the absolute value symbol to a positive. • Some students may decide to compare values by expressing them as fractions. Encourage students to use the method they find easier.
Example 3 Have students do the Your Turn related to Example 3.	• For part c), remind students to apply the order of operations.
Example 4 Have students do the Your Turn related to Example 4.	• It may benefit some students to model the change in floors using vertical number lines. • Some students may wish to mark each change using an arrow. Encourage them to label the arrow with the corresponding positive or negative value. • Before calculating, have students show the corresponding algebraic expression.

Check Your Understanding

Practise

These questions allow students to build basic skills in working with absolute value. Students can complete the questions individually or with a partner. Consider allowing students some choice in the questions they do. After completing a question, remind students to check the answer. In cases where there is a difference between their answer and the one in the answer key, encourage them to get help from a classmate or from you.

For #2 and 3, remind students to use the original notation in their final answer.

For #6, prompt students:

- At what point do you take the absolute value of a numerical expression? (after evaluating the expression inside the absolute value symbol)

Apply

These questions provide opportunities for students to apply absolute value to real-life situations. Use leading questions to help students choose the correct process that is required for a specific question. Allow students to work in pairs to solve some of the problems.

You might involve the class to solve #11. Ask students:

- What is the difference between the total change and the net change in her bank balance?

Extend

These questions require students to extend their knowledge by using new and previously learned skills and processes to solve problems.

For #15 and 16, prompt students to realize that they need to work backward to determine numbers that will satisfy the total amount.

Create Connections

Consider having students work in pairs or small groups to brainstorm solutions to these problems before developing an individual response.

For #23, students need to determine the average absolute value and describe its meaning. You might extend this question by talking about standard deviation. Once students have determined the difference between all the values, they can square each difference and then determine the total. Ask students:

- What is the difference between the standard deviation and the average absolute value?

Ask students to develop the equation for #24 by applying the square root principle to the equation $x = p \pm \sqrt{\left|\frac{q}{a}\right|}$.

Project Corner

The Project Corner provides students with information about space vehicles. As a class, you might have students brainstorm possible applications of space tourism related to absolute value functions, such as current spaceship designs, payload calculations for flights that NASA plans, life support system designs, and calculations related to the International Space Station. Students may find the related Web Links at the end of this section helpful.

Meeting Student Needs

- Encourage students to draw diagrams to model problems.
- Ensure that students spend enough time on the concepts related to absolute value in this section as they will build on these concepts in the next section.
- Have students use their graphic organizer to record an example of determining absolute value.
- Remind students to use BEDMAS when evaluating expressions involving absolute values.
- For #6e), encourage students to highlight the values within each set of absolute value symbols. This may help students decide where each calculation begins and ends.
- Allow students some choice in the Apply problems based on their interests and/or familiarity with the contexts.
- For #11a), check that students have written an expression that represents the total change before making the appropriate calculations.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 7–4 Section 7.1 Extra Practice** to students who would benefit from more practice.

ELL

- Use a combination of visuals, descriptions, and examples to help students understand such terms as *grain storage facility*, *amplitude of a wave*, *crest height*, *trough height*, *trading stock*, *scavenger hunt*, and *suspended*.
- For #10, use the related Did You Know? to help explain the term *mileposts*.
- For #13, explain that the Festival du Voyageur is a 10-day province-wide celebration of French-Canadian culture in Manitoba that has taken place every year since 1970.
- For #14, use the related Did You Know? to help explain the Yukon Quest.
- The language in the Project Corner may be challenging for some students. Consider having English language learners work with a partner who can help them understand the following terms: *aerodynamics*, *weight*, *payload capacity*, and *life support systems*.

Enrichment

- Present the following situation: A gasoline company must determine where to build a distribution centre for its fuel trucks. Suppose that n stations must be serviced along a highway and that the trucks can carry only enough fuel for one station at a time. Since the cost of transporting gasoline from the centre to the different stations is quite high and is proportional to the distance travelled, the company would like to select a location that minimizes the total distance. Challenge students to show how absolute value can be used to solve this problem and give an example of its use.
- Ask students to research weather records, and to develop and solve a related absolute value problem.
- For #20, invite students to research Robert Millikan's oil drop experiment. Have them explain how his work relates to absolute value.
- Invite students to research how aerodynamics might involve absolute values. They may find the related Web Link in this section interesting. Have students present their findings to the class to encourage other students who might be considering space tourism for their unit project.

Gifted

- Challenge students to consider the mathematics of rounding dollar amounts, such as in the case of interest payments on a loan. Ask them what could happen if the rounding procedures used absolute values to siphon fractions of cents from each transaction that occurred in their bank account over one year. Who would notice?
- The Cross Lake First Nation have been hunting and fishing the Minago River, which drains into Cross Lake and its surrounding area, for thousands of years. Manitoba Hydro's generating station at Jenpeg has had a great impact on these traditional hunting and fishing grounds, because the Minago River is flooded or drained as needed to operate the generating station. Challenge students to investigate the water level fluctuations of the Minago River. Invite them to express the data they find in terms of absolute value.
- Challenge students to research weather-related factors that produce changes in sea level. For example, they might explore storm surges, evaporation and precipitation, or atmospheric pressure. Have them develop an absolute value problem and a solution using data they find.

Web Link

For a video that explains the Millikan oil drop experiment, go to www.mhrprecalc11.ca and follow the links.

For a basic explanation about launch windows, go to www.mhrprecalc11.ca and follow the links.

For topics related to aerodynamics and propulsion, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2, 4 to 7, and two of 8 to 12. Students who have no problems with these questions can go on to the remaining questions.	• For #1, students should recognize that all the values are positive, with the exception of zero. • For #2, remind students that only the values inside an absolute value symbol may change sign. Encourage students to express the values in #2 in the form (decimal or fraction) they find easiest to compare. If needed, coach students through #2, and then assign #3 to check for understanding. • Some students may be confused about the order of subtraction for #5. Ask them if the order of subtraction changes the solution when they use absolute value. You may wish to use #4 as a model and ask students if reversing the order of the values changes the solution. • For #7, encourage students to plot the coordinates on grid paper and use the visual to assist them in answering the question. • For #11, ensure students understand the difference between total change and net change. After discussion, have students use their graphic organizer to summarize these terms in their own words.
Assessment *as* Learning	
Create Connections Have all students complete #21 and 23. Students who do not have any difficulty with these questions can go on to the remaining questions in this section.	• For #21, explain that *signed value* means either a plus or a minus sign. • In #23, students apply absolute value to measuring heights of players on a team, leading to an application of *standard deviation*. Students may not be familiar with the term *standard deviation* and its implications. Consider taking the opportunity to discuss this real-world application of absolute value.

7.2 Absolute Value Functions

***Pre-Calculus 11*, pages 368–379**

Suggested Timing

90–130 min

Materials

- grid paper
- ruler
- Mira™ (optional)

Blackline Masters

Master 2 Centimetre Grid Paper
BLM 7–3 Chapter 7 Warm-Up
BLM 7–5 Section 7.2 Extra Practice

Mathematical Processes

- [x] Communication (C)
- [x] Connections (CN)
- [x] Mental Math and Estimation (ME)
- [x] Problem Solving (PS)
- [x] Reasoning (R)
- [ ] Technology (T)
- [x] Visualization (V)

Specific Outcomes

RF2 Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–5, 6a), c), f), 7, 8a), b), f), 9, 10a), b), 11a), b), 12, 13, 27
Typical	#1–5, 7, 8a), b), f), 9, 10b), c), 12, 14–17, 19, 20, 27, 29
Extension/Enrichment	#9, 11, 15, 18, 20–27, 29

Planning Notes

Have students complete the warm-up questions on **BLM 7–3 Chapter 7 Warm-Up** to reinforce prerequisite skills needed for this section.

As a class, introduce the investigation by talking about the stroboscopic photo that shows the path of a bouncing ball over time. Explain that this is an example of a real-life situation that can be represented with an absolute value function. You might ask students to think of other real-life situations that can be represented using an absolute value function. (In navigation and transportation, for example, suppose you go 5 km east, then 8 km west, and then 10 km east again. You can determine where you are located relative to where you started. If you were driving a vehicle, you can determine whether you have enough fuel to get there and back.)

Investigate Absolute Value Functions

In this investigation, students explore the similarities and differences between linear, quadratic, and absolute value functions. In Part A, they compare linear functions and absolute value functions. In Part B, they compare quadratic functions and absolute value functions.

Provide students with grid paper or **Master 2 Centimetre Grid Paper**.

For Part A, you might have students work individually or in pairs, but they should record their own response to all questions. For #1, students may be confused about function notation. Remind students that $f(x)$ is simply the set of y-values or the output or range of a function.

Have students share their response to Reflect and Respond #3 to 5 with the class. For #4, help student reactivate their knowledge about functions. Ask them:

- What is a function? (a relation in which there is a unique value in the range for each value of the domain)
- What is the difference between a relation and a function? (A relation is an association between two quantities. All functions are relations but not all relations are functions.)
- How do you determine a function? (A relation is a function only if each value in the domain corresponds to exactly one value in the range.)

For #5b), students may benefit from recalling restrictions. Ask students if there are any values that x cannot be equal to. After students verbalize the restrictions, they may more easily write the restrictions using an inequality. The restrictions on domain and range are easier to identify from a graph.

For Part B, have students work individually or in pairs but record their own response to all questions.

When completed, have students share their response to Reflect and Respond #8 to 10 with the class. For #9b), you might coach students to divide the graph into two separate sections: a wide parabola opening upward and a narrow parabola opening downward.

Meeting Student Needs

- Some students may benefit from recalling the meaning of *domain* and *range*.
- Invite students to discuss the conditions necessary for an absolute value function to be located in quadrants I and II, I and III, I and IV, or II and III. Ask which ones are possible.
- Encourage students to use a different colour to represent each graph in parts A and B of the investigation.
- After students have completed the investigation, have them reflect on their findings and the discussion in the section opener. Ask them to reflect on how they would respond to the opening questions now and what revisions they might make to their earlier thinking.

Web Link

For some examples of stroboscopic photos, go to www.mhrprecalc11.ca and follow the links.

