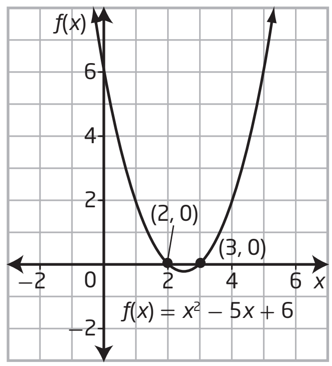
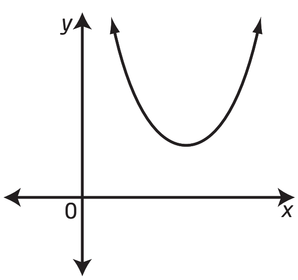
Chapter 4 Quadratic Equations

4.1 Graphical Solutions of Quadratic Equations

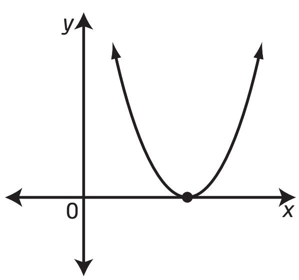
KEY IDEAS

Definition		
Term	**Description**	**Examples**
Quadratic equation	• an equation in which one of the terms is squared, and no other term is raised to a higher power: $ax^2 + bx + c = 0, a \neq 0$	$x^2 - 5x + 6 = 0$ and $-x^2 + 5x + 1 = 0$
Root(s) of a quadratic equation	• the solution(s) to a quadratic equation • correspond to the point(s) where the graph of the corresponding quadratic function intersects with the x-axis, the x-intercepts of the graph or the zeros of the quadratic function • Check the solution(s) by substituting them into the original equation.	For $x^2 - 5x + 6 = 0$, graph the corresponding function, $f(x) = x^2 - 5x + 6$.  The x-intercepts occur at (2, 0) and (3, 0), so $x = 2$ or $x = 3$ satisfies the equation $x^2 - 5x + 6 = 0$. Check: $(\mathbf{2})^2 - 5(\mathbf{2}) + 6 = 0$ ✓ $(\mathbf{3})^2 - 5(\mathbf{3}) + 6 = 0$ ✓

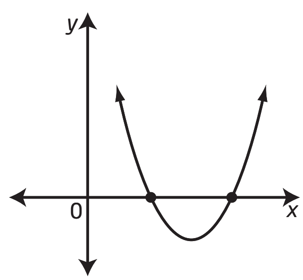
The graph of a quadratic function can have zero, one, or two x-intercepts (or zeros of the quadratic function). The zeros represent the solutions to the corresponding quadratic equation, so a quadratic equation can have zero, one, or two solutions.



No real x-intercepts
No real zeros
No real root



One real x-intercept
One real zero
One real root



Two real x-intercepts
Two real zeros
Two distinct real roots

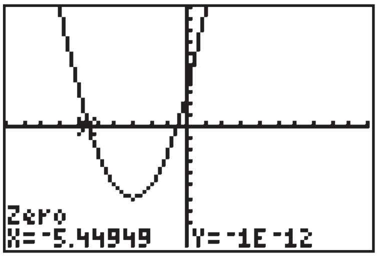
Working Example 1: Solve a Quadratic Equation Graphically

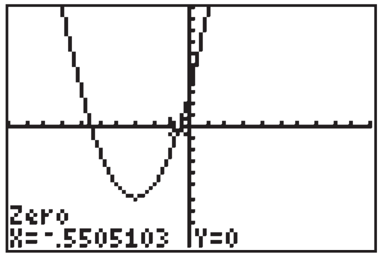
Determine the roots of each equation by graphing the corresponding function. Use either paper and pencil or technology. If necessary, round answers to the nearest tenth.

a) $x^2 + 6x + 3 = 0$ **b)** $-x^2 = 2x + 1$ **c)** $2x^2 + 22 = 12x$

Solution

a) Use technology to graph the function $f(x) = x^2 + 6x + 3$. Adjust the window settings, if necessary, so that all x-intercepts are visible. Use the trace or zero function of your technology to find the x-intercepts of the graph.





The x-intercepts of the graph correspond to the solutions to the equation. Thus, the solutions to $x^2 + 6x + 3 = 0$ expressed to the nearest tenth are $x =$ ________ and $x =$ ________.

