Chapter 1 Sequences and Series

1.1 Arithmetic Sequences

KEY IDEAS

Concept	Definition	Examples
Sequence	• an ordered list of elements that follows a pattern or rule	$-3, -1, 1, 3, 5, 7, \ldots$ Pattern: increase by 2
Terms of a sequence	• the elements in a sequence • the first term of a sequence is t_1, the second term is t_2, the third is t_3, and so on	$t_1 = -3$, $t_2 = -1$, $t_3 = 1$, $t_4 = 3$, $t_5 = 5$, $t_6 = 7$
Common difference in an arithmetic sequence	• to form each successive term, add a *constant* called the *common difference* • to determine the common difference, d, take any two consecutive terms in the sequence and subtract the first term from the second term: $d = t_2 - t_1 = t_3 - t_2 = t_n - t_{n-1}$ • if there is a common difference, the sequence is an *arithmetic sequence*	For the above sequence, $t_2 - t_1 = d$ $\mathbf{-3} - (\mathbf{-1}) = 2$ So, $d = 2$. This relationship is the same for any two consecutive terms. So, this is an arithmetic sequence.
Terms of a general arithmetic sequence	$t_1 = t_1$ $t_2 = t_1 + d$ $t_3 = t_1 + 2d$ $t_4 = t_1 + 3d$ . . . $t_n = t_1 + (n - 1)d$	$t_1 = -3$ and $d = 2$ $t_2 = -3 + 1(\mathbf{2}) = -1$ $t_3 = -3 + 2(\mathbf{2}) = 1$ $t_4 = -3 + 3(\mathbf{2}) = 3$ . . . $t_n = -3 + (n - 1)(\mathbf{2})$
General term of an arithmetic sequence	• t_n is the *general term* • to find the value of any term in the sequence, use the formula for the general term • the formula for the general term is $t_n = t_1 + (n - 1)d$, where t_1 is the first term of the sequence n is the number of terms ($n \in \mathrm{N}$) d is the common difference t_n is the general term or nth term	To find the 126th term in the above sequence, substitute known values into the formula for the general term: $t_1 = -3$, $d = 2$, $n = 126$ $t_n = t_1 + (n - 1)d$ $t_{\mathbf{126}} = \mathbf{-3} + (\mathbf{126} - 1)(\mathbf{2})$ $t_{126} = 247$

Working Example 1: Determine a Particular Term

A construction company hires a plumber to install pipes in new homes. The plumber will be paid \$65 for the first hour of work, \$110 for two hours of work, \$155 for three hours of work, and so on.

a) State the first three terms of the sequence. Use these terms to determine the common difference, d. Then, use this information to state the next three terms of the sequence.

b) Write the general term that you could use to determine the pay for any number of hours worked.

c) What will the plumber get paid for 10 h of work?

Solution

a) The first three terms of the sequence for this problem are given: $t_1 = 65$, $t_2 = 110$, $t_3 = 155$.

> How do you know that this is an arithmetic sequence?
> How many terms do you need before you know that a sequence is arithmetic?

Determine the common difference by subtracting the first term from the second term:

$d = t_2 - $ ________

$= \mathbf{110} - $ ________

$=$ ________

> What other terms can be subtracted to find the common difference?
> Does the order of subtraction matter?

The next three terms are

$t_4 = t_3 + d =$ ________ $+$ ________ $=$ ________

$t_5 = t_3 +$ ________$(45) = \mathbf{155} +$ ________ $=$ ________

$t_6 =$ ________ $+ 3($________$) =$ ________ $+ 135 =$ ________

b) Substitute t_1 and d into the formula for the general term of an arithmetic sequence:

$t_1 = 65$ and $d =$ ________.

$t_n = t_1 + (n - 1)d$

$t_n =$ ________ $+ (n - 1)$________

$t_n = 65 +$ ________$n - 45$

> What is the expanded form of $(n - 1)45$?

$t_n = 45n +$ ________

The general term of the sequence is $t_n =$ ________________.

> What is your interpretation of "20" in this scenario? Why might a tradesperson charge an additional amount for the first hour of work?

c) For 10 h of work, the amount the plumber gets paid is the ________ term in the sequence. Determine t_{10} by using the general term found in part b).

$t_n = 45n + 20$

$t_{10} = 45(\_\_\_\_\_\_\_\_) + 20$ Substitute $n =$ ________.

$t_{10} =$ ________ $+ 20$

$t_{10} =$ ________

The value of the tenth term is ________. Therefore, for 10 h of work, the plumber will be paid $470.

Compare this method with those on pages 10–11 of *Pre-Calculus 11*.

Working Example 2: Determine the Number of Terms

A farmer decides to plant an apple orchard. She plants 24 apple trees in the first year and 15 apple trees in each subsequent year. How many years will it take to have 204 apple trees in the orchard?

Solution

This sequence is arithmetic because the terms in this sequence have a common difference of ________. Begin by listing the values you know.

common difference: $d =$ ________

first term: $t_1 =$ ________

nth term: $t_n = 204$

Solve for n by substituting the known values into the formula for the general term.

Why are you solving for n?

$t_n = t_1 + (n - 1)d$

________ = ________ $+ (n - 1)($________$)$

________ $= 24 +$ ________$n - 15$

$204 =$ ________ $+ 15n$

$195 = 15n$

________ $= n$

There are ________ terms in the sequence. Therefore, it will take 13 years to have 204 trees in the orchard.

Working Example 3: Determine t_1, t_n, and n

An amphitheatre has 25 seats in the second row and 65 seats in the seventh row. The last row has 209 seats. The numbers of seats in the rows produce an arithmetic sequence.

a) How many seats are in the first row?

b) Determine the general term, t_n, for the sequence.

c) How many rows of seats are in the amphitheatre?

Solution

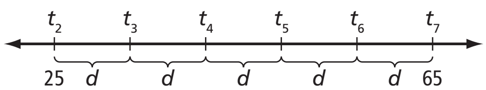
a) The number of seats in the first row is equivalent to finding the ________ term of this arithmetic sequence. To find the number of seats in the first row, subtract the common difference from t_2.

Find the common difference, d, to solve for the number of seats in any row.

Method 1: Use Logical Reasoning

Information you know: $t_2 =$ ________ and $t_7 =$ ________.

If you think about this sequence on a number line, it might help determine the number of common differences between terms.



There are 5 common differences between the second and seventh term.

So, $25 + d + d + d + d + d = 65$, or $25 +$ ________$d = 65$.

$25 +$ ________$d = 65$

________$d = 65 -$ ________

$d = \frac{\square}{5}$

$d =$ ________

Method 2: Use a System of Equations

Create a system of equations. Use the formula for the general term to write an equation for t_2 and t_7.

For $n = 2$

$t_n = t_1 + (n - 1)d$

$t_2 = t_1 + ($________$- 1)d$

________ $= t_1 - d$ ①

For $n = 7$

$t_n = t_1 + (n - 1)d$

________ $= t_1 + ($________$- 1)d$

$65 = t_1 -$ ________d ②

Solve the system of equations by subtracting the two equations: ① − ②.

$25 = t_1 - d$ ①

$65 = t_1 - 6d$ ②

________ $=$ ________d ① − ②

________ $= d$

Why does subtracting the equations help you solve for d?
Does the order of subtraction make a difference to the value of d?

Once you know d, the common difference, you can determine the number of seats in the first row.

$$t_2 = t_1 + d$$
$$\_\_\_\_\_\_\_\_ = t_1 + \_\_\_\_\_\_\_\_$$
$$\_\_\_\_\_\_\_\_ = t_1$$

Will the value of t_1 be different if you substituted the value of d into the second equation instead of the first?

Use $t_1 =$ ________ and $d =$ ________ to complete the sequence for the number of seats in each row: ________, 25, ________, ________, ________, ________, 65.

b) To find the general term of the sequence, substitute the known values, $t_1 =$ ________ and $d = 8$, in the formula for the general term.

$$t_n = t_1 + (\_\_\_\_\_\_\_\_ - 1)\_\_\_\_\_\_\_\_$$
$$t_n = \_\_\_\_\_\_\_\_ + (n - 1)\mathbf{8}$$
$$t_n = \mathbf{17} + 8n - \_\_\_\_\_\_\_\_$$
$$t_n = \_\_\_\_\_\_\_\_ + 8n$$

The general term is $t_n = 9 + 8n$.

c) The last row has 209 seats. Use the general term and solve for n to find the number of rows.

$$t_n = 9 + 8n$$
$$\_\_\_\_\_\_\_\_ = 9 + 8n$$
$$200 = 8n$$
$$\_\_\_\_\_\_\_\_ = n$$

There are 25 rows of seats in the amphitheatre.