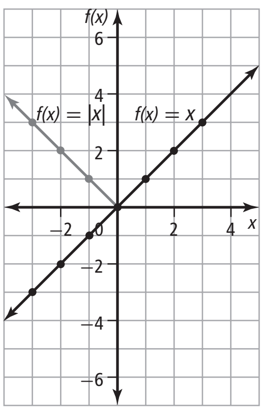
Answers

Investigate Absolute Value Functions

1.

x	−3	−2	−1	0	1	2	3
f(x)	−3	−2	−1	0	1	2	3
g(x)	3	2	1	0	1	2	3

2. Example:



3. Example: Similarity: The part of the graph of $g(x) = |x|$ that is above the x-axis is the same as the graph of $f(x) = x$.
Difference: The other part of the graph of $g(x) = |x|$ appears to be a reflection of the part of the graph of $f(x) = x$ that is below the x-axis.

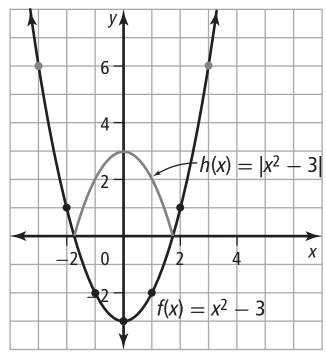
4. Using the vertical line test, there is only one y-value for each x-value.

5. a) The graph of $g(x)$ is a V shape.
b) $y = x, x \geq 0; y = -x, x \leq 0$

6.

x	−3	−2	−1	0	1	2	3
f(x)	6	1	−2	−3	−2	1	6
g(x)	6	1	2	3	2	1	6

7. Example:



8. Example: Similarity: Both graphs begin and end high.
Difference: The graph of $h(x) = |x^2 - 3|$ seems to bounce off the x-axis twice and then continue upward.

9. a) The values are the same for $x = -3, -2, 2,$ and 3. The values are different when $x = -1, 0,$ and 1.
b) $y = x^2 - 3, x < -1.732$ or $x > 1.732, y \geq 0$
$y = -(x^2 - 3), -1.732 < x < 1.732, y \geq 0$

10. Example: The parts of the graph that are above the x-axis are the same. The parts normally below the x-axis appear as a reflection in the x-axis.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• For #3 and 8, have students record the similarities and differences of the respective graphs in their graphic organizer. • Encourage students to explain how to determine whether each graph is a function. • For #5b) and 9b), some students may benefit from reactivating their knowledge of domain and range. • You might use the response to #10 as an assessment tool.

Link the Ideas

Talk about the definitions for *absolute value function* and *piecewise function*. Explain to students that piecewise functions are used to describe functions that contain distinct intervals, which are represented by different functions. The notation of piecewise functions is $y = \{$ with each expression for $f(x)$ on a separate line, along with the domain for that interval. You might refer back to Parts A and B of the investigation, point out a reflection and explain how it occurs. You might use a Mira™ to illustrate this.

By the end of this section, encourage students to produce and record their own example of a graph of an absolute value function and highlight the two distinct intervals on the graph. Have them record the related piecewise function.

Note that students will need grid paper or **Master 2 Centimetre Grid Paper** to sketch graphs for Examples 1 and 2.

Example 1

In this example, students graph an absolute value function of the form $y = |ax + b|$. For part b), have students attempt both methods shown. Using more than one strategy may help students come to a better understanding.

For Method 1, you might point out that when creating a table of values it is important to include some negative values. Ask:

- How might using only positive values mislead you about what the graph looks like? (The Web Link at the end of this section might be helpful for students.)

Direct students to the definition of *invariant point* in Method 2. Ask them:

- For $y = |2x - 3|$, why does the portion of the graph for $x > \frac{3}{2}$ above the x-axis stay the same as the graph of $y = 2x - 3$?
- Why are the x-intercepts considered invariant points but the y-intercepts are not considered as such?

Help students come to understand that in absolute value functions x can be negative but y cannot.

Have students work through the solution before assigning the Your Turn questions. Have students attempt the questions individually and compare their solutions with those of a classmate who used a different method for part b). Have them decide which method they prefer and explain why.

Example 2

In this example, students graph an absolute value function of the form $f(x) = |ax^2 + bx + c|$.

As students work through the solution for part b), prompt them to recall completing the square. Ask what other methods they might use to determine the coordinates of the vertex. Students might mention using the formula $x = \frac{-b}{2a}$. Consider asking students to determine the vertex using the x-intercepts of the parabola. Ask students how they would find the x-intercepts of the function if the quadratic could not be factored.

Have students work through the solution before assigning the Your Turn questions. Have students work in pairs and then compare their solutions with those of another pair.

Key Ideas

As a class, review the Key Ideas about analysing the graphs of absolute value functions. You might provide an absolute value function and have students analyse it graphically and algebraically. Have them record the example for review purposes.

You might explain a piecewise function as follows: When you take the absolute value of any function, the result is always positive. The original function, however, can be positive or negative, and therefore both cases must be considered. So, if $|x| = a$, then $x = a$ or $x = -a$.

Ask students:

- Why is the domain of the function $y = f(x)$ the same as the domain of the absolute value function $y = |f(x)|$?
- Why is the range different?

Have students record their own summary of the Key Ideas. Have them store the summaries of the Key Ideas for each chapter section together for review purposes.

Meeting Student Needs

- Encourage students to record their own explanation and examples for the terms *absolute value function*, and *invariant point*.
- Have students use their graphic organizer to record an example of an absolute value function and show two methods for graphing the function.
- For Example 1, encourage students to graph $y = 2x - 3$ using one colour and graph $y = |2x - 3|$ using a different colour. Have them circle the x-intercept and label the invariant points.
- For Example 2, have students use different colours to create each graph. Have them verbalize the pattern they observe when graphing the absolute value of a quadratic function.

- Pair students to review the Key Ideas. Have students develop an example to illustrate the Key Ideas and take turns explaining the example to their partner.

Enrichment

- Invite students to use the Internet and investigate piecewise functions.

Web Link

For an explanation of inputting negative values when creating a table of values for an absolute value function, go to www.mhrprecalc11.ca and follow the links.

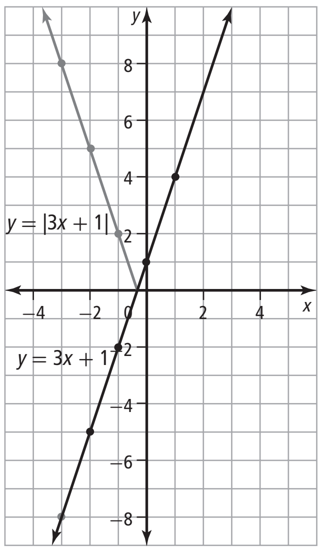
For a step-by-step tutorial on how to graph absolute value functions, including a discussion of domain, range, *x*-intercepts, and *y*-intercepts, go to www.mhrprecalc11.ca and follow the links.

Answers

Example 1: Your Turn

a) x-intercept: $\left(-\frac{1}{3}, 0\right)$; y-intercept: $(0, 1)$

b) Example:



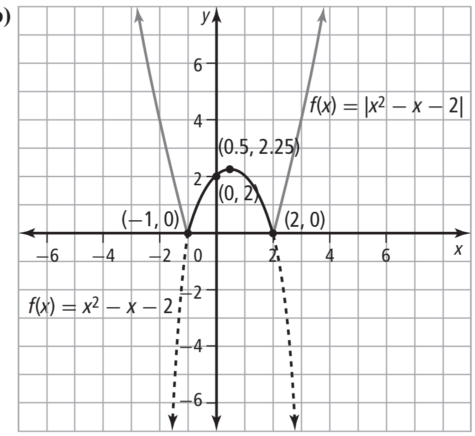
c) The domain is $\{x \mid x \in \mathrm{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

d) $y = \begin{cases} 3x + 1, \text{ if } x \geq -\frac{1}{3} \\ -(3x + 1), \text{ if } x < -\frac{1}{3} \end{cases}$

Example 2: Your Turn

a) x-intercepts $(2, 0)$ and $(-1, 0)$; y-intercept $(0, 2)$

b)



c) The domain is $\{x \mid x \in \mathrm{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

d) $y = \begin{cases} x^2 - x - 2, \text{ if } x \leq -1 \text{ or } x \geq 2 \\ -x^2 + x + 2, \text{ if } -1 < x < 2 \end{cases}$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Some students may benefit from using a table of values for $y = 3x + 1$ and adding a third column for $y = \lvert 3x + 1 \rvert$, so they can visualize the absolute values. • Ask students to identify which method (creating a table of values or using the graph of $y = 3x + 1$) is easier for them to understand. Encourage them to use their preferred method. • For piecewise notation, ensure that students understand that they need to perform two calculations: one for the positive value of x and one for the negative value of x. For the negative value, coach students to multiply the expression inside the absolute value symbol by -1.
Example 2 Have students do the Your Turn related to Example 2.	• Emphasize that the solution requires graphing. • For part b), you may need to coach students through the process of factoring. If so, ask them what the factors represent, or tell them once the factors have been determined. Some students may need additional practice factoring and identifying zeros. • The worked example provides a step-by-step approach. Some students may benefit from completing the Your Turn, handing it in for review, and then storing the corrected solution in their graphic organizer.

Check Your Understanding

Note that students will need grid paper or **Master 2 Centimetre Grid Paper** to sketch graphs.

Practise

These questions allow students to build basic skills in graphing and analysing graphs of absolute value functions. Students can complete the questions individually or with a partner. After they complete a question, remind students to check the answer. In cases where there is a difference between their answer and the one in the answer key, encourage them to get help from a classmate or from you.

For #9, note that students have not determined piecewise functions using a graph before. Consider having students talk about how to approach this question. Encourage students to try one of the methods they mentioned and compare their results with those of a classmate.

Apply

Allow students to work in pairs and then record an individual response.

For #15, as a class, have students explain their answer.

For #20, students may need to recall how to determine a linear equation given the x-intercept and y-intercept.

Likewise for #21, students may need to recall how to determine a quadratic equation given the x-intercepts and y-intercept.

Extend

You may wish to have students work through the problems in pairs and record their solutions individually.

Create Connections

Consider having students work in pairs or small groups to brainstorm answers to these questions before developing an individual response.