Check by substituting your solutions in the equation, ________________.
For $x = -5.5$:

Left Side	Right Side
$x^2 + 6x + 3$	0
= (________)^2 + 6(________) + 3	
= ________ + (−32.4) + 3	
= −0.24	

Left Side ≈ Right Side

Note that the two sides are not exactly equal in this case. Why? Why is this acceptable? What could you do to be more certain that −5.4 is a solution to the equation?

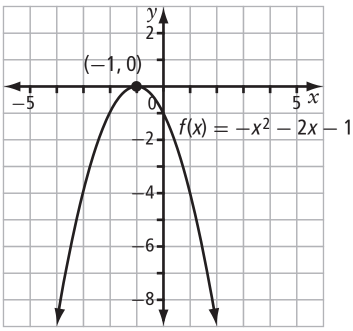
For $x = -0.6$:

Left Side	Right Side
$x^2 + 6x + 3$	0
= (________)^2 + 6(________) + 3	
= 0.36 + (________) + 3	
= −0.24	

Left Side ≈ Right Side

b) Use pencil and paper to graph the function. Begin by rewriting the equation in the form $ax^2 + bx + c = 0$.
Create a table of values. Plot the coordinate pairs and use them to sketch the graph of the function.
$-x^2 - 2x - 1 = 0$

x	$f(x)$
−4	−9
−3	−4
−2	−1
−1	0
0	−1
1	−4
2	−9



Since the graph intersects the x-axis at (−1, 0), the solution to the equation is $x = -1$.

Check by substituting $x = -1$:
$-x^2 = 2x + 1$

Left Side	Right Side
x^2 $= -(\mathbf{-1})^2$ $= -1$	$2x + 1$ $= 2(\mathbf{-1}) + 1$ $= -1$

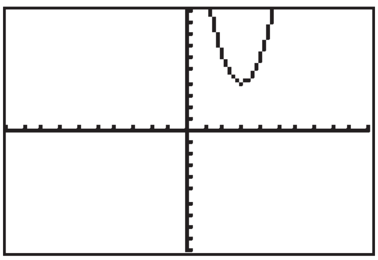
Left Side = Right Side

The solution is $x = -1$.

c) Begin by rewriting the equation in the form $ax^2 + bx + c = 0$.

Then, use technology to graph the corresponding function.

$2x^2 - 12x + 22 = 0$



Because the graph does not intersect the x-axis, you know

that the equation has ______________ solution(s).
(*one*, *two*, or *no*)

Compare this example with the Examples on pages 208–212 of *Pre-Calculus 11*.

Working Example 2: Solve a Quadratic Equation That Models a Situation

The profit, p, a particular company makes from selling its product is modelled by $p(n) = -0.1n^2 + 100n - 21\ 000$, where n is the number of products sold each month.

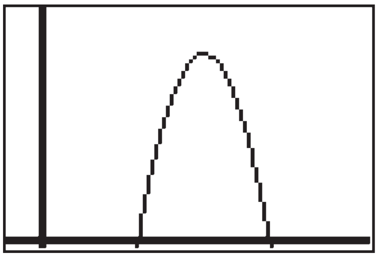
a) Graph the business's profit function. Explain why it is reasonable to model this situation with a quadratic equation.

b) A business hits a break-even point when its profits are \$0. At this point, the business neither makes nor loses money. Determine how many sales the company needs to reach this break-even point.

c) What should the business owner conclude from the solution to this quadratic equation?

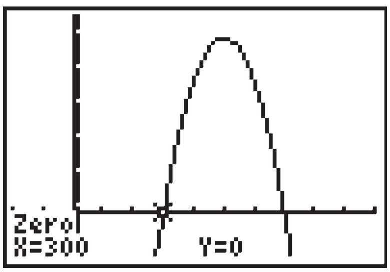
Solution

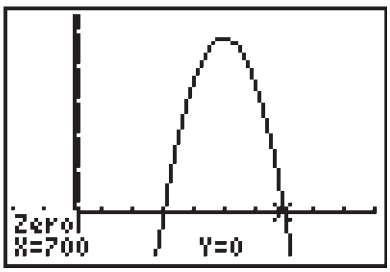
a)



You can assume that the company initially sets a price so that it makes a profit on each item. This would mean that the more product it sells, the greater the profit. However, as sales increase, the company may find that it has to hire more workers, buy more equipment, or move into a bigger facility. These expenses would cause profits to decrease.

b)





How can you check these solutions?