See a similar example on pages 13–14 of *Pre-Calculus 11*.

Check Your Understanding

Practise

1. State whether each sequence is arithmetic. If it is, state the common difference and the next three terms.

   a) 9, 14, 19, 24, …

   14 − ________ = ________ and ________ − 14 = 5

   There is a common difference of ________, so this ______________ (*is* or *is not*) an arithmetic sequence.

   The next three terms:

   t_5: 24 + **5** = ________

   t_6: ________ + 5 = ________

   t_7: ________ + ________ = ________

   b) 1, 1, 2, 3, 5, …

   1 − ________ = 0 and 2 − ________ = ________

   There ______________ (*is* or *is not*) a common difference, so this ______________ (*is* or *is not*) an arithmetic sequence.

   c) 11, 7, 3, −1, …

   d) 2, 4, 8, 16, 32, …

   e) −8, −5, −2, 1, 4, …

   f) 35, 22, 9, −4, −17, …

2. Given the first term and common difference, use mental math to write the first four terms. Then, write the general term for each sequence in simplified form.

a) $t_1 = 5, d = 6$

The first four terms of the sequence are 5, ________, ________, ________.

To determine the general term, substitute the given values into $t_n = t_1 + (n - 1)d$.

$t_n =$ ________ $+ (n - 1)$ ________

$=$ ________ $+ \mathbf{6}n -$ ________

$=$ ________ $+ 6n$

The general term is $t_n =$ ______________.

b) $t_1 = 50, d = -9$

c) $t_1 = 4.5, d = -1.5$

d) $t_1 = \frac{1}{5}, d = \frac{2}{5}$

3. The general term of an arithmetic sequence is given. Find the indicated term.

a) $t_n = 7n - 3, t_1$

Why do you substitute $n = 1$ in this case?

Substitute $n = 1$ in $t_n = 7n - 3$.

b) $t_n = -2n + 5, t_8$

Substitute $n =$ ________ in $t_n = -2n + 5$.

c) $t_n = 6n - 9.5, t_{15}$

d) $t_n = -\frac{1}{7}n + \frac{2}{7}, t_{20}$

4. For each of the following arithmetic sequences, determine the values of d and t_1. Then, fill in the missing terms of the sequence.

a) ________, ________, ________, 27, 39

$d = 39 -$ ________

$=$ ________

$t_3 =$ ________ $- 12$

$=$ ________

b) ________, ________, 6, −2, ________

$d = -2 -$ ________

$=$ ________

$t_5 = t_4 + d =$ ________________

$= -10$

$t_2 = t_3 - d =$ ________________

$=$ ________

c) ________, 19.4, ________, ________, 29

How many differences are there between 19.4 and 29? Explain.

$19.4 +$ ________ $d =$ ________

$3d = 29 -$ ________

$d =$ ________

d) ________, $-\frac{1}{2}$, ________, ________, $\frac{5}{2}$

5. For each sequence, determine the position, n, of the given value.

a) $-3, -8, -13, \cdots, -58$

$t_1 =$ ________ and $d =$ ________

Substitute the known values in $t_n = t_1 + (n - 1)d$. Solve for n.

b) $-19, -13, -7, \cdots, 119$

$t_1 =$ ________ and $d =$ ________

$119 = -25 +$ ________

$n =$ ________

c) $1.8, 2.2, 2.6, \cdots, 8.6$

d) $5, 4\frac{7}{8}, 4\frac{6}{8}, \cdots, \frac{3}{8}$

Apply

6. State whether the term 89 is part of each arithmetic sequence. Justify your answers.

a) $t_n = 9 + (n - 1)5$

$89 = 9 +$ ________ $n - 5$

n is a whole number, so ________________.

b) $t_1 = -8, d = 23$

$t_n =$ ________ $+ (n - 1)$________

c) $t_n = 107 - 3n$

d) $-9, 0.8, 10.6, \ldots$

7. Determine the common difference, first term, and general term for each arithmetic sequence.

a) $t_8 = 33$ and $t_{14} = 57$

Subtract the two equations and solve for d. Then, determine t_1.

$\mathbf{33} = t_1 + (\mathbf{8} - 1)d$ ①

$\mathbf{57} =$ ________________ ②

b) $t_{10} = 50$ and $t_{27} = 152$

c) $t_5 = -20$ and $t_{18} = -59$

d) $t_7 = 3 + 5x$ and $t_{11} = 3 + 23x$

For a similar example to #7, see pages 13 and 14 of *Pre-calculus 11*.

8. At the end of the second week after opening, a new fitness club has 870 members. At the end of the seventh week, there are 1110 members. If the increase in membership is arithmetic, how many members were there in the first week?

9. Five fence posts are equally spaced between two corner posts that are 42 m apart. How far apart are the five fence posts?

10. The terms x, $0.5x + 7$, and $3x - 1$ are consecutive terms of an arithmetic sequence. Determine the value of x and state the three terms.

Explain why, for an arithmetic sequence, $t_3 - t_2 = t_2 - t_1$. How might this knowledge help you solve for x in this case?

The concepts in #10 are similar to those used in #11 on page 17 of *Pre-Calculus 11*.

11. An engineer's salary is $90 000. The company has guaranteed a raise of $5230 every year with satisfactory performance. Assuming satisfactory performance in each year, when will the engineer's salary be $168 450?

12. The sum of the first two terms of an arithmetic sequence is 15, and the sum of the next two terms is 43. Write the first four terms of the sequence.

Create and solve a series of equations to find d.

$\_\_\_\_\_\_\_\_ + (t_1 + \_\_\_\_\_\_\_\_) = 15$ ①

$(t_1 + \_\_\_\_\_\_\_\_ d) + (t_1 + \_\_\_\_\_\_\_\_ d) = 43$ ②

Once you determine d, what will substituting this value in the equation give you?

13. Consider the calendar month shown.

MONTH						
M	**T**	**W**	**T**	**F**	**S**	**S**
1	2	3	4	5	6	7
8	9	10	11	12	13	14
15	16	17	18	19	20	21
22	23	24	25	26	27	28
29	30	31				

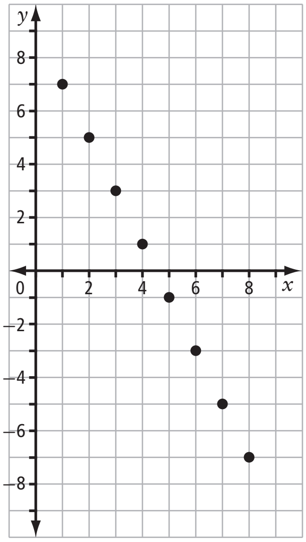
a) What type of sequence do the numbers in each row form? Justify your answer. Write the general term for the sequence.

b) What type of sequence do the numbers in each column form? Justify your answer. Write the general term for the third column.

c) Choose numbers in any diagonal sloping downward to the right. Do these numbers form a sequence? Justify your answer. If they do form a sequence, write the general term for the diagonal with $t_1 = 2$.

Connect

14. Consider the graph.



How are the terms in a sequence related to the points (1, 7), (2, ________), (3, ________), …? Why does it make sense that an arithmetic sequence would be in a straight line (linear)?

a) Do the points on the graph represent a sequence? Justify your answer, stating the sequence represented.

b) Describe how you can find the general term for this sequence. Then, use your answer to find the general term.

What is the formula for the general term of an arithmetic sequence?

c) What is t_{60}? t_{300}?

d) Describe the relationship between the slope of the graph and your formula from part b).

e) Describe the relationship between the *y*-intercept and your formula in part b).