For #29, which is a Mini Lab, have students work in pairs. Encourage students to sketch their own diagram to help them. If the Mini Lab is used for summative assessment, ensure that you present your expectations for the completed work and provide a marking rubric for the assignment. If the Mini Lab is used for formative assessment, meet with the class as a whole and ask a couple of student pairs to lead a discussion of the results.

Meeting Student Needs

- For #1, students may find it helpful to create three-column tables. They can enter the x-values and $f(x)$ values as given, and then enter the $|f(x)|$ values using a different colour. Students might use the same colour code to graph the functions in #5.
- Invite students to suggest shortcuts for graphing absolute value functions.
- For #15, have students work in pairs. Assign each student to play the role of Raza or Michael. Have them prepare a debate on the question, and then invite volunteers to present the debate to the class. Ask students:
  – Who provided the most compelling argument? Why?
  – Who is correct? Explain why.
- For #29, consider having students work in groups to create a presentation including illustrations and calculations. Invite groups to present their work to the class. In lieu of other questions, you might assign the Mini Lab to students who demonstrate an understanding of the content of section 7.2. Have students present their report to the class. Invite the rest of the class to provide constructive feedback about the accuracy of the presentation.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 7–5 Section 7.2 Extra Practice** to students who would benefit from more practice.

Web Link

For an applet that allows students to explore absolute value functions, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 5, 6a), c), f), 7, 8a), b), f), 9, 10a), b), 11a), b), 12, and 13. Students who have no problems with these questions can go on to the remaining questions.	• For #1 and 5, ensure that students can determine the absolute value algebraically and graphically. They should be proficient with these skills before moving on to #6. • For #2 to 4, remind students that the *x*-axis acts as a mirror. Students may find it easier to graph the points or equations before answering the questions. • For #6, encourage students to use a table of values for $y = f(x)$ before determining the absolute value. Encourage them to follow the same process for #7 and 8. • For #9, remind students to write both cases for the piecewise function. • It is important that students are able to determine the important points for sketching an absolute value graph without initially using a calculator. Ensure that they are able to use an appropriate method for writing a function in piecewise notation, and solving for the domain by determining the *x*-intercepts. • Coach students who have difficulty with #11 by asking them to describe the differences between the absolute value of a linear function and a quadratic function. Check their understanding by having them complete #11c) and d) before moving on. • After they work through #11, you might suggest that students complete #27.
Assessment *as* Learning	
Create Connections Have all students complete #27.	• For #27, encourage students to use an example to develop their explanation. • You might write an absolute value function on the board and have students use it to answer #27. Afterward, as a class, have volunteers present their response. This may help clarify understanding for students. • Encourage students to share their responses to #27 with a classmate. Have them record their own example for each question in their graphic organizer or notebook. • Consider collecting students' responses to #27 and checking for weaknesses in their thinking. Provide coaching to students as needed.

7.3 Absolute Value Equations

***Pre-Calculus 11*, pages 380–391**

Suggested Timing

130–180 min

Materials

- grid paper
- ruler
- graphing calculator or computer with graphing software

Blackline Masters

Master 2 Centimetre Grid Paper
BLM 7–3 Chapter 7 Warm-Up
BLM 7–6 Section 7.3 Extra Practice

Mathematical Processes

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Specific Outcomes

RF2 Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1, 2b), c), 3, 4a), d), 5a), b), 6a)–d), 7, two of 8–12, 15, 23
Typical	#1a), b), 2b), c), 3, 4a), d), 5a), b), 6a)–d), 7, three of 8–14, 15, 23, 24
Extension/Enrichment	#5d), e), 6d), e), 13, 14, 16–20, 22, 23

Planning Notes

Have students complete the warm-up questions on **BLM 7–3 Chapter 7 Warm-Up** to reinforce prerequisite skills needed for this section.

As a class, have students respond to the opening questions related to the speed of light and absolute maximum velocity. Use prompts such as the following to promote discussion:

- Do you think it is possible that humans could travel faster than the speed of light? Why or why not?
- How would being able to exceed the speed of light affect future space travel?

Investigate Absolute Value Equations

In this investigation, students explore the solutions to absolute value equations.

Have students work individually or in pairs to complete the investigation.

Have students answer Reflect and Respond #6 to 8. For #6, prompt students to make a connection with the definition of absolute value to help them.

For #7, you might use a case to help explain that when taking the absolute value of a function, the answer is a positive value even though the original function (before taking the absolute value) could have been positive or negative. Reinforce that both cases must be considered.

Meeting Student Needs

- Ask students to describe *Star Trek* and Mr. Spock for students who may not be familiar with this television show.
- For the investigation, students may wish to refer to a number line on their desk or on the floor. For #1 and 3, have students use the number line to visualize the numbers that are the indicated number of units away from zero.

Gifted

- Invite students to research whether the speed of light is the absolute maximum velocity possible. Have them present their findings to the class using a format of their choice.

Answers

Investigate Absolute Value Equations

2. There are two solutions: $x = 10$, $x = -10$

3. There are two solutions for each equation.
For $|x| = 15$: $x = 15$ and $x = -15$. For $|x| = 5$: $x = 5$ and $x = -5$.
For $|x| = b$, $b \neq 0$: $x = b$ and $x = -b$.

4. Example: There are two solutions for an absolute value equation.

5. $x = 0$

6. Example: Yes, it is possible to have no solutions. The equation $|x| = -10$ has no solution in the set of real numbers. The absolute value of a number is always non-negative.

7. Example: When x is positive or zero, the first half of the equation holds true, so x is x. When x is negative, the second half of the equation holds true, so x is $-x$. Using a numerical example, $|-5|$ would lead to $-(-5)$, which is equal to $+5$ or 5.

8. a) When $|A| = b$, $b \geq 0$, $A = b$.
b) When $|A| = b$, $b < 0$, $-A = b$.

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• Consider having students present their response to #6 to 8 to the class, in order to help students clarify their understanding. • The response to #8 is particularly important because it forms the basis of understanding whether a solution is possible or acceptable. • For #8, you may wish to substitute a value for *b*, such as $\|x\| = -2$. Ask students: – What number can you substitute for *x* that will result in −2?

Link the Ideas

Direct students to the definition of *absolute value equations*. Reinforce that there are always two cases to consider. Ensure that they understand why if $|a| = b$, then $a = b$ or $-a = b$. By the end of this section, encourage students to develop and record their own example of solving an absolute value equation.

Note that students will need grid paper or **Master 2 Centimetre Grid Paper** to sketch graphs for the worked examples.

Example 1

This example models how to solve an absolute value equation using two methods. Encourage students to attempt both methods before moving on.

In Method 1, students use algebra to solve each case.

Method 2 has students graph each side of the equation and determine the point of intersection. The x-coordinates of the intersection points are the solutions to the equation. Ask students:

- Can you find another method involving x-intercepts to solve the equation graphically? (Some students may equate the equation to zero and determine the zeros of the function to determine the solutions.)

Have students complete the Your Turn question with a partner and compare their solution with that of another student pair who may have used a different method.

Example 2

In this example, students solve an absolute value problem involving a computerized process.

Have students work through both methods: using a number line and using algebra. Afterward, ask them which method they prefer and why. Students who are visual learners may prefer Method 1.

Although students should verify their solution through substitution into the original equation each time they solve an absolute value equation, you may need to remind them to do so.

The Your Turn problem requires students to solve for a different computerized process. Have students work individually to solve the equation and compare their solution with that of a classmate.

Example 3

This example demonstrates an absolute value equation with an extraneous solution. Consider having students work through the solution as a class.

For Case 1, ask students to compare the potential solution to the domain of the case. Students should realize that the solution $x = 2$ does not satisfy the domain of $x \geq \frac{5}{2}$ and is therefore an extraneous root.

Comparing the possible solution to the domain of each case is a valid method to identify an extraneous root, but students should get in the habit of checking the correctness of their solutions through substitution.

Have students complete the Your Turn question in pairs. Remind students to check for any extraneous roots. Ask:
- How can you identify any extraneous roots?

Have them compare their solution with that of another student pair.

Example 4

In this example, students try to solve an absolute value equation that has no solution.

You might ask students what the given equation would look like graphically. Ask them:
- If you graphed each side of the equation, what would happen?
- If you equated the equation to zero and then graphed it, what would you notice? (Students should be able to make a connection between an equation with no solution and its graph. Depending on the method used, the graph would have no point of intersection or no x-intercepts.)

After working through the given solution, direct students to the Did You Know? on page 385 in the student resource, which shows the symbol for the empty set.

Have students complete the question individually and compare their solution with that of a classmate.

Example 5

This example models solving a quadratic absolute value equation.

Consider working through the solution as a class. Have students state the domains for Case 1 and Case 2. You may wish to have students verify the domain for each case using a graph of the corresponding function. Encourage them to test values in each interval on the graph and to connect their observations to the given domains. For Case 1, ask students:
- Can you solve the quadratic equation using another method? If so, which method could you use?
- Is it necessary to solve this algebraically? Explain.
- How can you check that the solutions are correct?
- What methods can you use to verify your solution graphically?

Consider demonstrating how to enter an equation as a system of equations or as one equation. Ask students which method they prefer and why.

Have students complete the Your Turn question with a partner and compare their solution with that of another student pair.

Example 6

This example models solving a quadratic absolute value equation involving linear and quadratic expressions.

Before starting to solve the equation, direct students to the given equation. Ask:
- Might this equation have extraneous roots? How do you know? (Absolute value equations with variables on both sides of the equation have potential extraneous solutions.)

After working through Cases 1 and 2, ask students:
- Do both solutions satisfy the conditions of the domain? Explain.

For Case 1, students should identify that the solution $x = 1$ does not fit the domain of $x \geq 10$. For Case 2, students should identify that $x = 10$ does not fit the domain of $x < 10$. Encourage students to check the domain to identify any extraneous roots. Remind students to verify the correctness of the remaining possible solutions through substitution into the original equation.

Have students complete the Your Turn question individually and compare solutions as a class. You may need to remind students to identify and reject any extraneous roots.

Key Ideas

You might provide an example of an absolute value equation and have students solve it graphically and algebraically. Check that they verify the solution. Ask volunteers to present the solution to the class.

Remind students they can also solve an absolute value equation by equating it to zero and determining the zeros (x-intercepts) of the function.