By using the trace or zero features of a graphing calculator, you can determine that the solutions to the equation $-0.1n^2 + 100n - 21\ 000 = 0$ are $n =$ ________ and $n =$ ________.
The business will break even if they sell 300 products or 700 products each month.

c) The business owner should conclude that the company will make a profit if it sells more than ________ products, but fewer than ________ products each month.

Assume that because of competition, the company cannot charge more for its product. Why, then, might the point at the vertex of this parabola be of special interest to the owner?

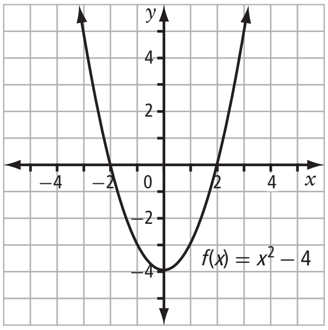
See pages 208–213 of *Pre-Calculus 11* for more examples.

Check Your Understanding

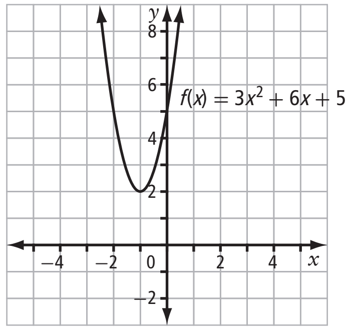
Practise

1. How many roots does each graph have? Explain how you know. State the root(s).

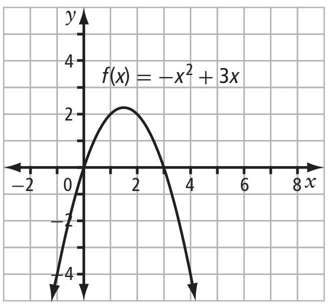
a)



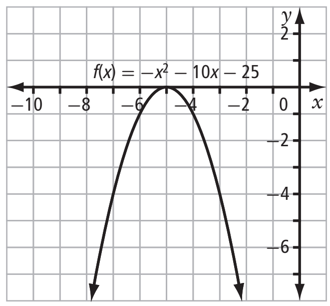
b)



c)



d)

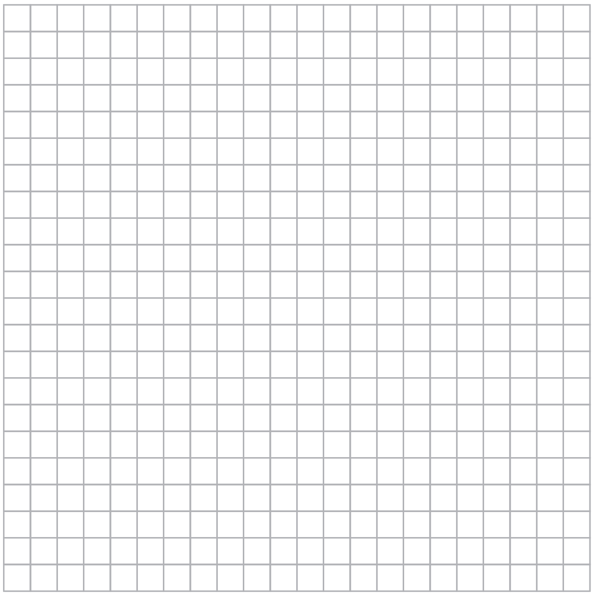


2. Solve each equation by graphing.

a) $x^2 - 9x + 8 = 0$

b) $x^2 + 14x + 45 = 0$

c) $2x^2 - 16x = 0$



3. Rewrite each quadratic equation in the form $ax^2 + bx + c = 0$. Then, use technology to solve each by graphing. Round your answers to the nearest hundredth, where necessary.

a) $3x^2 + 30 = -19x$

b) $12x^2 + 23x = 24$

c) $6x^2 = 25x - 24$

d) $-33 - 23x = 4x^2$

Apply

4. Two numbers have a sum of 12 and a product of 32. Determine the two numbers.

Step 1: Assign variables.

Let x represent one number.

The second number is $\underline{\hspace{1.5cm}} - x$.

Step 2: Set up an equation for the product.