1.2 Arithmetic Series

KEY IDEAS

Concept	Definition	Examples
Series	• the sum when you add the terms of a sequence together • for any sequence with terms $t_1, t_2, t_3, t_4, \ldots, t_n$, the associated series is $S_n = t_1 + t_2 + t_3 + t_4 + \cdots + t_n$	For the sequence −1, 1, 3, 5, 7, the series is (−1) + 1 + 3 + 5 + 7
Arithmetic series	• the sum when you add the terms of an arithmetic sequence together • write the general arithmetic series as $S_n = t_1 + (t_1 + d) + (t_1 + 2d) + (t_1 + 3d) + \cdots + [t_1 + (n - 1)d]$ or $S_n = t_1 + (t_1 + d) + (t_1 + 2d) + (t_1 + 3d) + \cdots + (t_n - d) + t_n$. • You can find the value of any term in a series by substituting known values into the formula for the general term.	For the above series, $d = 2$. To find the 126th term in the above series, substitute $d = 2$ and $n = 126$ into the general term. $t_{\mathbf{126}} = -1 + (\mathbf{126} - 1)(\mathbf{2})$ $t_{126} = 249$
Sum of an arithmetic series	• to determine the sum of an arithmetic series, use one of these formulas, depending on what is known – if you know n, t_1, and d, use $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$, where t_1 is the first term n is the number of terms d is the common difference S_n is the sum of the first n terms – if you know n, t_1, and t_n, use $S_n = \frac{n}{2}(t_1 + t_n)$, where t_1 is the first term n is the number of terms t_n is the nth term S_n is the sum of the first n terms	For the above series, if you know d, $S_{\mathbf{126}} = \frac{\mathbf{126}}{2}[2(\mathbf{-1}) + (\mathbf{126} - 1)\mathbf{2}]$ $S_{126} = 63[248]$ $S_{126} = 15\,624$ If you know $t_{126} = 249$, $S_{\mathbf{126}} = \frac{\mathbf{126}}{2}(\mathbf{-1} + \mathbf{249})$ $S_{126} = 63(248)$ $S_{126} = 15\,624$

Working Example 1: Determine the Sum of an Arithmetic Series

A toy car is rolling down an inclined track, picking up speed as it goes. The car travels 4 cm in the 1st second, 8 cm in the 2nd second, 12 cm in the 3rd second, and so on.

a) How far does the toy car travel in the 50th second?

b) Determine the total distance travelled by the car in 50 s.

> What assumptions are you making about the acceleration of the car in this scenario?

Solution

a) For this arithmetic sequence, $t_1 =$ ________, $d =$ ________, and $n =$ ________.

Substitute these values into the formula for the general term.

$t_n = t_1 + (n - 1)d$

$t_{50} =$ ________ + (________ − 1)________

$t_{50} = 4 +$ (________) 4

$t_{50} =$ ________

> Considering the car increases its speed by 4 cm each second, how could you use logical reasoning to arrive at this solution?

The toy car travels ________ cm in the 50th second.

b) You can use two methods to find the total distance travelled in 50 s.

Method 1: Use the Formula $S_n = \frac{n}{2}(t_1 + t_n)$

> What information do you need to use this formula?

S_{50} represents the total distance travelled by the car in 50 s.

Substitute $n =$ ________, $t_1 =$ ________, and $t_{50} =$ ________.

$S_{50} =$ ________(________ + ________)

$S_{50} = 25$(________)

$S_{50} =$ ________

Method 2: Use the Formula $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$

> What information do you need to use this formula?

Substitute $t_1 =$ ________, $d =$ ________, and $n =$ ________.

$S_{50} = \frac{\square}{2}$ [2(**4**) + (________ − 1) ________]

$S_{50} = 25$ [8 + (________)4]

$S_{50} = 25$ [8 + ________]

$S_{50} = 25$ (________)

$S_{50} =$ ________

The toy car travels ________ cm in 50 s.

> Which formula is most efficient in this case? Why?

Compare these methods with the methods shown on page 25 of *Pre-Calculus 11*.

Working Example 2: Determine the Terms of an Arithmetic Series

The sum of the first six terms of an arithmetic series is 297. The sum of the first eight terms is 500.

a) Determine the first ten terms of the series.

b) Use two different methods to find the sum to ten terms.

Solution

a) For this series, there is information provided to create two equations:

① $n = 6, S_6 =$ ________ ② $n =$ ________, $S_8 =$ ________

Substitute the given information into the formula $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$.

① $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$

297 = ________ [$2t_1$ + (________ − 1)d]

297 = 3[$2t_1$ + ________d]

________ = $2t_1 + 5d$

Why do you divide each side by 3?

② $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$

S_8 = ________[$2t_1$ + (________ − 1)d]

________ = 4[$2t_1$ + ________d]

________ = $2t_1 + 7d$

Solve the system of two equations.

$99 = 2t_1 + 5d$ ①

$125 = 2t_1 + 7d$ ②

______ = ______d

______ = d

Why do you subtract the two equations?

Substitute d into one of the equations to solve for t_1.

$99 = 2t_1 + 5$(________)

$99 = 2t_1 +$ ________

$34 = 2t_1$

________ = t_1

Since t_1 = ________ and $d = 13$, the first ten terms of the series are ________ + 30 + ________ + ________ + 69 + ________ + ________ + 108 + ________ + 134.

b) Find the sum of the first ten terms.

Method 1: Use the Formula $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$

When is it better to use this formula to find the sum of the series?

Substitute $n =$ ________, $t_1 =$ ________, and $d =$ ________.

$S_{10} = \frac{10}{2}[2(\_\_\_\_\_\_\_\_) + (\_\_\_\_\_\_\_\_ - 1)\_\_\_\_\_\_\_\_]$

$S_{10} = 5[\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_]$

$S_{10} = 5(\_\_\_\_\_\_\_\_)$

$S_{10} = 755$

Method 2: Use the Formula $S_n = \frac{n}{2}(t_1 + t_n)$

When is it better to use this formula to find the sum of the series?

Substitute $n =$ ________, $t_1 =$ ________, and $t_{10} = 128$.

$S_{10} = \frac{10}{2}(\_\_\_\_\_\_\_\_ + 134)$

$S_{10} = 5(\_\_\_\_\_\_\_\_)$

$S_{10} = 755$

Which formula do you prefer to use?

See page 26 of *Pre-Calculus 11* for similar examples.

Working Example 3: Determine an Unknown Value Given the Sum of an Arithmetic Series

Determine the indicated value given the characteristics of each arithmetic series.

a) Find t_1 given $d = 4$, $S_n = 1830$, and $n = 30$.

Substitute into the formula $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$.

$S_n = \frac{n}{2}[2t_1 + (n - 1)d]$

$1830 = \_\_\_\_\_\_\_\_[2t_1 + (\_\_\_\_\_\_\_\_ - 1)(4)]$

$\_\_\_\_\_\_\_\_ = [2t_1 + \_\_\_\_\_\_\_\_]$

$\_\_\_\_\_\_\_\_ = 2t_1$

$\_\_\_\_\_\_\_\_ = t_1$

Why is this formula the best to find t_1 in this case?

b) Find n given $t_1 = 6$, $t_n = 69$, and $S_n = 375$.

Substitute into the formula $S_n = \frac{n}{2}(t_1 + t_n)$.

Why is this formula the best to find t_1 in this case?

$S_n = \frac{n}{2}(t_1 + t_n)$

$\_\_\_\_\_\_\_\_ = \frac{n}{2}(\_\_\_\_\_\_\_\_ + \_\_\_\_\_\_\_\_)$

$\_\_\_\_\_\_\_\_ = n(\_\_\_\_\_\_\_\_)$

$\_\_\_\_\_\_\_\_ = n$

Why do you multiply each side by 2?

Check Your Understanding

Practise

1. Determine the sum of each arithmetic series.

   a) $3 + 12 + 21 + 30 + 39 + 48$

   $S_n =$ ________$(t_1 +$ ________$)$

   b) $19 + 31 + 43 + 55 + 67 + 79 + 91$

   c) $t_1 = 25$, $t_n = 73$, $n = 9$

   d) $t_1 = -20$, $t_n = -2$, $n = 10$

2. For each arithmetic series, determine S_{30}.

   a) $5 + 5.3 + 5.6 + 5.9 + \cdots$

   $t_1 =$ ________, $d =$ ________, and $n =$ ________

   Substitute in the formula $S_n = \frac{n}{2}[2t_1 + ($________$)d]$.