Ask students how else extraneous roots can be identified and rejected. (by checking the domain for each case)

Have students record their own summary of the Key Ideas. Have them store the summaries of the Key Ideas for each chapter section together for review purposes.

Meeting Student Needs

- Encourage students to create their own definition and an example of an *absolute value equation*.
- Have students work through both methods shown in Example 1. Ask them to summarize when they would use a graph and when they would use algebra to solve an absolute value equation. Allow students to use their method of choice throughout the section.
- Post $|x| = \begin{cases} x \text{ if } x \geq 0 \\ -x \text{ if } x < 0 \end{cases}$

  in the classroom for students to reference as they work through this section. Some students may benefit from recording this information in their graphic organizer.

- Invite students to study the given equations in Examples 3 to 5 and determine whether they can predict the number of solutions (no solution, one solution, more than one solution).
- Ensure that students understand that all solutions must be verified. Emphasize that they can use the method they prefer to solve the equation.
- Encourage students to add an example of solving an absolute value equation to their graphic organizer.

ELL

- For Example 2, clarify for students the meaning of *tolerance* in the given context.

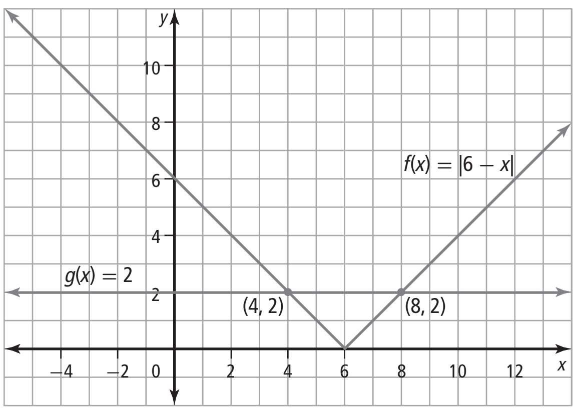
Common Errors

- When considering both cases in solving absolute value equations, some students may consider only the variable as positive or negative instead of considering the entire expression within the absolute value symbol. For example, $|2x + 2| = 5$:
  Case 1: $2x + 2 = 5$
  Case 2: $-2x + 2 = 5$

R_x Coach students to consider the entire expression within the absolute value symbol. For example, $|2x + 2| = 5$:
Case 1: $+(2x + 2) = 5$
Case 2: $-(2x + 2) = 5$

Answers

Example 1: Your Turn



$x = 4$ or $x = 8$

Example 2: Your Turn

The minimum mass of fish is 164 g, and the maximum mass is 176 g.

Example 3: Your Turn

$x = 2$

Example 4: Your Turn

the empty set or { }

Example 5: Your Turn

$x = \frac{3 - \sqrt{17}}{2}, x = 1, x = 2, x = \frac{3 + \sqrt{17}}{2}$

Example 6: Your Turn

$x = 2$ or $x = 5$

Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• Provide a similar question before assigning the Your Turn. • Ensure students solve the positive and the negative cases algebraically. It may be helpful for students to set up two columns, with the positive case and the negative case side by side. This visual approach may help them make the comparisons between the solutions for the two cases. • Remind students that they must show the distance of 2 from 0 graphically.
Example 2 Have students do the Your Turn related to Example 2.	• Encourage students to solve using a number line before using algebra. • Have students refer to the worked example, since it provides a model for solving a similar absolute value problem.
Example 3 Have students do the Your Turn related to Example 3.	• Ensure students solve the positive and the negative cases algebraically. It may be helpful for students to set up two columns, with the positive case and the negative case side by side. This visual approach may help them make the comparisons between the solutions and the verification of the roots for the two cases.
Example 4 Have students do the Your Turn related to Example 4.	• If students are unclear about the concepts, work with them to review the responses to the Reflect and Respond questions they completed earlier. • You may wish to provide similar equations so students can practise visually identifying which ones would have extraneous roots.

Assessment	Supporting Learning
Assessment *for* Learning	
Example 5 Have students do the Your Turn related to Example 5.	• Provide a similar question before assigning the Your Turn. • Ensure students solve the positive and the negative cases algebraically. It may be helpful for students to set up two columns, one for each of the positive and the negative case. They might further subdivide each column to determine the roots of any quadratics. This visual organizer may make it easier for students to compare the cases and verify the roots.
Example 6 Have students do the Your Turn related to Example 6.	• Provide a similar question before assigning the Your Turn. • Check that students move all values to one side of the equal sign for both cases. • Ensure students solve the positive and the negative cases algebraically. Students may wish to set up two columns, one for each of the positive and the negative case. They might further subdivide each column to determine the roots of any quadratics. This visual organizer may make it easier for students to compare the cases and verify the roots.

Check Your Understanding

Practise

These questions allow students to build basic skills in solving absolute value equations.

Consider allowing students some choice in the equations they solve. Remind students to verify all solutions. Provide students with grid paper or **Master 2 Centimetre Grid Paper**.

For #5, ask students whether they prefer to verify the solutions algebraically or graphically, and why. Then, have them do so.

For #6, ask students:

- What are two methods for verifying your solutions graphically?

Apply

Consider allowing students to work in pairs to solve at least some of the problems before recording their own solutions. Tell students to sketch diagrams whenever possible.

Clarify for #7 that *acceptance limits* refers to an acceptable range of values.

Clarify the meaning of *measurement uncertainty* in #8 and 11. Explain that there is always a degree of error in taking measurements. The measurement uncertainty of 1.1 m/s in #8 quantifies the doubt about the exact accuracy of the measured speed of light.

Note that #11 is an application of concepts students have learned in science class.

For #13, encourage students to verify their answers by substituting values for n, and to justify their answers using the definition of absolute value.

Consider using #14 for enrichment, as it involves equations with more than one variable. As you circulate, observe whether students understand the process of solving these absolute value equations. Consider allowing students who can readily solve these equations some choice in solving other problems. It is likely unnecessary for them to complete all of the questions.

Extend

For #18, although there is no Project Corner in this section, you may wish to point out that launching a rocket is an application of absolute value.

For #20, students need to work backward. You might use this question to check whether students understand the link between solutions and the graphical representation of absolute value equations. Students could use the connection between solutions and x-intercepts to solve this problem.

Create Connections

Consider having students work in pairs or small groups to brainstorm solutions to #22 and answers to #23 and 24 before developing an individual response.

Meeting Student Needs

- Encourage students to attempt all of the assigned questions and use the method of their choice for solving problems.
- For #12, use the related Did You Know? to clarify the terms *perigee* and *apogee*.
- For #15, have students share their response with at least one other student. This may lead them to understand that they could either verify each solution or work through each solution and look for the errors. If students verified the solutions, encourage them to look for the mistake Erin made.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome

addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.

- Provide **BLM 7–6 Section 7.3 Extra Practice** to students who would benefit from more practice.

ELL

- Use a combination of descriptions, visuals, and examples to help students understand such terms as *aviation fuels*, *spawning*, *vegetation*, and *launch window*.

Enrichment

- Invite students to create their own written strategy for solving an absolute value equation. Set a space limitation for the strategy, such as one side of an index card. Encourage students to be as succinct and clear as possible.
- Challenge students to develop a problem involving absolute value equations in a context of their choice, and provide a solution. Have them exchange their problem with that of a classmate, and attempt the solution.
- Ask students to research measurement uncertainty. Have them prepare a report that includes an explanation of the concept, some examples, the importance of measurement uncertainty, and the careers that need to consider measurement uncertainty (e.g., technicians in testing and calibration laboratories, technicians in manufacturing, research scientists, and teachers).
- Have students test each of the following statements by showing an example of each:
  - $|a| \geq 0$
  - $|a| = 0 \Leftrightarrow a = 0$
  - $|ab| = |a||b|$
  - $|a + b| = |a| + |b|$

Gifted

- One way to quantify the spread of values in a set of measurements is standard deviation. Challenge students to research standard deviation. Have them develop an example and calculate an estimate of standard deviation. The related Web Link at the end of this section may be helpful.
- Challenge students to summarize the following information in their own words.

Statement	Property
$\|a\| \geq 0$	Non-negative
$\|a\| = 0 \Leftrightarrow a = 0$	Positive definite
$\|ab\| = \|a\|\|b\|$	Multiplicative
$\|a + b\| = \|a\| + \|b\|$	Subadditive

Web Link

For a tutorial on solving absolute value equations, including solutions and explanations, go to www.mhrprecalc11.ca and follow the links.

For information about measurement uncertainty and standard deviation, go to www.mhrprecalc11.ca and follow the links.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1, 2b), c), 3, 4a), d), 5a), b), 6a) to d), 7, two of 8 to 12, and 15. Students who have no problems with these questions can go on to the remaining questions.	• For #1, students should be able to identify whether an equation has a solution. Some students may need coaching to solve the equations. Then, provide a few additional questions to check for understanding or have them complete the related questions on **BLM 7–6 Section 7.3 Extra Practice**. Ensure understanding before moving on to #2 and 3. • Remind students to solve for positive and negative values of *x*, and to verify their solutions. • You might suggest that students use a chart divided into two columns in order to record the solutions for the positive and negative case. • You may wish to suggest to students that they complete #15 after 4. • Coach students through Example 2 to assist them with #7 to 9. Have students identify similarities between the absolute value equation in the worked example and the values and variables presented in #7 to 9.
Assessment *as* Learning	
Create Connections Have all students complete #23.	• Some students may benefit from working with a partner to plan their responses. • Consider having students attempt to solve both equations in #23 and record their explanation beside the solution. Have students compare their solution with that of a classmate. After a class discussion, have students record the equation and explanation in their graphic organizer or notebook for review purposes. • Consider collecting students' response to #23 and checking for weaknesses in their thinking. Provide coaching as needed.

7.4 Reciprocal Functions

***Pre-Calculus 11*, pages 392–409**

Suggested Timing
130–180 min

Materials
- graphing calculator
- grid paper
- ruler

Blackline Masters
Master 2 Centimetre Grid Paper
BLM 7–3 Chapter 7 Warm-Up
BLM 7–7 Section 7.4 Extra Practice

Mathematical Processes
- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

Specific Outcomes
RF11 Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).