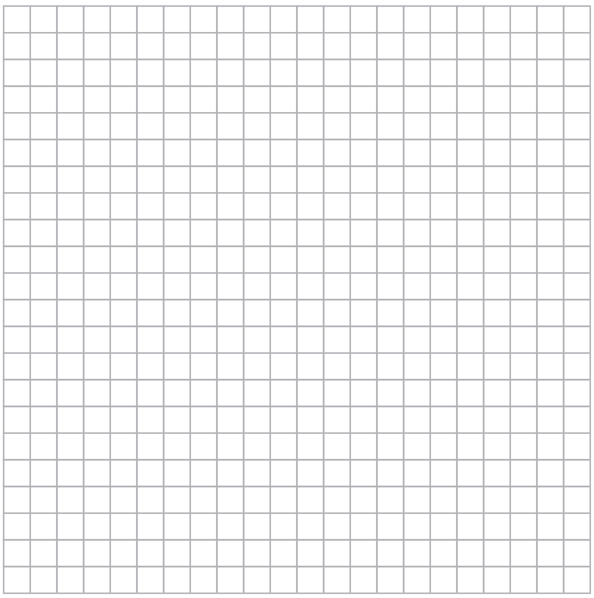
$$x(\underline{\hspace{1.5cm}}) = \underline{\hspace{1.5cm}}$$

$$x(\underline{\hspace{1.5cm}}) = 32$$

$$\underline{\hspace{1.5cm}} - x^2 = 32$$

$$\underline{\hspace{3cm}} = 0$$

Step 3: Solve for x by graphing.



The two x-intercepts are ________ and ________, so the two numbers are ________ and ________.

You can use a similar approach to solve #6 on page 215 of *Pre-Calculus 11*.

5. When a basketball is thrown, its height can be modelled by the function $h(t) = -4.9t^2 + 15t + 1$, where h is the height of the ball, in metres, and t is time, in seconds.

   a) Write a quadratic equation that you can use to determine how long the basketball is in the air.

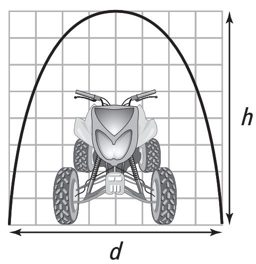
   b) Use graphing technology to solve the equation you wrote in part a). If necessary, round your answers to the nearest tenth of a second.

   What is the height of the ball when it lands on the ground?

   $t =$ ________ and $t =$ ________

   c) Do both solutions to the quadratic equation make sense in the context of this problem? Explain.

6. Madisen stores her all-terrain vehicles (ATVs) in a storage shed. The walls and roof of the shed form a parabolic shape, so the shape of the building can be represented by a quadratic function. The height of the building, h, in metres is modelled by $h = -\frac{1}{3}(d - 3)^2 + 3$, where d is the distance along the floor, in metres, from the edge of the building.



   a) Use graphing technology to determine the roots of the equation corresponding to the shape of the building.

   b) What is the width of the floor of Madisen's storage building, to the nearest tenth of a metre?

   What does the equation $-\frac{1}{3}(d - 3)^2 + 3 = 0$ represent in this situation?
   What do the roots of the equation represent?

See #9 on page 216 of *Pre-Calculus 11* for a similar question.

7. Parabolic arches are often used in architecture because they provide great strength and durability. The height, h, in metres, of a certain parabolic bridge support is modelled by $h = \frac{w^2}{9} + \frac{4v}{3}$, where w is the distance, in metres, from the side of the arch.

a) Write a quadratic equation that you can use to determine the width of the arch at the widest point.

b) Use graphing technology to graph the quadratic equation from part a). What is the width of the arch? Express your answer to the nearest tenth of a metre.

> What is the significance of the two x-intercepts of the graph, or zeros of the function?
>
> How could you use the zeros to determine the highest point of this bridge?

8. The height, h, in metres, of a BMX rider t seconds after leaving a jump can be modelled by the function $h(t) = -4.5t^2 + 8.3t + 2.1$.

a) State a quadratic equation that can be used to find the time that elapses before the rider lands.

b) Create a graph with technology to determine how long the rider remains in the air, to the nearest tenth of a second.

Connect

9. In your work with quadratics, you have used equations of the form $y = a(x - p)^2 + q$ in some places, and equations of the form $ax^2 + bx + c = 0$ in other places.

a) Are there any coefficients that are the same in both forms?

b) Describe a situation in which one of these forms of a quadratic equation is more useful than the other. Justify your answer.

10. **a)** Write a quadratic equation that has exactly one real root.

b) Explain how you know that it only has one root.