   $S_{\mathbf{30}} =$ ________$[2($________$) + (\mathbf{30} - 1)$________$]$

   $S_{30} = 15[$________$]$

   $S_{30} =$ ________

   b) $-21 - 15.5 - 10 - 4.5 - \cdots$

   c) $\frac{1}{9} + \frac{4}{9} + \frac{7}{9} + \frac{10}{9} + \cdots$

3. Determine the sum of each arithmetic series.

a) $t_1 = 42$, $d = -5$, and $n = 18$

b) $t_1 = -11.2$, $d = 7.8$, and $n = 23$

4. Determine the value of the first term, t_1, for each arithmetic series.

a) $d = 6$, $S_n = -256$, $n = 32$

b) $d = -2$, $S_n = -350$, $n = 25$

c) $d = 5$, $S_n = 1250$, $n = 20$

d) $d = -4$, $S_n = -345$, $n = 15$

5. For the arithmetic series, determine the value of n.

a) $t_1 = -42$, $t_n = 75$, $S_n = 330$

b) $t_1 = 4$, $t_n = 213$, $S_n = 2170$

c) $t_1 = 5$, $t_n = -190$, $S_n = -1480$

d) $t_1 = \frac{5}{3}$, $t_n = \frac{53}{3}$, $S_n = 87$

Apply

6. In a grocery store, cans of soup are stacked in a triangular display. There are 4 cans in the top row and 20 cans in the bottom row. How many cans are in the display if there are 17 rows?

7. The sum of the first nine terms of an arithmetic series is 162, and the sum of the first 12 terms is 288. Determine the first five terms of the series.

See page 26 of *Pre-Calculus 11* for similar examples.

8. Find the first five terms of the arithmetic series with $t_{12} = 35$ and $S_{20} = 610$.

$t_n = t_1 + (n - 1)d$

$\mathbf{35} = t_1 +$ ________ d

$t_1 =$ ________________

Substitute t_1 into $S_n = \frac{n}{2}[2t_1 + (n - 1)d]$. Then, solve for d.

$\mathbf{610} =$ ________$[2(35 - 11d) + ($________ $- 1)d]$

9. Determine an expression for the sum of the terms of an arithmetic series with general term $t_n = 3n - 2$.

Connect

10. Describe a method to determine the sum of all the multiples of 5 between 1 and 999. Use your method to find the sum.

> What key pieces of information do you need to answer this question? What formula can you use to find the sum?

11. A student has the choice between two summer jobs that will last 12 weeks (3 months).

- **Job A** pays $1350 per month with a monthly raise of $100.
- **Job B** pays $360 per week with a weekly raise of $5.

Assuming that the student wants to make the most money possible, which job should the student accept? Explain.

1.3 Geometric Sequences

KEY IDEAS

Concept	Definition	Examples
Geometric sequence	• a sequence in which the ratio between consecutive terms is constant • multiply the same value or variable with each term to create the next term	1, 3, 9, 27, 81, 243, … In this sequence, each term is 3 times the previous one. The next term in this sequence is 243(3) = 729.
Common ratio of a geometric sequence (r)	• the constant by which each term is multiplied • to determine r, take any two consecutive terms and divide the second term by the first term: $r = \frac{t_n}{t_{n-1}}$ that is, $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_n}{t_{n-1}}$ • can be negative or positive, but cannot be zero	For the above sequence, $r = \frac{t_2}{t_1} = \frac{\mathbf{3}}{\mathbf{1}} = 3$. This is true for any two consecutive terms in the sequence. For example, $\frac{t_6}{t_5} = \frac{\mathbf{243}}{\mathbf{81}} = 3$.
Terms of a general geometric sequence	$t_1 = t_1$ $t_2 = t_1r$ $t_3 = t_1r^2$ $t_4 = t_1r^3$ . . . $t_n = t_1r^{n-1}$	For the above sequence, $t_1 = 1$ $t_2 = 1(\mathbf{3}) = 3$ $t_3 = 1(\mathbf{3})^2 = 9$ $t_4 = 1(\mathbf{3})^3 = 27$ . . . $t_n = 1(\mathbf{3})^{n-1}$
General term of a geometric sequence	• when n is a positive integer, $t_n = t_1r^{n-1}$, where t_1 is the first term of the sequence n is the number of terms r is the common ratio t_n is the general term or nth term • use the general term to find the value of any term in the sequence	The 15th term in the above sequence is $t_{\mathbf{15}} = \mathbf{1}(\mathbf{3})^{\mathbf{15}-1}$ $t_{15} = 3^{14}$ $t_{15} = 4\ 782\ 969$

Working Example 1: Determine t_1, r, and t_n

Andrew and David organize a tree planting challenge. They have people plant a tree and then sign up three more people to join the challenge. Andrew and David are the first to join the challenge, each planting one tree and each signing up three more people.

a) State the values for t_1 and r for the geometric sequence that represents this situation. Then, state the geometric sequence.

b) Determine the general term that relates the number of trees planted to the number of people who sign up for the challenge.

Solution

a) The initial planters are Andrew and David, so $t_1 = 2$. Since they each sign up three other people, $t_2 =$ ________.

Determine the common ratio, r, by dividing the second term by the first term.

$r = \frac{t_2}{t_1}$

$r =$ ________

$t_1 = 2$, $t_2 =$ ________, $t_3 =$ ________, $t_4 =$ ________, $t_5 = 162, \ldots$

b) Substitute $t_1 = 2$ and $r =$ ________ into the formula for the general term.

$t_n = t_1 r^{n-1}$

$t_n = (\mathbf{2})(\_\_\_\_\_\_\_\_)^{n-1}$

Why is it incorrect to write the general term as 6^{n-1}?

The general term of the sequence is $t_n =$ ______________.

See page 34 of *Pre-Calculus 11* for a similar example.

Working Example 2: Determine a Particular Term

A company stores 5 kg of a radioactive material. After one year, 92% of the radioactive material remains. How much radioactive material will be left after ten years? State your answer to nearest tenth of a kilogram.

Solution

Why do you use 0.92 instead of 92% for the common ratio? Why is $n = 10$?

$t_1 = 5$, $r = 0.92$, $n = 10$

The general term is $t_n = (\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_)^{n-1}$.

Substitute $n = 10$ and solve.

$t_{\mathbf{10}} = 5(\_\_\_\_\_\_\_\_)^{\square - 1}$

$t_{10} = 5(\_\_\_\_\_\_\_\_)^{\square}$

$t_{10} \approx$ ________

After ten years, approximately ________ kg of radioactive material remains.

See page 35 of *Pre-Calculus 11* for a similar example.

Working Example 3: Determine t_1 and r

In a geometric sequence, the fifth term is 1050 and the seventh term is 26 250.

a) Find the values of t_1 and r.

b) List the first four terms of the sequence.

Solution

a) $t_5 =$ ________, $t_7 =$ ________

Create an equation for each of the known terms using the general term, $t_n = t_1r^{n-1}$.

$t_n = t_1r^{n-1}$ | $t_n = t_1r^{n-1}$

$\mathbf{1050} = t_1r^{\square - 1}$ | ________ $= t_1r^{\square - 1}$

$1050 = t_1r^{\square}$ ① | ________ $= t_1r^{\square}$ ②

Divide equation ② by equation ① to eliminate t_1.

$$\frac{26\ 250}{\square} = \frac{t_1r^6}{t_1r^4}$$

________ $= r^2$

$$\sqrt{\square} = \sqrt{r^2}$$

$$\pm 5 = r$$

Why does dividing the equations help you solve for r?
Does the order of division make a difference to the value of r?

Why is there a negative solution and a positive solution?

b) Substitute $r = +5$ in ① | Substitute $r = -5$ in ①

$1050 = t_1($________$)^4$ | $1050 = t_1($________$)^4$

$1050 = t_1($________$)$ | $1050 = t_1($________$)$

$1.68 = t_1$ | $1.68 = t_1$

Why are there two possible values for r, but only one value for t_1?

Since $r = \pm 5$, there are two possible sequences.

When $r = 5$, the first four terms of the sequence are 1.68, ________, 42, ________, ….

When $r = -5$, the first four terms of the sequence are 1.68, ________, 42, ________, ….

Compare this method with the methods shown on pages 36–37 of *Pre-Calculus 11*.

Working Example 4: Apply Geometric Sequences

Listeria monocytogenes are bacteria that are sometimes found in food. It takes about 7 h for the number of these organisms to double when the temperature is 10 °C. Suppose the bacteria count in a sample of food is 100.

a) Write the first five terms of the geometric sequence that models this situation.

b) How long will it take for the bacteria count to reach 1 638 400?

Solution

a) The food sample has 100 bacteria and they double in number every 7 h.

The first five terms of the geometric sequence that models this situation are

100, 200, ________, ________, ________.

b) The number of doubling periods is $n - 1$. This is the exponent of r in the general term, $t_n = t_1r^{n-1}$.

Find the general term by substituting the known values into the general term.