Category	Question Numbers
Essential (minimum questions to cover the outcomes)	#1–5, 7a), b), 8a), d), 9, 10, 12, 19, and 22
Typical	#2–9, 10 or 11, three of 13–16, 19, and 22
Extension/Enrichment	#3, 6, 9, 15–22

Planning Notes

Have students complete the warm-up questions on **BLM 7–3 Chapter 7 Warm-Up** to reinforce prerequisite skills needed for this section.

Introduce Newton and his law of universal gravitation to the class. Some students may need a more basic explanation of Newton's conclusions. Ask students:
- What prevents you from floating into space? (force of gravity)
- When you drop an object, why does it fall to the ground? (because of the gravitational force directed toward the centre of Earth)

Reinforce that the force of gravity attracts objects, even the moon, to the centre of Earth. Because of the force of gravity, the moon follows an orbit around Earth. Newton also concluded that any two objects exert gravitational attraction on each other. Use the diagram on page 392 of the student resource to help explain that every object attracts every other object with a force (F_1 and F_2) directed along the centres of the two objects that is proportional to the product of their masses (m_1 and m_2) and inversely proportional (or reciprocal) to the square of the distance (r) from the centres about which they revolve. The constant G is called a universal gravitational constant since it is assumed to be the same at all times and locations.

Investigate Exchange Rates

In this investigation, students develop an understanding of reciprocal functions by exploring exchange rates.

For #4, you may wish to prepare a more current table of foreign currency values in Canadian dollars for students to use. If so, provide them with the updated table.

You might have students work in pairs but record their own response to all questions.

Students will need graphing calculators.

For #3, help students interpret the table. Ask students:
- What is the difference between column 1 and column 2?

Help students realize that column 1 is the amount of US dollars they could buy with one Canadian dollar for different exchange rates. Column 2 is the cost, in Canadian dollars, to buy one US dollar, for different exchange rates.

For #3c), prompt students by asking:
- What does *extrapolate* mean?
- Compare *extrapolate* and *interpolate*.
- What does the x-intercept mean in the context of this question? the y-intercept?

For #7, you might have students graph the relationship of currency exchange between Canada and the U.S. to help them understand the relationship between the two calculations. Have students input the table from #3 into their graphing calculator, entering values into L1 and L2. Have students graph the relationship of L1 versus L2. Ask them what type of graph the scatter plot appears to follow.

As a class, have students share their response to Reflect and Respond #7 and 8.

Direct students to the Did You Know? about spread on page 393 of the student resource. Have students who have travelled recall their experience with buying and selling foreign currency and the cost of completing the exchange. Ask students the approximate amount of the spread and whether they thought the spread was reasonable.

Meeting Student Needs

- Provide students with a basic explanation of the universal law of gravitation. Students may find the related Web Link in this section interesting.

Enrichment

- Invite students to research Isaac Newton's contributions to mathematics by accessing the Web Link on page 392 in the student resource. Have them present a summary of their findings in a format of their choice.
- Invite students to research the current value of the Canadian dollar compared to the US dollar. Or, they might research the history of the highest and lowest values of the Canadian dollar in comparison to the US dollar. Ask students:
  - At the Canadian dollar's highest value, how many US dollars could you purchase with C$100?
  - At the Canadian dollar's lowest value, how many US dollars could you purchase with C$100?

Gifted

- Invite interested students to research the law of universal gravitation. Challenge them to apply the law in order to determine the distance between two objects of their choice. They may find the related Web Links in this section helpful.

Web Link

For information about the universal law of gravitation, go to www.mhrprecalc11.ca and follow the links.

For sample problems and solutions involving the universal law of gravitation, go to www.mhrprecalc11.ca and follow the links.

Answers

Investigate Exchange Rates

1. $0.80 = \frac{4}{5}$; $1.25 = \frac{5}{4}$. These fractions are reciprocals of each other. Example: You could use multiplication to convert one currency to the other.

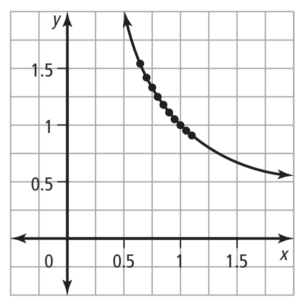
2. 1.25; You can buy C$1.25 with US$1.

3. a)

C$1 in US$	Purchase Price of US$1
0.65	1.54
0.70	1.43
0.75	1.33
0.80	1.25
0.85	1.18
0.90	1.11
0.95	1.05
1.00	1.00
1.05	0.95
1.10	0.91

b) Example: Divide 1 by the C$1 value in US$, or $\frac{1}{\text{C\$1 in US\$}}$. The method will work as long as a conversion rate is available.

c) Example:



You cannot convert from no amount of money in one currency to an amount of currency in the other currency, so neither the x-intercepts nor the y-intercepts exist.

4. In C$, 1 yen is worth 0.0108 dollars, or just slightly more than $0.01.

5. a) 92.59 yen with C$1; 18 518.52 yen with C$200
b) $54

6. a) Example: 1.02 Australian dollars can be purchased with C$1.
b) Example: It would cost $97.96 to purchase 100 units of Australian currency.

7. Example: If the Canadian-to-U.S. dollar exchange rate is 0.9645, then the U.S.-to-Canadian dollar exchange rate is $\frac{1}{0.9654}$. The two calculations are reciprocals of each other.

8. Yes

Assessment	Supporting Learning
Assessment *as* Learning	
Reflect and Respond Have students complete the Reflect and Respond questions with a partner or individually. Listen as students discuss what they learned during the Investigate. Encourage them to generalize and reach a conclusion about their findings.	• You may wish to have students compare their responses to #7. Ensure that students have made a connection to the concept of a reciprocal and how to determine a reciprocal. • Have students who did not understand #7 check several rows in the given table, and coach them to observe the pattern before moving on.

Link the Ideas

Emphasize the importance of identifying the non-permissible values (e.g., $f(x) \neq 0$). The non-permissible values are key in graphing reciprocal functions. Tell students that whenever they are working with reciprocal functions, they will need to consider the restrictions of the function.

Students need a graphing calculator and either grid paper or **Master 2 Centimetre Grid Paper** for the worked examples.

Example 1

In this example, students compare the graphs of a function and its reciprocal.

Direct students to the definition of *reciprocal function* and have them provide several examples of reciprocal functions.

Use the questions in green type on page 394 in the student resource to promote a class discussion.

As they view the table, ask students:

- Are there any points that are the same for the function and the reciprocal function?
- How important is it to identify the invariant points? Why?
- Does a reciprocal function always have invariant points? If so, how do you determine them?

As students view the graph, ask them:

- What do you think will happen on the graph when $y = 0$?
- Will this happen every time that $y = 0$? Explain.

Talk about asymptotes and the characteristics of a function and its reciprocal, as shown in the chart on page 395 in the student resource. Emphasize the following points:

- For reciprocal functions, the vertical asymptotes occur at the non-permissible values for the domain of the function. (The domain for the function is all real numbers except those that make the denominator zero.)
- For reciprocal functions, the horizontal asymptotes are defined by the fact that zero is not in the range of the function.

Remind students that when graphing rational functions (including reciprocal functions), they should set the mode of their calculator to Dot (and not to Connected). When the mode is set to Connected, the graphing calculator graphs the vertical asymptote as well and it appears to be part of the graph. When the mode is set to Dot, the asymptote does not appear.

Have students complete the Your Turn question in pairs. Have them compare their solution with that of another pair before you take up the solution as a class.

Example 2

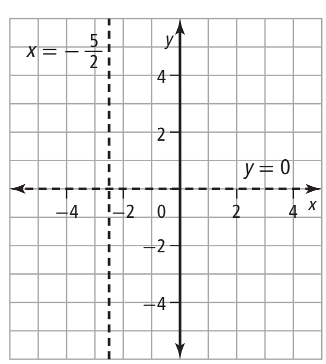
In this example, students graph the reciprocal of a linear function.

Consider working through the solution as a class. Have students try both methods for part c): using pencil and paper and using a graphing calculator.

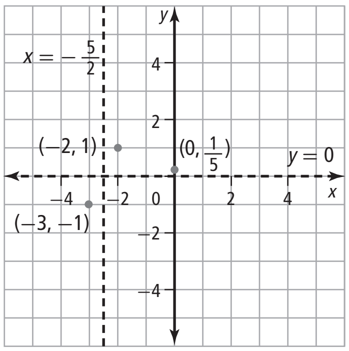
Encourage students to ask questions to clarify understanding.

For Method 1 of part c), consider the following step-by-step explanation for students who may benefit from a different approach to graphing the reciprocal function.

- First, mark the horizontal and vertical asymptotes as a reminder that the graph approaches these lines but never touches or crosses them. As shown in the student resource, the non-permissible value is $x = -\frac{5}{2}$, so the vertical asymptote is at $x = -\frac{5}{2}$. The horizontal asymptote is at $y = 0$.

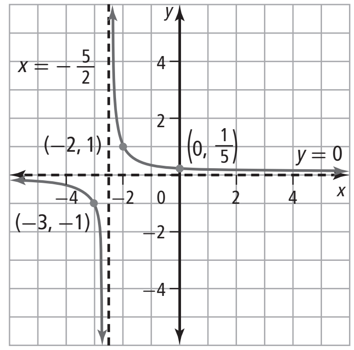


- Second, determine the invariant points, the x-intercepts, and y-intercepts. Points in the function that have a y-coordinate of 1 have a corresponding y-coordinate of 1 in the reciprocal function (since $\frac{1}{1} = 1$).
  – Determine the invariant points by solving the equations $2x + 5 = 1$ and $2x + 5 = -1$. The invariant points are $(-2, 1)$ and $(-3, -1)$. These points help determine the shape of the graph.
  – Since the horizontal asymptote is at $y = 0$, there is no x-intercept. The value of the y-intercept is $\frac{1}{5}$ (when $x = 0$ in the reciprocal function). Label these points on the graph.



- Third, consider the behaviour of the reciprocal function as the graph approaches the asymptote from the left and the behaviour as the graph approaches the asymptote from the right. To do this, compare the reciprocal function $\frac{1}{f(x)} = \frac{1}{2x + 5}$ to the original function $f(x) = 2x + 5$.
  – In the original function, as $x \to -\frac{5}{2}$ from the left, the y-values get progressively larger. Therefore, as $x \to -\frac{5}{2}$ from the left, the y-values of the reciprocal function get progressively smaller.
  – Similarly, in the original function, as $x \to -\frac{5}{2}$ from the right, the y-values get progressively smaller. Therefore, as $x \to -\frac{5}{2}$ from the right, the y-values of the reciprocal function get progressively larger.