4.2 Factoring Quadratic Equations

KEY IDEAS

Definition		
Term	**Description**	**Examples**
Zero product property	• If two real numbers have a product of zero, then one or both of the factors must be equal to zero. • If the factors of a quadratic equation have a product of zero, then one or both of the factors must be equal to zero.	• $(x)(y) = 0$, so $x = 0$ or $y = 0$ • $(x + 3)(x - 9) = 0$, so $(x + 3) = 0$ or $(x - 9) = 0$

Strategies for Factoring Quadratics		
Task	**Description**	**Examples**
Solve a quadratic equation by factoring	• Three steps for solving: Step 1: Write the equation in the form $ax^2 + bx + c = 0$. Step 2: Factor the left side. Step 3: Set each factor equal to zero and solve for the unknown.	$x^2 + 12x = 27$ $x^2 + 12x - 27 = 0$ $(x + 3)(x - 9) = 0$ Either $(x + 3) = 0$ or $(x - 9) = 0$, so $x = -3$ or 9.
Factor polynomials in quadratic form	• The squared term in a quadratic function may be an expression, such as $(x + 3)^2$. To solve this type of quadratic expression, Step 1: Replace the expression with a variable. Step 2: Factor. Step 3: Substitute the expression back into the factored expression. Step 4: Simplify the final factors, if possible.	Factor $2(x + 3)^2 - 11(x + 3) + 15$. Let $r = x + 3$. Replace $x + 3$ with r: $2\mathbf{r}^2 - 11\mathbf{r} + 15$ $= 2r^2 - 5r - 6r + 15$ $= (2r^2 - 5r) + (-6r + 15)$ $= r(2r - 5) - 3(2r - 5)$ $= (2r - 5)(r - 3)$ Substitute $(x + 3)$ back in for r: $= [2(\mathbf{x + 3}) - 5][\mathbf{x + 3} - 3]$ $= (2x + 1)(x)$ $= x(2x + 1)$
Factor a difference of squares	Consider the following: $9^2 - 5^2 = 81 - 25 = 56$ $(9 - 5)(9 + 5) = (4)(14) = 56$ This is true for any two expressions: $P^2 - Q^2 = [P - Q][P + Q]$.	$25x^2 - 100y^2$ $= (5x - 10y)(5x + 10y)$

Working Example 1: Factor Quadratic Polynomials

Factor each polynomial.

a) $\frac{x^2}{16} - 25y^2$

b) $9(x - 1)^2 - 100y^2$

c) $4(x + 3)^2 + 8(x + 3) - 5$

Solution

> You can factor a difference of squares, $(ax)^2 - (by)^2$, into $(ax - by)(ax + by)$.

a) This is a difference of squares:

$\sqrt{\frac{x^2}{16}} = \frac{x}{4}$ and $\sqrt{25y^2} =$ ________

So, $\frac{x^2}{16} - 25y^2 =$ (________ + ________)(________ − ________).

> How can you verify that the factored form is correct?

b) This is a difference of squares:

$\sqrt{9(x-1)^2} = 3($________$)$ and $\sqrt{100y^2} =$ ________ y.

So, $9(x - 1)^2 - 100y^2 = (3($________$) + 10y)($________$(x - 1) -$ ________$)$.

Simplify by using the distributive property.

$(3x - 3 +$ ________$)($________ $- 10y)$.

c) This is a quadratic in $(x + 3)$. This means that you can view $(x + 3)$ as a single variable. Replace $x + 3$ in the equation with a variable, such as a: $4\boldsymbol{a}^2 + 8\boldsymbol{a} - 5$.

Determine two factors of −20 that have a sum of 8: ________ and ________.

Write the equation using these factors:

> How do you factor a quadratic where the leading coefficient is something other 1?
> Explain why you find factors of −20 in this case.

$4a^2 + 8a - 5 = 4a^2 - 2a + 10a - 5$

$= (4a^2 - 2a) + (10a - 5)$

$=$ ________$(2a - 1) + 5($________$)$

$= (2a - 1)($________$)$

Replace a with $x + 3$ and simplify.

(________ + ________)$(2(\mathbf{x + 3}) - 1)$

$= ($________$)($________$)$

For additional factoring examples, see pages 220–222 of *Pre-Calculus 11*.

Working Example 2: Solve a Quadratic Equation by Factoring

Factor to determine the roots of each quadratic equation. Verify your solutions.

a) $x^2 + 2x - 24 = 0$ **b)** $6x^2 - 11x - 35 = 0$ **c)** $x^2 = 10x - 16$

Solution

a) The factors of −24 that have a sum of 2 are ________ and ________.