$t_1 =$ ________ and $r =$ ________

$t_n = t_1r^{n-1}$

$t_n = \mathbf{100}(\_\_\_\_\_\_\_\_)^{n-1}$

Solve for $t_n = 1\ 638\ 400$.

________ $= \mathbf{100(2)}^{n-1}$

$16\ 384 = (\_\_\_\_\_\_\_\_)^{n-1}$

To find n, express 16 384 as a power with the same base as the right side.

$16\ 384 = 2^{n-1}$

$2^{\square} = 2^{n-1}$

________ $= n - 1$

The bases are equal, so for the two sides to be equivalent, the exponents must also be equal.

There are 14 doubling periods, and each is 7 h long.

14(________) = ________

It takes 98 h for the bacteria count to reach 1 638 400.

Check Your Understanding

Practise

1. State if the sequence is geometric. If it is, state the common ratio and the next three terms.
   a) 5, 10, 15, 20, …
   b) 1, 3, 9, 27, …
   c) 3, 0.3, 0.03, 0.003, …
   d) 36, 30, 24, 18, …
   e) 25, 5, 1, $\frac{1}{5}$, …
   f) −8, 4, −2, 1, …

2. Given the first term and common ratio of each geometric sequence, write the general term.
   a) $t_1 = 3$, $r = 4$

   Substitute in $t_n =$ ________$^{n-1}$

   b) $t_1 = 36$, $r = -\frac{1}{3}$
   c) $t_1 = 4.5$, $r = -1.5$
   d) $t_1 = \frac{1}{5}$, $r = -\frac{2}{5}$

3. The general term of a geometric sequence is given. Determine the indicated term.
   a) $t_n = 2(-2)^{n-1}$, t_{10}

   Substitute $n =$ ________ into t_n.

   b) $t_n = 0.5(4)^{n-1}$, t_9
   c) $t_n = -1000(-0.1)^{n-1}$, t_{11}
   d) $t_n = -(-1)^{n-1}$, t_{200}

4. Match each geometric sequence to the correct corresponding general term.

a) 2, 10, 50, … **A** $t_n = 4(-3)^{n-1}$

b) 3, −12, 48, … **B** $t_n = 5(3)^{n-1}$

c) 4, −12, 36, … **C** $t_n = 3(3)^{n-1}$

d) 5, 15, 45, … **D** $t_n = 3(-4)^{n-1}$

e) 2, −4, 8, … **E** $t_n = 2(5)^{n-1}$

f) 3, 9, 27, … **F** $t_n = 2(-2)^{n-1}$

Apply

5. Given two terms of a geometric sequence, find the general term for the sequence.

a) $t_2 = 6$ and $t_3 = -12$

Substitute the known values into the formula for the general term, $t_n = t_1r^{n-1}$.

For $n = 2$

$\mathbf{6} = t_1r^{\square - 1}$

$6 =$ ________

For $n = 3$

________ $= t_1r^{\mathbf{3} - 1}$

$-12 =$ ________

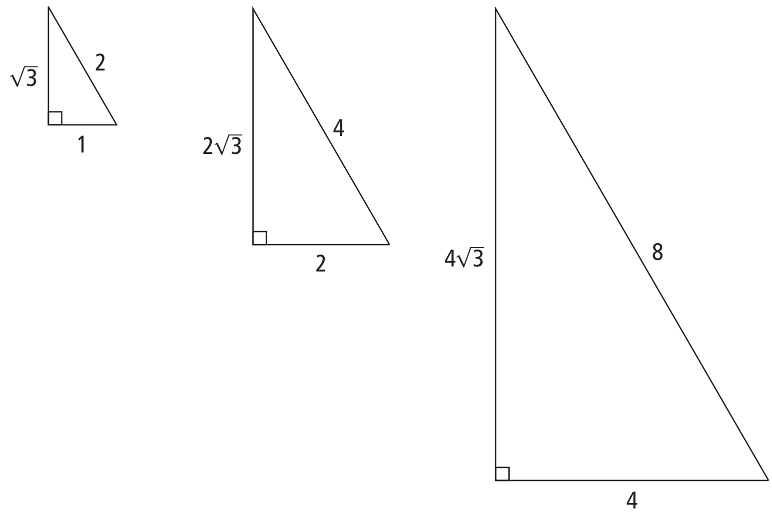
Divide the two equations to solve for r. Then, solve for t_1. Write the general term.

b) $t_2 = 4$ and $t_4 = 64$

c) $t_4 = 64$ and $t_7 = 8$

To see a similar problem, see pages 36–37 of *Pre-Calculus 11*.

6. The diagrams show the side lengths of three 30°-60°-90° triangles. Find the side lengths of the next triangle in the sequence.



7. Find the number of terms, n, in each of the following geometric sequences.

a) 4, 12, 36, …, 2916

b) 2, −4, 8, …, −1024

c) 4374, 1458, 486, …, 2

d) $\frac{1}{25}, \frac{1}{5}, 1, \ldots, 625$

8. Consider a geometric sequence with $t_3 = 18$ and $t_7 = 1458$.

a) Are there one or two values for the common ratio? Explain how this affects the sequence.

b) State the first five terms of the sequence(s).

9. Find the first four terms of a geometric sequence with $t_5 = 1536$ and $t_{10} = 48$.

10. Find the missing terms in each geometric sequence.

a) 4, ________, 16, ________

b) 2, ________, ________, 432

c) 3, ________, 12, ________

11. The population of a city increases by 6.5% each year for ten years. If the initial population is 200 000, what is the population after ten years?

12. Determine the value of x that makes the sequence 4, 8, 16, $3x + 2$, … geometric.

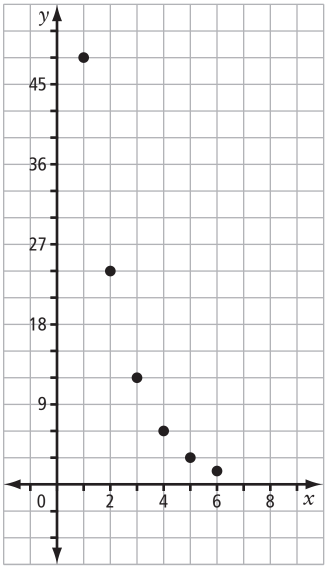
13. The value of a rare comic book is expected to follow a geometric sequence from year to year. It is presently worth \$800 and is expected to be worth \$1250 two years from now.

a) How much is the comic book expected to be worth one year from now?

b) How much is the comic book expected to be worth three years from now?

Connect

14. Consider the graph.



The points (1, 48) and (2, 24) correspond to the terms in the sequence $t_1 = 48$ and $t_2 = 24$.
How are the coordinate pairs related to the sequence?
How is the graph of a geometric sequence different from the graph of an arithmetic sequence?

a) Do the points on this graph represent a geometric sequence? If so, list the first five terms of the sequence.

b) Does a general term exist? Justify your answer. State the general term for this geometric sequence.

15. A balloon filled with helium has a volume of 20 000 cm^3. The balloon loses one tenth of its helium every 24 h.

a) What type of sequence does this situation represent? Justify your answer.

b) Describe how you can determine what volume of helium will be in the balloon at the start of the seventh day. What is the volume at this time?

1.4 Geometric Series

KEY IDEAS

Concept	Definition	Examples
Geometric series	• the sum when the terms of a geometric sequence are added together • write the general geometric series as $S_n = t_1 + t_1r + t_1r^2 + t_1r^3 + \cdots + t_1r^{n-1}$, where t_1 is the first term n is the number of terms r is the common ratio S_n is the sum of the first n terms of the series	The sequence 1, 3, 9, 27, 81, 243, … has a common ratio, r, of 3. It is a geometric sequence. When you add the terms of this sequence together, it becomes a geometric series: $1 + 3 + 9 + 27 + 81 + 243 + \cdots$ The general geometric series for this series is $S_n = 1 + 1(3) + 1(3)^2 + 1(3)^3 + \cdots + 1(3)^{n-1}$
Sum of a geometric series	• when you know t_1, r, and n, determine the sum of a geometric series using the formula $S_n = \frac{t_1(r^n - 1)}{r - 1}$, $r \neq 1$, where t_1 is the first term in the series n is the number of terms r is the common ratio S_n is the sum of the first n terms • when you know r, t_1, and t_n, use the formula $S_n = \frac{rt_n - t_1}{r - 1}$, $r \neq 1$, where t_1 is the first term n is the number of terms r is the common ratio t_n is the nth term S_n is the sum of the first n terms	$t_1 = 1$, $r = 3$, $n = 15$, $\mathbf{S_{15}} = \frac{\mathbf{1(3^{15} - 1)}}{\mathbf{3 - 1}}$, $r \neq 1$ $S_{15} = \frac{1(14\ 348\ 907 - 1)}{2}$ $S_{15} = 7\ 174\ 453$ $t_1 = 1$, $r = 3$, $t_{15} = 4\ 782\ 969$ $\mathbf{S_{15}} = \frac{\mathbf{3(4\ 782\ 969) - 1}}{\mathbf{3 - 1}}$ $S_{15} = \frac{14\ 348\ 906}{2}$ $S_{15} = 7\ 174\ 453$

Working Example 1: Determine the Sum of a Geometric Series

Determine the sum of each geometric series.

a) $12 - 6 + 3 - 1.5 + \cdots (S_{11})$

b) $2 + 8 + 32 + \cdots (S_8)$

Solution

a) For this geometric series, $t_1 =$ ________, $r =$ ________, and $n =$ ________. Substitute these values into the formula to determine the sum of the series.