To help students visualize the shape of the graph, consider the end behaviour of the reciprocal function.
  – In the original function, as $x \to +\infty$, y is positive. Therefore, the reciprocal function will be positive and approach 0 from above, since all values of $\frac{1}{f(x)}$ are positive.
  – In the original function, as $x \to -\infty$, y is negative. Therefore, the reciprocal function will be negative and approach 0 from below, since all values of $\frac{1}{f(x)}$ are negative.



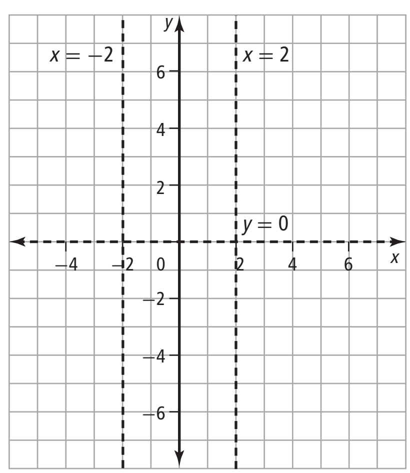
Example 3

In this example, students graph the reciprocal of a quadratic function.

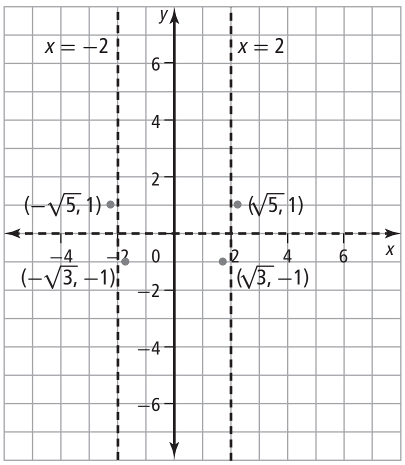
Consider working through the solution as a class. Have students attempt both methods for part d): using pencil and paper and using a graphing calculator. While two methods are presented, students need to have a basic understanding of reciprocal functions and their graphs in order to help them interpret the graph shown on the graphing calculator. Therefore, it is recommended that students use pencil and paper before using a graphing calculator. Encourage students to ask questions to clarify understanding.

Consider using the following step-by-step explanation for students who may benefit from a different approach to graphing the reciprocal function.

- First, mark the horizontal and vertical asymptotes as a reminder that the graph approaches these lines but never touches or crosses them. As shown on page 399 in the student resource, the non-permissible values are $x = 2$ and $x = -2$, so the vertical asymptotes are at $x = 2$ and $x = -2$. The horizontal asymptote is at $y = 0$.

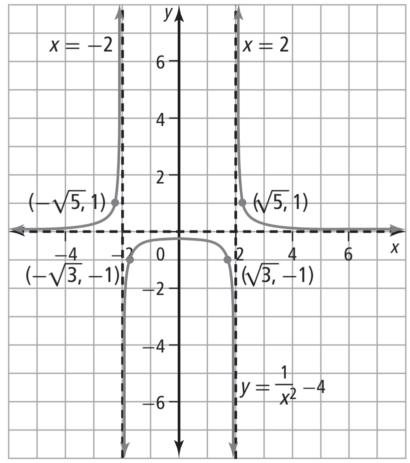


- Second, determine the invariant points and the x-intercepts and y-intercepts. Points in the function that have a y-coordinate of 1 have a corresponding y-coordinate of 1 in the reciprocal function (since $\frac{1}{1} = 1$).
  – Determine the invariant points by solving the equations $x^2 - 4 = 1$ and $x^2 - 4 = -1$. The invariant points are $(\sqrt{5}, 1)$, $(-\sqrt{5}, 1)$, $(\sqrt{3}, -1)$, and $(-\sqrt{3}, -1)$. These points help determine the shape of the graph.
  – Since the horizontal asymptote is at $y = 0$, there is no x-intercept. The y-intercept has a value of $-\frac{1}{4}$ (when $x = 0$ in the reciprocal function).
  Label these points on the graph.



- Third, since there are two vertical asymptotes, there are three sections of the graph to consider: the behaviour of the reciprocal function as the graph approaches the asymptote from the left, the behaviour as the graph approaches the asymptote from the right, and the behaviour in between the two asymptotes. To do this, compare the reciprocal function $f(x) = \frac{1}{x^2 - 4}$ to the original function $f(x) = x^2 - 4$.
  – In the original function, as $x \to -2$ from the left, the y-values get progressively smaller. Therefore, as $x \to -2$ from the left, the y-values of the reciprocal function get progressively larger.
  – Similarly, in the original function, as $x \to 2$ from the right, the y-values get progressively smaller. Therefore, as $x \to 2$ from the right, the y-values of the reciprocal function get progressively larger.
  – In the original function, when x is in between -2 and 2, the y-values decrease to a minimum value and then increase. Therefore, when x is in between -2 and 2, the y-values of the reciprocal function increase to a maximum value and then decrease.

To help students visualize the shape of the graph, consider the end behaviour of the reciprocal function.
– In the original function, as $x \to +\infty$, y is positive. Therefore, the reciprocal function will be positive and approach 0 from above.
– In the original function, as $x \to -\infty$, y is positive. Therefore, the reciprocal function will be positive and approach 0 from above.



Have students complete the Your Turn questions in pairs. Have them compare their solution with that of another student pair, before taking up the solution as a class.

Example 4

In this example, students sketch the graph of the original function given the graph of a reciprocal function. Two methods are modelled: using slope and the y-intercept, and using the x-intercept.

Have students work through the solution by attempting both methods.

For Method 1, students may benefit from recalling how to determine the slope of a linear function given two points. Ask students:

- What formula do you use? $\left(m = \frac{y_2 - y_1}{x_2 - x_1}\right)$
- When you solve for b, does it matter which point you use? Why?

Have students try using both points to illustrate that the value of b will remain the same, regardless of which point is used.

After working through each method, ask students which method they prefer. Ask them to share any other strategies for determining the equation of the original function.

Have students complete the Your Turn questions. Have them compare their solution with that of a classmate.

Key Ideas

As a class, read the guidelines about graphs of reciprocal functions. Remind students to always state the restrictions of reciprocal functions. Remind students that the domain of the reciprocal function is the same as the original function except for the non-permissible values. The range of a reciprocal function is $\{y \mid y \neq 0, y \in R\}$. The horizontal asymptote of a reciprocal function is at $y = 0$.

Have students write their own summary of the guidelines for determining the graph of the reciprocal function given the graph of a function. Have students use the method they prefer to determine the original function given the graph of a reciprocal function they choose. Have them store the summaries of the Key Ideas for each chapter section together for review purposes.

Meeting Student Needs

- Encourage students to create their own definition for and an example of a *reciprocal function* and an *asymptote*.
- For Example 1, some students may benefit from writing the whole numbers in the table of values as fractions with a denominator of 1 to help them understand how the reciprocal was obtained.
- For Example 1, consider displaying a poster of the chart of characteristics of a function and its reciprocal. Encourage students to refer to the poster as they work through the chapter.
- Some students may find it difficult to see the graph of a function and its reciprocal on the same grid. Have them use a different colour to highlight each function on the same grid.
- Example 2, Method 1 provides key information about the graphs of reciprocal functions. Encourage students to add this information to their graphic organizer.
- Pair students to work through the Key Ideas. Have them develop an example to illustrate the guidelines and then take turns explaining the process to their partner.
- Invite student pairs to talk about the connection between the graphs of a function and its reciprocal function. Challenge them to recall as many characteristics as possible without referring to the Key Ideas.

ELL

- For the Key Ideas, allow students to write the summary of the guidelines in their first language, illustrate their thoughts using a sketch of a graph, and then use the graph to explain their thoughts orally in English.

Common Errors

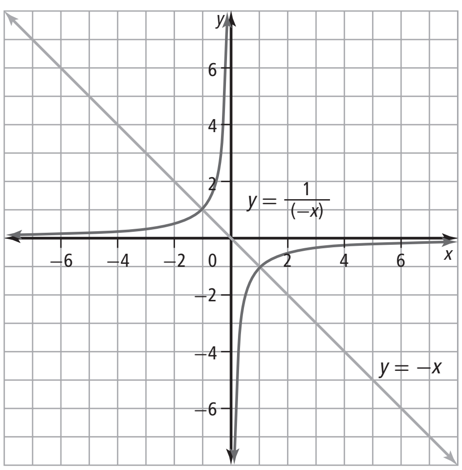
- Students may have part of the graph of these reciprocal functions cross or touch the *x*-axis.

R_x Reinforce that the reciprocal functions students are working with always have a horizontal asymptote at $y = 0$. Therefore, the graph of a reciprocal function will never cross or touch the *x*-axis. Have students highlight the horizontal asymptote on a graph as a reminder.

Example 1: Your Turn

Example:

x	$y = -x$	$y = \frac{1}{-x}$
-10	10	$\frac{1}{10}$
-5	5	$\frac{1}{5}$
-2	2	$\frac{1}{2}$
-1	1	1
-0.5	0.5	2
-0.2	0.2	5
-0.1	0.1	10
0	0	undefined
0.1	-0.1	-10
0.2	-0.2	-5
0.5	-0.5	-2
1	-1	-1
2	-2	$-\frac{1}{2}$
5	-5	$-\frac{1}{5}$
10	-10	$-\frac{1}{10}$



Example:

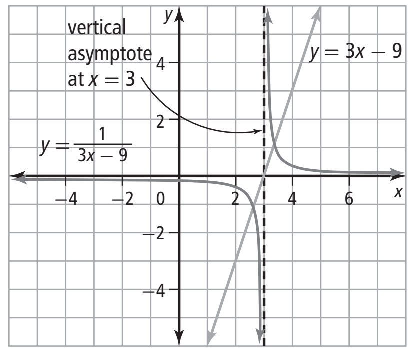
Characteristic	$y = -x$	$y = \frac{1}{-x}$
Domain	$\{x \mid x \in R\}$	$\{x \mid x \neq 0, x \in R\}$
Range	$\{y \mid y \in R\}$	$\{y \mid y \neq 0, y \in R\}$
End behaviour	• as x approaches negative infinity, y approaches positive infinity • as x approaches positive infinity, y approaches negative infinity	• as x approaches negative infinity, y approaches 0 from above • as x approaches positive infinity, y approaches 0 from below • horizontal asymptote at $y = 0$
Behaviour at $x = 0$	$y = 0$	undefined, vertical asymptote at $x = 0$
Invariant points	$(-1, 1)$ and $(1, -1)$	

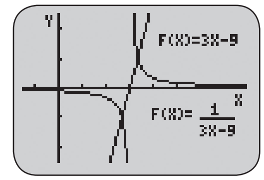
Example 2: Your Turn

a) $y = \frac{1}{3x - 9}$

b) $x = 3$

c) Students should provide a graph with and without technology.