$x^2 + 2x - 24 = 0$

(________)(________) = 0

Either $(x + 6) = 0$ or ________ = 0. So, $x =$ ________ or $x =$ ________.

Check your solutions.

For $x = -6$:

Left Side	Right Side
$x^2 + 2x - 24$	0
$= (-6)^2 + 2(-6) - 24$	
$= 0$	

Left Side = Right Side

For $x =$ ____:

Left Side	Right Side
$x^2 + 2x - 24$	0
= (________)^2 + 2(________) − 24	
$= 0$	

Left Side = Right Side

b) The factors of −210 that have a sum of −11 are ________ and ________.

Why do you use factors of −210?

$6x^2 - 11x - 35 = 0$

$6x^2 - 21x +$ ________ $x - 35 = 0$

________________ $(2x - 7) + 5($________$) = 0$

(________________)(________________) = 0

$2x - 7 = 0$ or ________________ = 0

$x =$ ________ $x =$ ________

Check your solutions.

For $x = \frac{7}{2}$:

Left Side	Right Side
$6x^2 - 11x - 35$	0
$= 6\left(\frac{7}{2}\right)^2 - 11\left(\frac{7}{2}\right) - 35$	
$= \frac{147}{2} - \frac{77}{2} - \frac{70}{2}$	
$= 0$	

Left Side = Right Side

For $x =$ ____:

Left Side	Right Side
$6x^2 - 11x - 35$	0
= 6(________)^2 − 11(________) − 35	
= ________ + ________ − ________	
= ________	

Left Side ________ Right Side

c) Begin by writing the equation as $x^2 -$ ________________ $= 0$. Then, factor the left-hand side.

$(x -$ ________$)(x$________________$) = 0$

(________________) $= 0$ or (________________) $= 0$

$x =$ ________ $x =$ ________

Check your solutions.

For $x =$ ____:

Left Side	Right Side
x^2	$10x - 16$
$= ($________$)^2$	$= -10($________$) - 16$
$=$ ________	$=$ ________

Left Side = Right Side

For $x =$ ____:

Left Side	Right Side
x^2	$10x - 16$
$= ($________$)^2$	$= 10($________$) - 16$
$=$ ________	$=$

Left Side = Right Side

Working Example 3: Solve a Quadratic Equation That Models a Situation

Bobbi is building a rectangular concrete patio. Her budget allows her to buy enough cement to fill an area 48 m^2 at a depth of 3 inches. She wants the length of the patio to be 2 m longer than its width.

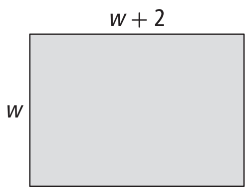
a) Sketch a diagram of this situation.

b) Write a quadratic equation that models this situation.

c) Solve the quadratic equation. What are the dimensions of the patio that Bobbi should build?

Solution

a)



b) The area of a rectangle is given by $A = wl$.

$A = wl$

________ $=$ ________(________)

c) Expand and rearrange the equation so that it equals zero.

$48 = w(w + 2)$

$0 =$ ________ $+$ ________ $- 48$

$0 = ($________$)($________$)$ Factor this equation.

________ $= 0$ or ________ $= 0$

$w =$ ________ or $w =$ ________

Why do you reject one of the solutions for the quadratic equation?

Bobbi should build a patio that is ________ m wide and ________ m long.

See pages 223–228 of *Pre-Calculus 11* for more examples.

Check Your Understanding

Practise

1. Factor completely.

a) $x^2 - 9x + 18$

b) $5b^2 - 5b - 30$

First remove the common factor.

2. Factor

a) $3n^2 - 11n - 4$

b) $4x^2 + 11x + 6$

c) $2t^2 - 17t + 30$

d) $12x^2 - x - 6$

3. Factor each quadratic equation.

a) $\frac{1}{2}x^2 - 2x - 6$

$= \frac{1}{2}(x^2 - \_\_\_\_\_\_\_\_ x - \_\_\_\_\_\_\_\_)$

$= \frac{1}{2}(\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_)$

b) $\frac{1}{4}x^2 + \frac{1}{2}x - 6$

c) $0.1a^2 - 0.1a - 3$

d) $0.5z^2 - 5.4z + 4$

4. Factor, using a difference of squares.

a) $0.81x^2 - 0.25y^2$

b) $1.21k^2 - 0.01x^2$

c) $\frac{1}{25}d^2 - \frac{1}{49}f^2$

d) $8a^2 - 18b^2$

5. Factor completely.

a) $(x + 1)^2 + 2(x + 1) - 15$

Treat $x + 1$ as a single variable, m.
Substitute m for $x + 1$.