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{\mathbf{11}} = \frac{\square[(\mathbf{-0.5})^{\square} - 1]}{\mathbf{-0.5} - 1}$$

What is the order of operations for evaluating the sum?

$S_{11} =$ ________________

$S_{11} \approx$ ________

The sum of the first 11 terms of the geometric series is approximately ________.

b) For this geometric series, $t_1 =$ ________, $r =$ ________, and $n =$ ________. Substitute these values into the formula for finding the sum of a series.

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{\mathbf{8}} = \frac{\mathbf{2}\left[(\square)^{\mathbf{8}} - 1\right]}{\square - 1}$$

$$S_8 = \frac{2\square}{3}$$

$S_8 =$ ________

The sum of the first eight terms of the geometric series is ________.

Working Example 2: Determine the Sum of a Geometric Series for an Unspecified Number of Terms

Determine the sum of each geometric series.

a) $10 + 5 + \frac{5}{2} + \cdots + \frac{5}{64}$

b) $1 - 3 + 9 - \cdots - 243$

Solution

a) First determine the number of terms in the series by substituting the known values in the general term of a geometric sequence.

$t_n =$ ________, $t_1 =$ ________, $r = \frac{1}{2}$.

$$t_n = t_1 r^{n-1}$$

$$\mathbf{\frac{5}{64}} = \text{\_\_\_\_\_\_\_\_}(\text{\_\_\_\_\_\_\_\_})^{n-1}$$

$$\frac{5}{640} = \left(\frac{1}{2}\right)^{n-1}$$

Why do you divide both sides by 10?

$$\frac{1}{128} = \left(\frac{1}{2}\right)^{n-1}$$

$$\left(\frac{1}{2}\right)^7 = \left(\frac{1}{2}\right)^{n-1}$$ Express the left side as a power.

When the base is the same on each side of the equal sign, what does this tell you about exponents?

________ $= n - 1$

________ $= n$

There are eight terms in the series.

Use the formula for the sum of a geometric series to find the sum of the series.

Substitute $n =$ ________, $t_1 =$ ________, $r = \frac{1}{2}$.

$$S_n = \frac{t_1(r^n - 1)}{r - 1}$$

$$S_{\mathbf{8}} = \frac{\mathbf{10}\left[\left(\square\right) - 1\right]}{\square}$$

$$S_8 = \frac{\square}{\square}$$

$S_8 =$ ________

The sum of the geometric series is ________.

Compare this method with the methods shown on pages 50–51 of *Pre-Calculus 11*.

b) Use the formula $S_n = \frac{rt_n - t_1}{r - 1}$.

$t_1 = 1$, $r =$ ________, $t_n =$ ________

$$S_n = \frac{(\square)(\square) - 1}{\square - 1}$$

$S_n =$ ________

$S_n =$ ________

The sum of the geometric series is ________.

> Compare the solutions in parts a) and b). Which formula do you prefer for the sum of a geometric series with an unspecified number of terms? Why?

Working Example 3: Apply Geometric Series

A tennis tournament has 128 players. When players win their match, they go on to play another match. If they lose their match, they are out of the tournament. What is the total number of matches that will be played in this tournament?

Solution

Since there are two players per match, the first term is ________ ÷ 2. So, $t_1 =$ ________.

After each round of matches, half the players are eliminated due to loss, so $r =$ ________.

At the end of the tournament a single match is played to decide the winner, so $t_n =$ ________.

Substitute the known values into the formula.

$$S_n = \frac{rt_n - t_1}{r - 1}$$

$$S_n = \frac{\mathbf{\frac{1}{2}}(\square) - \mathbf{64}}{(\square) - 1}$$

$$S_n = \frac{\frac{1}{2} - \frac{128}{2}}{(\square)}$$

$$S_n = \frac{-\frac{127}{2}}{-\frac{1}{2}}$$

$S_n =$ ______________

There will be ________ matches in the tournament.

Compare this working example with the example on page 52 of *Pre-Calculus 11*.

Check Your Understanding

Practise

1. State whether each series is geometric. If it is, state the common ratio and the sum of the first seven terms of the series.

   a) $6 + 18 + 54 + \cdots$

   b) $8 - 24 + 72 - \cdots$

   c) $3 - 9 + 18 - 54 + \cdots$

   d) $\frac{1}{3} + \frac{1}{6} + \frac{1}{12} + \cdots$

2. Determine the indicated sum for each geometric series.

   a) $2 + 6 + 18 + \cdots (S_8)$

   $t_1 =$ ________ and $r =$ ________

   b) $2.1 - 4.2 + 8.4 - \cdots (S_9)$

   c) $30 - 5 + \frac{5}{6} - \cdots (S_7)$

   d) $24 - 18 + \frac{27}{2} - \cdots (S_6)$

3. What is S_n for each geometric series?

   a) $t_1 = 2, r = -2, n = 12$

   b) $t_1 = 2700, r = 10, n = 8$

4. Calculate the sum of each geometric series.

a) $1 + 6 + 36 + \cdots + 279\,936$

b) $1200 + 120 + 12 + \cdots + 0.0012$

c) $-6 + 24 - 96 + \cdots + 98\,304$

d) $\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \cdots + \frac{128}{6571}$

5. Determine S_n for each geometric series with the given general term t_n.

a) $t_n = 2(3)^{n-1}$

$t_1 =$ ________, $r =$ ________

b) $t_n = 5(2)^{n-1}$

c) $t_n = 3(4)^{n-1}$

d) $t_n = 4(2)^{n-1}$

6. Determine the number of terms in each geometric series. Then, find the sum of the series.

a) $3 + 6 + 12 + \cdots + 3072$

$t_n =$ ________, $t_1 =$ ________, $r =$ ________

Substitute in $t_n = t_1r^{n-1}$.

Then, substitute in $S_n = \dfrac{rt_n - t_1}{r - 1}$.

$S_{11} =$ ________________

b) $3.4 - 7.14 + 14.994 + \cdots - 138.859$

Apply

7. A large company has a strategy for contacting employees in case of an emergency shutdown. When an emergency occurs, each of the five senior managers calls three employees. Then, each of these employees calls another three employees, and so on. This is sometimes referred to as a phone tree. If the tree consists of exactly seven levels, how many employees does the company have?

8. In a geometric series, $t_1 = 12$, $t_3 = 92$, and the sum of all of the terms of the series is 62 813. How many terms are in the series?

9. On the first swing, a pendulum swings through an arc of 40 cm. On each successive swing, the length of the arc is 0.98 times the previous length. In the first 20 swings, what is the total distance that the lower end of the pendulum swings, to the nearest hundredth of a centimetre?

10. A doctor prescribes 200 mg of medication on the first day of treatment. The dosage is reduced by one half on each successive day for one week. What is the total amount of medication prescribed, to the nearest milligram?

11. Every person has two parents, four grandparents, eight great-grandparents, and so on. Determine the number of ancestors a person has back through ten generations.

12. When you shut off a circular saw, it continues to turn. Each second after shut-off, the speed of the blade is $\frac{2}{3}$ of the speed in the previous second. After the first 8 s, the saw has turned 258 times. What was the speed of the saw before the motor was shut off, to the nearest tenth of a turn per second?

13. Dalla is rock climbing. She climbs 60 m up a cliff in the first hour. In each of the next four hours, she climbs 75% of the distance of the previous hour. What is the total distance Dalla climbs after five hours?