As x approaches 3 from the left, the graph of $y = \frac{1}{3x - 9}$ approaches negative infinity.

As x approaches 3 from the right, the graph of $y = \frac{1}{3x - 9}$ approaches positive infinity.

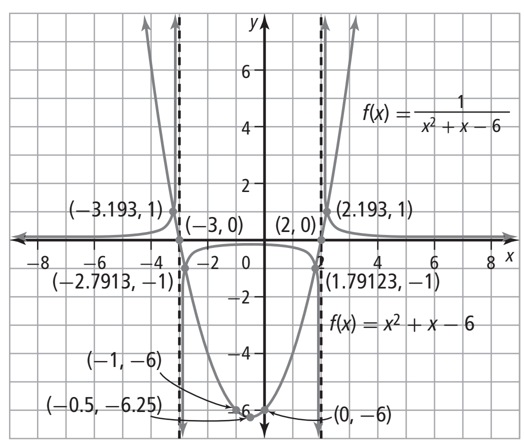
Example 3: Your Turn

a) $y = \frac{1}{x^2 + x - 6}$

b) Non-permissible values of x: $x = -3$, $x = 2$
Vertical asymptotes: $x = -3$ and $x = 2$

c) x-intercepts: none; y-intercept: $-\frac{1}{6}$

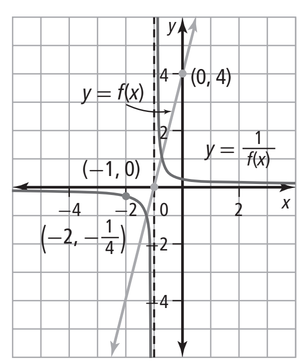
d)



Example 4: Your Turn

a) Example: Draw a line passing through $(-2, -4)$ and $(-1, 0)$.

b) $y = 4x + 4$



Assessment	Supporting Learning
Assessment *for* Learning	
Example 1 Have students do the Your Turn related to Example 1.	• You may wish students to work in pairs. • Ensure students select test values that include positive, negative, and zero values.
Example 2 Have students do the Your Turn related to Example 2.	• Encourage students to use the method they find easiest to understand. However, encourage them to attempt a second method. • Invite a student pair who used pencil and paper to record their solution on the board, and then use their work for a class discussion.
Example 3 Have students do the Your Turn related to Example 3.	• Encourage students to list all the information that they know or can determine, and then use the information to sketch a graph, even if they have a graphing calculator. • Note that part d) asks students to sketch the graph. (It does not ask for an accurate graph. Encourage students to make a sketch and use a calculator only for verifying their work.)
Example 4 Have students do the Your Turn related to Example 4.	• Have students refer to the worked example, which models an approach for graphing the original function given the graph of its reciprocal.

Check Your Understanding

Students need grid paper or **Master 2 Centimetre Grid Paper** and a graphing calculator to complete these questions.

Practise

These questions allow students to build skills in graphing and analysing graphs of reciprocal functions. Students can complete the questions individually or with a partner.

Remind students to state the restrictions in #1. Tell them to adopt the habit of stating restrictions whenever they work with reciprocal functions.

For #2 and 3, ask students:

- Are there any horizontal asymptotes? If so, what are they?

For #5, ask students:

- How can you use the x-intercepts and y-intercepts of the given functions to determine the intercepts of the corresponding reciprocal functions?
- How do the intercepts of a given function relate to the asymptotes of the reciprocal function?

For #7 and 8, encourage students to use different colours for the original function and the reciprocal function to help differentiate the two functions on the graph.

Apply

Allow students to work in pairs to solve the problems.

For #10, students work backward to determine the original function given the reciprocal function. Encourage students to identify asymptotes and given points on the reciprocal functions and how they relate to points on the original function.

Students may answer #11c) and d) by substituting the given values into the original equation and solving for the remaining variable. Alternatively, they may apply their knowledge of reciprocals.

For #12, you might use the following prompts:

- What are the x-intercepts and y-intercepts of the function?
- How do the x-intercepts and y-intercepts relate to the original context of the question?

Extend

For #17, tell students they might use a similar table to help them graph functions and their reciprocals.

After students complete #19, have them explain their response to the class.

Create Connections

Consider having students work in pairs or small groups to brainstorm responses to these questions before developing an individual response.

For #21, which is a Mini Lab, you might have students work in pairs. If the Mini Lab is used for summative assessment, ensure that you present your expectations for the completed work and provide a marking rubric for the assignment. If the Mini Lab is used for formative assessment, meet with the class as a whole and ask a couple of student pairs to lead a discussion of the results.

For #22, have volunteers present their completed chart to the whole class. Have students provide constructive feedback on how to improve each chart.

Meeting Student Needs

- Encourage students to attempt each type of question.
- Invite students to refer to the Key Ideas and their own summary of the Key Ideas to help them understand the characteristics of reciprocal functions.
- Invite students to work in pairs or small groups to solve one problem in the Apply section. Have them prepare a short presentation to the class that includes any challenges in the problem, the solution, and the strategies used to determine the solution.
- Have students refer to their own list of learning outcomes for the chapter and assess their progress. Tell them to highlight in yellow each outcome addressed in this section. Once they have completed the Check Your Understanding questions, have them use a different colour to highlight the outcomes that they have no problem with. Have students identify the outcomes that remain highlighted in yellow and require more work. Provide any needed coaching.
- Provide **BLM 7–7 Section 7.4 Extra Practice** to students who would benefit from more practice.

ELL

- Use the Did You Know? related to #11 and a diagram to elaborate on the motion of a pendulum.
- Use the Did You Know? related to #12, descriptions, and examples to help explain the terms *scuba diver* and *decompression*.
- For #16, use the photo of the spiral tracks to help explain the Spiral Tunnels.
- Use a combination of visuals, descriptions, and examples to help students understand such terms as *pitch*, *Hertz*, *period*, *air vibrations*, *inverted image*, *convex lens*, *focal length*, and *zoom lens*.
- If several students share a common first language, consider having them discuss the responses to #19 and 22 in their first language. This offers students the opportunity to activate their knowledge using familiar language. Afterward, it may be easier for students to ask for the missing vocabulary to express their thinking in English.

Enrichment

- Ask students to make a statement that connects the idea of an asymptote to the notion of dividing by zero.

Gifted

- Have students explain how it is possible for a line to get closer and closer to an asymptote and yet never touch it.

Assessment	Supporting Learning
Assessment *for* Learning	
Practise and Apply Have students do #1 to 5, 7a), b), 8a), d), 9, 10, and 12. Students who have no problems with these questions can go on to the remaining questions.	• As students work, circulate and check that they are confident in solving #2, which deals with the key features of a reciprocal function. Provide coaching to students, as needed, before having them move on. You may wish to assign some of the related questions from **BLM 7–7 Section 7.4 Extra Practice**. • Some students may need coaching about asymptotes to help them answer #3. Then, use #4 to check for understanding. After making any needed revisions to their response to #4, have students include the response in their graphic organizer or notebook. • For #6 to 8, have students use different colours for the graph of a function and its reciprocal. Check for students' understanding before having them move on to #9. • It is important for students to complete #10 successfully and show their understanding of how to determine the original function given its reciprocal function.
Assessment *as* Learning	
Create Connections Have all students complete #22.	• For #22, encourage students to refer to their own summary of the Key Ideas in this section to help them complete the chart. Have students walk through their chart with a classmate, and make revisions based on constructive feedback they receive. As you circulate, clarify any misunderstandings. • Have students store the chart for #22 in their graphic organizer or notebook for review purposes.

Chapter 7 Review

7

***Pre-Calculus 11*, pages 410–412**

Suggested Timing
90—120 min

Materials
- grid paper
- ruler

Blackline Masters
Master 2 Centimetre Grid Paper
BLM 7–4 Section 7.1 Extra Practice
BLM 7–5 Section 7.2 Extra Practice
BLM 7–6 Section 7.3 Extra Practice
BLM 7–7 Section 7.4 Extra Practice

Planning Notes

Have students who are not confident discuss strategies with you or a classmate. Encourage them to refer to their summary notes, worked examples, and previously completed questions in the related sections of the student resource.

Have students make a list of questions that they need no help with, a little help with, and a lot of help with. They can use this list to help them prepare for the practice test.

Make grid paper or **Master 2 Centimetre Grid Paper** available.

Meeting Student Needs

- Students who require more practice on a particular topic may refer to **BLM 7–4 Section 7.1 Extra Practice**, **BLM 7–5 Section 7.2 Extra Practice**, **BLM 7–6 Section 7.3 Extra Practice**, or **BLM 7–7 Section 7.4 Extra Practice**.
- Encourage students to use technology to assist them in their work.
- Invite students to refer to their summary of the Key Ideas for each section. Encourage them to clarify any misunderstandings. Also, display any posters created during the chapter for students to refer to as they complete the review.
- If it has not been done earlier, post the learning outcomes. Invite students to ask questions about any outcomes that they do not understand. You might group students according to the learning outcomes they identified as needing more work.
- Encourage students to try each type of question, but have them spend the most time on those questions that they identified as difficult.
- Individualize the chapter review. Give students some choice in the questions they do from each section. Correct the questions and analyse the errors. Encourage students to get help for the questions they are unable to complete successfully. Students can then choose additional practice questions based on their results.