________ + ________ − 15

Factor the resulting quadratic.

$m^2 + 2m - 15 =$ (________)(________).

Replace m with $x + 1$ and simplify.

(________)(________) = ________________

b) $(2x - 1)^2 + 16(2x - 1) + 63$

c) $2(x + 2)^2 + 3(x + 2) - 20$

d) $4(5x - 1)^2 - 12(5x - 1) + 5$

6. Solve each equation. Note that they are already factored for you.

a) $(x - 5)(x + 9) = 0$

b) $(2x + 9)(x - 4) = 0$

c) $\left(x + \frac{3}{4}\right)\left(x - \frac{11}{2}\right) = 0$

d) $x(3x - 14) = 0$

7. Solve each equation by factoring. Verify your answers.

a) $x^2 - 9x + 20 = 0$

b) $x^2 + 12x = -36$

c) $3x^2 + 11x = 42$

d) $8x^2 = 18x - 9$

e) $2x^2 + 12x = 0$

f) $\frac{1}{3}x^2 = -5x - 18$

See *Pre-Calculus 11* page 230 for more practice factoring and solving quadratic equations.

Apply

8. A rectangular picture frame has dimensions w and $w + 3$. The area of the glass in the frame is 154 in.^{2}.

a) What equation could you use to determine the width of the frame?

b) What are the dimensions of the frame, in inches?

9. Two consecutive integers have a product of 156.

a) Assign a variable for one number. Express the second number in terms of the variable.

Let x represent the first number. The second number is ________________.

b) Write a single-variable quadratic equation that represents the product of these numbers.

c) Solve the quadratic equation to determine the two numbers.

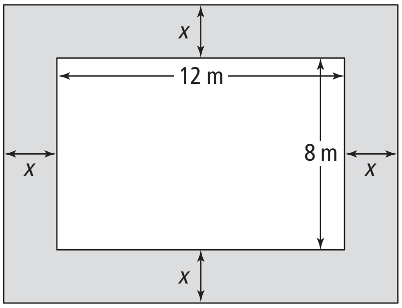
10. When a football is kicked, its height can be modelled by the function $h(d) = -0.1d^2 + 4.8d$, where d is the horizontal distance that the ball has travelled, in metres, and h is the height of the ball, in metres.

a) Write a quadratic equation that can be used to determine the distance that the ball has been kicked.

> What is the height of the ball when it lands on the ground?

b) Solve the quadratic equation to determine the distance the ball travels.

11. Aaryn has built a play area for his children that is 12 m long and 8 m wide. He wants to install a rubberized safety border around the area. This border will be the same width all the way around, as shown in the diagram. He has enough of the safety material to construct a border with a total area of 44 m^2.



a) What is the total area of the play area plus the border in square metres?

________ + 44 = ________

b) Let x be the width of the border. Write a quadratic equation to represent the total area of the play area plus the border. Then, solve your equation to determine the width of the border Aaryn can install.

See #23 on page 232 of *Pre-Calculus 11* for a similar question.

Connect

12. How can you tell when a quadratic equation of the form $ax^2 + bx + c = 0$ has 0 as one solution? Justify your answer.

13. In Section 4.2, you verified the solutions to quadratic equations by substituting the roots into the original quadratic. How can you use your work in Section 4.1 to verify your solutions in another way?

4.3 Solving Quadratic Equations by Completing the Square

KEY IDEAS

Definition	
Term	**Description**
Completing the square	The process of rewriting a quadratic polynomial from the standard form, $ax^2 + bx + c$, to the vertex form, $a(x - p)^2 + q$.