Connect

14. The second term of a geometric series is 15. The sum of the first three terms is 93. Describe how you would find the terms of the series. Use your method to determine the series.

15. The air in a hot-air balloon cools as the balloon rises. If the air is not reheated, the balloon rises more slowly every minute. Suppose that a hot-air balloon rises 50 m in the first minute. In each succeeding minute, the balloon rises 70% as far as it did in the previous minute.

a) What type of series does this situation represent? Justify your answer.

b) Describe a method to determine how far the balloon rises in n minutes. Use your method to determine how far the balloon rises in 7 minutes, to the nearest metre.

1.5 Infinite Geometric Series

KEY IDEAS

Concept	Definition	Examples
Infinite series	• a series with an unlimited, or infinite, number of terms • ∞ is the symbol for infinity; something that is infinite has no ending—it goes on forever • a *partial sum*, S_n, of an infinite series is the sum of the first n terms in the series	You can continue adding 2 to the last term of the sequence 2, 4, 6, 8, 10, 12, … forever. Adding the terms of this sequence creates an infinite series: $2 + 4 + 6 + 8 + 10 + 12 + \cdots$ $S_1 = 2, S_2 = 6, S_3 = 12, S_4 = 20$
Infinite geometric series	• a geometric series that goes on forever • to converge on something means to approach it; an infinite series is *convergent* if the sequence of partial sums approaches a fixed value—a sum • if an infinite geometric series is convergent, the value of the common ratio is a proper fraction between −1 and 1 ($-1 < r < 1$) • to find the sum of an infinite geometric series, where $-1 < r < 1$, use the formula $S_\infty = \frac{t_1}{1 - r}$, where t_1 is the first term of the series r is the common difference S_∞ represents the sum of an infinite number of terms • an infinite series is *divergent* if the sequence of partial sums does not approach a fixed value	$4 + 2 + 1 + 0.5 + 0.25 + \cdots$ Convergence test: $r = \frac{t_2}{t_1}$ $= \frac{\mathbf{2}}{\mathbf{4}}$ $= \frac{1}{2}$ Since r is between −1 and 1 the sequence is convergent. $S_\infty = \frac{\mathbf{4}}{1 - \frac{\mathbf{1}}{\mathbf{2}}}$ $= 8$ $4 + 8 + 16 + 32 + 64 + \cdots$ Convergence test: $r = \frac{t_2}{t_1}$ $= \frac{\mathbf{8}}{\mathbf{4}}$ $= 2$ r is not between −1 and 1, so this series is divergent.

Working Example 1: Sum of an Infinite Geometric Series

Decide whether each infinite geometric series is convergent or divergent. State the sum of the series, if it exists.

a) $0.5 - 1 + 2 - 4 + \cdots$

b) $12 + 3 + \frac{3}{4} + \frac{3}{16} + \cdots$

Solution

a) $t_1 =$ ________, $r =$ ________

The series is ________________ if the value of r is between −1 and 1;
(*convergent* or *divergent*)

otherwise, it is ________________.
(*convergent* or *divergent*)

In this case, $r =$ ________.

This series is ________________ and the sum ______________ exist.
(*convergent* or *divergent*) (*does* or *does not*)

b) $t_1 =$ ________, $r =$ ________

The series is ________________ because ________ $< r <$ ________.

To find the sum of the series, substitute the known values into the formula for the sum of an infinite geometric series.

$$S_\infty = \frac{t_1}{1 - r}$$

$S_\infty =$ ________

$$S_\infty = \frac{12}{\frac{3}{4}}$$

$S_\infty =$ ________

Compare with Example 1 on page 61 of *Pre-Calculus 11*.

Working Example 2: Apply the Sum of an Infinite Geometric Series

In the first month, an oil well in Manitoba produced 36 000 barrels of crude oil. Every month after that, it produced 95% of the previous month's production.

a) Write the series of terms that represents this situation.

b) If this trend continues, what will be the lifetime production of this well?

c) What assumption are you making about the well and its production? Is your assumption reasonable?

Solution

a) The series is 36 000 + [________(0.95)] + t_2(________) + ________ ⋯.

So, the series is ________ + ________ + 32 490 + ________ + ⋯.

b) This is a geometric series where t_1 = ________ and r = ________.

The series is ________________ because ________ $< r <$ ________.
(*convergent* or *divergent*)

The lifetime production of the well is equal to the sum of the series. Substitute the known values into the formula for the sum of an infinite geometric series.

$$S_\infty = \frac{t_1}{1 - r}$$

$$S_\infty = \frac{\mathbf{36\ 000}}{1 - \square}$$

$$S_\infty = \frac{36\ 000}{\square}$$

S_∞ = ________

The lifetime production of the well is ________ barrels of crude.

c) This calculation assumes that the well will always produce 95% of the previous month, not more and not less, which is probably unrealistic. At some point the company will shut down operations because it will no longer make financial sense to continue operations.

Compare with Example 2 on page 62 of *Pre-Calculus 11*.

Working Example 3: Apply the Sum of an Infinite Geometric Series to Express a Repeating Decimal as a Fraction

Apply the sum of an infinite geometric series to express the repeating decimal $2.1\overline{35}$ as a fraction.

Solution

This repeating decimal expands to $2.1\overline{35} = 2.135\ 353\ 535....$

Write the repeating decimal as an infinite series.

$$2.1\overline{35} = 2.1 + \frac{35}{1000} + \frac{35}{\boxed{}} + \frac{35}{\boxed{}} + \cdots$$

After the term 2.1, the series $\frac{35}{1000} + \frac{35}{100\ 000} + \frac{35}{10\ 000\ 000} + \cdots$ is geometric with

$t_1 =$ ________ and $r = \frac{1}{100}$. The series is convergent because ________ $< r <$ ________.

To find the sum of the infinite geometric series, substitute the known values in $S_\infty = \frac{t_1}{1 - r}$.

$$S_\infty = \frac{\boxed{}}{1 - \frac{1}{100}}$$

$S_\infty =$ ________

$S_\infty =$ ________

Add the sum of the series to 2.1.

$2.1\overline{35} = 2.1 + S_\infty$

$$2.1\overline{35} = \frac{\boxed{}}{10} + \frac{\boxed{}}{\boxed{}}$$

$$2.1\overline{35} = \frac{\boxed{}}{990} + \frac{35}{990}$$

$$2.1\overline{35} = \frac{\boxed{}}{990}$$

$2.1\overline{35} =$ ________

Therefore, $2.1\overline{35}$ is equivalent to the fraction ________.

Working Example 4: Given the Sum and Common Ratio of an Infinite Geometric Series, Find the First Term

The sum of an infinite series is 92 and the common ratio is $\frac{1}{4}$.

a) What is the value of the first term?

b) Write the first three terms of the series.

Solution

a) To solve for t_1, substitute the known values $S_\infty = 92$ and $r = \frac{1}{4}$ into the formula

$S_\infty =$ ________________.

$$\_\_\_\_\_\_\_\_ = \frac{t_1}{1 - \mathbf{\frac{1}{4}}}$$

$$92 = \frac{t_1}{\boxed{}}$$

$$92\left(\frac{3}{4}\right) = t_1$$

$$\_\_\_\_\_\_\_\_ = t_1$$

The first term is ________.

b) $t_1 =$ ________, $r = \frac{1}{4}$

The first three terms of the series are ________ + ________ + ________.

Check Your Understanding

Practise

1. Decide whether each infinite geometric series is convergent or divergent. State the sum of the series, if it exists.

   a) $-81 + 27 - 9 + 3 - \cdots$

   $t_1 =$ ________, $r =$ ________

   b) $24 + 6 + \frac{3}{2} + \frac{3}{8} + \cdots$

   c) $\frac{1}{4} - \frac{5}{16} + \frac{25}{64} - \frac{125}{256} + \cdots$

   d) $0.1 + 0.05 + 0.025 + 0.0125 + \cdots$

   e) $\frac{2}{7} + 2 + 14 + 28 + \cdots$

   f) $-5 + 5 - 5 + 5 - \cdots$

2. Determine the sum of each infinite geometric series, if it exists.

   a) $t_1 = -1$, $r = \frac{3}{4}$

   b) $t_1 = -4$, $r = \frac{4}{3}$

   c) $t_1 = -36$, $r = \frac{2}{3}$

   d) $t_1 = 144$, $r = -\frac{7}{12}$

   e) $t_1 = 8$, $r = -\sqrt{2}$

   f) $t_1 = 8$, $r = -\frac{5}{\sqrt{6}}$

3. Express each of the following repeating decimals as a fraction.

a) $0.\overline{3}$

b) $0.\overline{25}$

c) $4.\overline{51}$

d) $1.1\overline{23}$

4. Does 3.9999... = 4? Support your answer.

$3.9999 = 3 +$ ________

$0.9999 = \frac{9}{10} +$ ________ $+$ ________ $+$ ________ $\cdots$

For this infinite series, $t_1 =$ ________ and $r =$ ________.