ELL

- Use a combination of visuals, descriptions, and examples to help students understand such terms as *suspension bridge*, *Toronto Stock Exchange*, *tides*, *high tide*, *low tide*, *bushel*, *moisture content*, *lever*, *fulcrum*, and *pivots*.

Enrichment

- Encourage students to explore the impact of the domain on a reciprocal function. For example, ask them:
  - Can zero be included in the domain of a reciprocal function? Why or why not?
  - Are there other domain-related issues that affect the reciprocal function? If so, what are they?

Gifted

- Ask students to explore the idea of an asymptotic curve. Under what circumstances would an asymptote be curved rather than straight?

Assessment	Supporting Learning
Assessment *for* Learning	
Chapter 7 Review The Chapter 7 Review is an opportunity for students to assess themselves by completing selected questions in each section and checking their answers against answers in the student resource.	• Consider allowing students to work in pairs to answer the questions. • Have students revisit any section that they are having difficulty with prior to working on the chapter test. • Encourage students to refer to the summary notes they completed throughout the chapter.

7 Chapter 7 Practice Test

***Pre-Calculus 11*, pages 413–414**

Suggested Timing

45—60 min

Materials

- grid paper
- ruler

Blackline Masters

Master 2 Centimetre Grid Paper

BLM 7–8 Chapter 7 Test

Planning Notes

Have students start the practice test by writing the question numbers in their notebook. Have them indicate which questions they need no help with, a little help with, and a lot of help with. Have students first complete the questions they know they can do, followed by the questions they know something about. Finally, suggest to students that they do their best on the remaining questions.

Make grid paper or **Master 2 Centimetre Grid Paper** available.

This practice test can be assigned as an in-class or take-home assignment. Provide students with the number of questions they can comfortably do in one class. These are the minimum questions that will meet the related curriculum outcomes: #1–10.

Study Guide

Question(s)	Section(s)	Refer to	The student can …
#1	7.1	Link the Ideas Example 3	✓ determine the absolute values of numbers and expressions
#2	7.2	Example 1	✓ sketch the graph of $y = \|f(x)\|$ and determine its intercept(s), domain, and range
#3, 7, 9, 11	7.3	Example 1 Example 4	✓ solve algebraically an equation with a single absolute value and verify the solution
#5	7.4	Example 2	✓ graph the reciprocal of a given function
#4, 12	7.4	Example 4	✓ identify the values of x for which the graph of $y = \frac{1}{f(x)}$ has vertical asymptotes
#6	7.2	Link the Ideas Example 1	✓ generalize a rule for writing absolute value functions in piecewise notation
#8	7.3	Example 1	✓ solve an absolute value equation graphically, with or without technology
#10	7.4	Example 2	✓ graph the reciprocal of a given function ✓ analyse the graph of the reciprocal of a given function

Assessment	Supporting Learning
Assessment *as* Learning	
Chapter 7 Self-Assessment Have students use their responses on the practice test and work they completed earlier in the chapter to identify skills or concepts they may need to reinforce.	• Encourage students to study their chapter summary notes before they begin the practice test. They can use these to identify any areas of weakness. • Before the chapter test, coach students in areas in which they are having difficulties.
Assessment *of* Learning	
Chapter 7 Test After students complete the practice test, you may wish to use **BLM 7–8 Chapter 7 Test** as a summative assessment.	• Consider allowing students to use their chapter summary notes to complete the practice test.

UNIT 3

Unit 3 Project Wrap-Up

***Pre-Calculus 11*, page 415**

Suggested Timing

60—90 min plus individual time

Blackline Masters

Master 1 Project Rubric
BLM U3–1 Unit 3 Project Checklist
BLM U3–2 Unit 3 Project Rubric — Option 1
BLM U3–3 Unit 3 Project Rubric — Option 2
BLM U3–4 Unit 3 Project Rubric — Option 3

Mathematical Processes

- ✓ Communication (C)
- ✓ Connections (CN)
- ✓ Mental Math and Estimation (ME)
- ✓ Problem Solving (PS)
- ✓ Reasoning (R)
- ✓ Technology (T)
- ✓ Visualization (V)

General Outcome

Develop algebraic reasoning and number sense.

Specific Outcomes

AN1 Demonstrate an understanding of the absolute value of real numbers.

AN2 Solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

AN3 Solve problems that involve radical equations (limited to square roots).

AN4 Determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

AN5 Perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials or trinomials).

AN6 Solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials or trinomials).

General Outcome

Develop algebraic and graphical reasoning through the study of relations.

Specific Outcomes

RF2 Graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.

RF11 Graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).

Planning Notes

At the beginning of Unit 3, students were challenged to explore how functions and relations, including radical, rational, absolute value, and reciprocal, relate to our understanding of space and its exploration. The Project Corner boxes at the end of some sections in Chapters 5, 6, and 7 provided information and notes about space. Students were encouraged to choose at least one of the three options described on page 269 in the student resource for their final project.

Students are now asked to develop the final report, which involves choosing one of the following options:

- researching a radical equation or a formula related to space exploration or the historical contributions of an astronomer
- researching rational expressions related to space anomalies
- designing a logo for a company specializing in space tourism

Make sure students are aware of the project information provided on page 269 and the Unit 3 Project Wrap-Up on page 415.

Have students use **BLM U3–1 Unit 3 Project Checklist** to make sure that all parts of their project have been completed. As a class, brainstorm different ways students can do their presentations. You may wish to limit the time each student is allowed to present.

Students do best if they know exactly how they will be evaluated. One way to increase student motivation is to work with the class to create a rubric for the project. You may wish to use the **Master 1 Project Rubric** template and review the general holistic points within the 1 to 5 scoring levels. Discuss with students how they might achieve each of these levels in the Unit 3 project. You may wish to begin with the level 3 (acceptable) score.

Ask questions such as the following:

- What are the big ideas in the unit? (for example, understanding radical, rational, absolute value, and reciprocal functions and equations)
- Which of the big ideas are involved in the project?

- What part of the project could you complete or get partially correct to indicate that you have a basic understanding of what was learned in the three chapters? Should you get a pass mark if you can solve absolute value equations but struggle with reciprocal functions? The answer is yes. Help students to understand that completed projects and partial understandings of concepts, in which most of the concepts are started correctly, should merit a level 3 score.
- What would constitute a level 1 project? What might you start on correctly? What could be considered a significant start?
- What would be expected for a level 5 project? What should it include? Try to help students realize that a level 5 project may have a minor error or omission that does not affect the final result.
- Knowing the expectations of levels 1, 3, and 5 projects, what would be expected for a level 4? Help students to understand that this is still an honours level and therefore the work should be reflective of this. However, even an honours project may have a minor error or omission. Discuss the difference between a major conceptual error and a minor miscalculation or omission. Understanding this point will help clarify for students the expectations and differences between a pass and above-average result, and may encourage some students to work toward the highest level. Repeat the process for a level 2 project.

BLM U3–2 Unit 3 Project Rubric — Option 1, **BLM U3–3 Unit 3 Project Rubric — Option 2**, and **BLM U3–4 Unit 3 Project Rubric — Option 3** model completed rubrics for this project. Note that each one of these rubrics is one idea for completing a rubric. Your class rubrics may have more detail. Use what your students have developed or these examples to ensure that students understand the criteria for an acceptable level, as well as what would warrant either an unacceptable or an honours grading.

Assessment	Supporting Learning
Assessment *of* Learning	
Unit 3 Project This unit project gives students an opportunity to apply and demonstrate their knowledge of the following: • radical equations • rational equations • absolute value functions and equations • reciprocal functions and equations Work with students to develop assessment criteria for this project. **Master 1 Project Rubric** provides a holistic descriptor that will assist you in assessing students' work on the Unit 3 project.	• You may wish to have students use **BLM U3–1 Unit 3 Project Checklist**, which provides a list of the required components for the Unit 3 project. • Reviewing the Project Corner boxes at the end of certain sections of Chapters 5, 6, and 7 will assist students in developing appropriate data presentations. • Make sure students recognize what is expected for the minimum requirements for an acceptable project, as well as the difference between levels 5 and 4. • Clarify the expectations and the scoring with students using **Master 1 Project Rubric** or the rubrics you developed as a class. It is recommended to review the scoring rubrics at the beginning of the project and intermittently throughout the project to refresh student understanding of the project assessment.

UNIT 3

Cumulative Review and Test

***Pre-Calculus 11*, pages 416–419**

Suggested Timing
60–90 min

Materials
- grid paper
- ruler

Blackline Masters
Master 2 Centimetre Grid Paper
BLM U3–5 Unit 3 Test

Planning Notes

Have students work independently to complete the review and then compare their solutions with those of a classmate. Alternatively, you may wish to assign the cumulative review to reinforce the concepts, skills, and processes learned so far; if students encounter difficulties, provide an opportunity for them to discuss strategies with other students. Encourage them to refer to their notes, and then to the specific section in the student resource. Once they have determined a suitable strategy, have students add it to their notes. Consider having students make a list of the questions that they found difficult. They can then use the list to help them prepare for the unit test.

Meeting Student Needs

- Encourage students to sketch and label diagrams, when appropriate.

ELL

- Use a combination of visuals, descriptions, and examples to explain any unfamiliar terms.
- The description of curling in #21 may be challenging for students who are not familiar with this sport. Ask a student who has knowledge of curling to explain the given information using words and a diagram.

Enrichment

- Have students develop questions that are similar to the questions in the review, test, or chapters, or that are completely original. Have them exchange questions and answer them for further practice.

Assessment	Supporting Learning
Assessment *for* Learning	
Cumulative Review, Chapters 5 to 7 The cumulative review provides an opportunity for students to assess themselves by completing selected questions pertaining to each chapter and checking their answers against the answers in the back of the student resource.	• Have students review their notes from each chapter to identify items that they had problems with, and do the questions related to those items. Have students do at least one question that tests skills from each chapter. • Have students revisit any chapter section they are having difficulty with.
Assessment *of* Learning	
Unit 3 Test After students complete the cumulative review, you may wish to use the unit test on pages 418 and 419 as a summative assessment.	• Consider allowing students to use their graphic organizers and summary notes. • You may wish to have students complete **BLM U3–5 Unit 3 Test**, which provides a sample unit test. You may wish to use it as written or adapt it to meet the needs of your students.