What is the sum of this infinite series?

5. What is the sum of each infinite geometric series?

a) $2 + 2\left(\frac{3}{5}\right) + 2\left(\frac{3}{5}\right)^2 + 2\left(\frac{3}{5}\right)^3 + \cdots$

b) $3 + 3\left(-\frac{4}{7}\right) + 3\left(-\frac{4}{7}\right)^2 + 3\left(-\frac{4}{7}\right)^3 + \cdots$

Apply

6. Each of the following represents an infinite geometric series. For what values of x will each series be convergent?

a) $1 + x + x^2 + x^3 + \cdots, x \neq 0, 1$

b) $4 + 2x + x^2 + \frac{x^3}{2} + \cdots, x \neq 0, 1$

7. The length of an initial swing of a pendulum is 65 cm. Each successive swing is 0.75 times the length of the previous swing. If this process continues forever, how far will the pendulum swing?

$t_1 =$ ________, $r =$ ________

$$S_\infty = \frac{65}{1 - \square}$$

8. A rubber ball is dropped from a height of 32 m. Each time it bounces it rebounds to 60% of its previous height. If the ball bounces indefinitely, what is the total vertical distance it travels?

$t_1 =$ ________, $r =$ ________

$$S_\infty = \frac{\square}{1 - 0.60}$$

Connect

9. The sum of an infinite series is 105 and the common ratio is $-\frac{2}{3}$.

a) Explain how you can use the given information to find the first term. Justify your answer.

b) Write the first three terms of the series.

10. The first term of an infinite geometric series is −18, and its sum is −45.

a) Explain how you can use the given information to find the common ratio. Justify your answer.

b) Write the first four terms of the series.

Chapter 1 Review

1.1 Arithmetic Sequences, pages 1–13

1. Identify which of the following are arithmetic sequences. For each arithmetic sequence, state the common difference and find the general term.

a) 1, 5, 10, 15, …

b) $2, 2\frac{1}{2}, 3, 3\frac{1}{2}, ...$

c) 1, 4, 9, 16, …

d) $-2x^2, -5x^2, -8x^2, -11x^2, \ldots$

2. State the first five terms of the arithmetic sequence with the given t_1 and d.

a) $t_1 = 1, d = -4$

b) $t_1 = -6, d = 6$

c) $t_1 = 5m, d = 3$

d) $t_1 = c + 1, d = c - 2$

3. Determine the first term and common difference of the arithmetic sequence with the given terms.

a) $t_5 = 16$ and $t_8 = 25$

b) $t_{50} = 140$ and $t_{70} = 180$

c) $t_2 = -12$ and $t_5 = 9$

d) $t_7 = 37$ and $t_{10} = 22$

4. Determine the number of terms in the arithmetic sequences.

a) 3, 5, 7, …, 129

b) −1, 2, 5, …, 164

c) −29, −24, −19, …, 126

d) $p + 3q, p + 7q, p + 11q, \ldots, p + 111q$

5. A museum purchases a painting for \$15 000. The painting increases in value each year by 10% of the original price. What is the value of the painting after ten years?

1.2 Arithmetic Series, pages 14–21

6. Find the sum of the first ten terms of each arithmetic series.

a) $2 + 8 + 14 + \cdots$

b) $6 + 18 + 30 + \cdots$

c) $45 + 39 + 33 + \cdots$

d) $6 + 13 + 20 + \cdots$

7. Find the sum of each arithmetic series.

a) $2 + 7 + 12 + \cdots + 92$

b) $3 + 5.5 + 8 + \cdots + 133$

c) $20 + 14 + 8 + \cdots - 70$

d) $100 + 90 + 80 + \cdots - 100$

8. Find the first five terms of an arithmetic series with $S_{10} = 210$ and $S_{20} = 820$.

9. $S_n = -441$ for the series $19 + 15 + 11 + \cdots + t_n$. Determine the value of n.

1.3 Geometric Sequences, pages 22–31

10. Find the missing terms in each geometric sequence.

a) ________, 5, ________, 125

b) 3, ________, ________, 375

11. Find the number of terms, n, in each of the geometric sequences.

a) 3, 6, 12, …, 1536

b) −409.6, 102.4, −25.6, …, 0.025

c) $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots, \frac{1}{2048}$

d) $\frac{2}{81}, \frac{4}{27}, \frac{8}{9}, \ldots, 6912$

12. In a geometric sequence, $t_1 = 2$ and $t_5 = 162$. Find the common ratio, r, and the terms between t_1 and t_5.

13. Given two terms of each geometric sequence, find the general term for the sequence.

a) $t_3 = 36$ and $t_4 = 108$

b) $t_3 = 99$ and $t_5 = 11$

14. Most photocopiers can reduce the size an image by a maximum of 64% of the original dimensions. How many reductions, at the maximum setting, would it take to reduce an image to less than 10% of its original dimensions?

1.4 Geometric Series, pages 32–40

15. For each geometric series, state the values of t_1 and r. Then, determine each indicated sum. Express your answers as exact values in fraction form.

a) $24 - 12 + 6 - \cdots\ (S_{10})$

b) $0.3 + 0.003 + 0.000\,03 + \cdots\ (S_{15})$

c) $8 - 8 + 8 - \cdots\ (S_{40})$

d) $1 - \frac{1}{3} + \frac{1}{9} - \cdots\ (S_{12})$

16. What is S_n for each geometric series?

a) $t_1 = 6, r = 2, n = 9$

b) $t_1 = \frac{1}{2}, r = 4, n = 8$

17. Calculate the sum of each geometric series.

a) $960 + 480 + 240 + \cdots + 15$

b) $17 - 51 + 153 - \cdots - 334\,611$

18. Determine the number of terms in each geometric series.

a) $7\,971\,615 + 5\,314\,410 + 3\,542\,940 + \cdots + 92\,160$

b) $1 + 3x^2 + 9x^4 + \cdots + 243x^{10}$

19. A sweepstakes gives away \$1 000 000, by giving away \$25 in the first week, \$75 in the second week, \$225 in the third week, and so on. How many weeks will it take to give away all of the prize money?

1.5 Infinite Geometric Series, pages 41–49

20. Decide whether each infinite geometric series is convergent or divergent. State the sum of the series, if it exists.

a) $-64 + 16 - 4 + 2 - \cdots$

b) $\frac{5}{12} - \frac{5}{6} + \frac{5}{3} - \frac{10}{3} + \cdots$

c) $6.1 + 1.22 + 0.244 + 0.0488 + \cdots$

d) $\frac{24}{5} - 12 + 30 - 75 + \cdots$

21. The sum of an infinite series is 120 and the common ratio is $-\frac{2}{5}$. State t_1 and the first three terms of the series.

22. The length of an initial swing of a pendulum is 90 cm. Each successive swing is 0.70 times the length of the previous swing. If this process continues forever, how far will the pendulum swing?

Chapter 1 Skills Organizer

Complete the missing information in the chart.

<table>
<tr><th colspan="2">Sequences</th></tr>
<tr><th>Arithmetic</th><th>Geometric</th></tr>
<tr><td>A general arithmetic sequence is …

Example:</td><td>A general geometric sequence is …

Example:</td></tr>
<tr><td>Formula for Common Difference</td><td>Formula for Common Ratio</td></tr>
<tr><td>Formula for General Term</td><td>Formula for General Term</td></tr>
<tr><th colspan="2">Series</th></tr>
<tr><th>Arithmetic</th><th>Geometric</th></tr>
<tr><td>A general arithmetic series is …

Example:</td><td>A general geometric series is …

Example:</td></tr>
<tr><td>Formula for Sum of Series</td><td>Formulas for Sum of Series</td></tr>
<tr><th colspan="2">Infinite Geometric Series</th></tr>
<tr><td colspan="2">A general infinite geometric series is …</td></tr>
<tr><td colspan="2">Condition for Convergent Geometric Series</td></tr>
<tr><td colspan="2">Formula for Sum of Series</td></tr>
</table>