Strategy for Applying Completing the Square	Example
Step 1: If the coefficient of x^2 is not 1, check for a common factor. In the example, remove the common factor, 2.	$2x^2 - 12x - 4 = 0$ $x^2 - 6x - 2 = 0$
Step 2: Isolate the variable terms on the left side.	$x^2 - 6x = 2$
Step 3: Determine what value to add to the left side to make it a perfect square. Do this by taking half of b (the coefficient of x) and squaring it.	$\left(\frac{1}{2}(6)\right)^2 = 3^2$ $= 9$
Step 4: Add the value you calculated in step 3 to both sides of the equation.	$x^2 - 6x + 9 = 2 + 9$ $x^2 - 6x + 9 = 11$
Step 5: Factor the left side.	$(x - 3)^2 = 11$
Step 6: Take the square root of both sides.	$x - 3 = \pm\sqrt{11}$
Step 7: Solve for x. Express roots of quadratic equations as exact roots or as decimal approximations.	$x - 3 = \sqrt{11}$ $x = 3 + \sqrt{11}$ $x \approx 6.32$ or $x - 3 = -\sqrt{11}$ $x = 3 - \sqrt{11}$ $x \approx -0.32$

Working Example 1: Take Square Roots to Solve Quadratic Equations

Solve each equation.

a) $x^2 = 36$

b) $(x - 4)^2 = 64$

Solution

a) Take the square root of both sides of the equation.

$x^2 = 36$

$x = \pm\sqrt{36}$

$x = \pm$ ________

b) Take the square root of each side of the equation.

$(x - 4)^2 = 64$

$x - 4 = \pm\sqrt{64}$

________ = ________

Solve for x.

$x - 4 =$ ________ or $x - 4 =$ ________

$x =$ ________ $x =$ ________

Working Example 2: Solve Quadratic Equations by Completing the Square

Complete the square to solve each of the following. Express your answers as exact values.

a) $a^2 + 18a = -32$

b) $t^2 - 8t = 4$

Solution

How do you use the coefficient of a to find the value needed to complete the square?

a) $a^2 + 18a = -32$

$a^2 + 18a +$ ________ $= -32 +$ ________

$(a + 9)^2 =$ ________

$(a + 9) = \pm\sqrt{\square}$

________________ $= \pm 7$

Solve for a.

$a + 9 =$ ________ or $a + 9 =$ ________

$a = 7 -$ ________ $a =$ ________ $- 9$

$a =$ ________ $a =$ ________

b) $t^2 - 8t = 4$

$t^2 - 8t +$ ________ $= 4 +$ ________

$($________$)^2 = 20$

________ $= \pm\sqrt{20}$

________ $= \pm 2\sqrt{5}$

Why is $\pm\sqrt{20}$ rewritten as $\pm 2\sqrt{5}$?

Solve for t.

$t - 4 = 2\sqrt{5}$ or $t - 4 = -2\sqrt{5}$

$t = 2\sqrt{5} +$ ________ $t = -$________ $+ 4$

Working Example 3: Apply Completing the Square When $a \neq 1$

Pumpkin chunking is a competition in which competitors use various ways to throw a pumpkin as far as possible. Winning teams often fling their pumpkins more than a thousand metres. A pumpkin chunking team has determined that the height, h, in metres, of their pumpkin is modelled by $h(d) = -0.000\,25d^2 + 0.3d + 2.5$, where d is the horizontal distance, in metres. How far does the team's pumpkin travel, to the nearest tenth of a metre?

Solution

When the pumpkin hits the ground, its height is ________ m. So, at this point

$-0.000\,25d^2 + 0.3d + 2.5 =$ ________.

Begin by moving the constant to the other side of the equal sign.

$-0.000\,25d^2 + 0.3d + 2.5 - 2.5 = 0 - 2.5$

$-0.000\,25d^2 + 0.3d = -2.5$

Since the quadratic term has a coefficient other than 1, divide through by the coefficient.

$d^2 -$ ________$d = 10\,000$

$d^2 -$ ________ $+ 360\,000 = 10\,000 +$ ________

$(d -$ ________$)^2 =$ ________

$d - 600 = \pm\sqrt{\square}$

Solve for d.

$d -$ ________ $= \sqrt{370\,000}$ or $d -$ ________ $=$ ________

$d = \sqrt{370\,000} +$ ________ $d =$ ________ $+ 600$

$d \approx$ ________ $d \approx$ ________

Since a negative horizontal distance is not reasonable in this problem, the pumpkin travels a horizontal distance of ________ m.

For additional examples, see pages 236–239 of *Pre-Calculus 11*.

Check Your Understanding

Practise

1. State the value of c that makes each of the following a perfect square.

   a) $x^2 - 12x + c$

   b) $x^2 - 3x + c$

   c) $x^2 + \frac{1}{4}x + c$

   d) $x^2 + 0.8x + c$

2. Solve each of the following by completing the square. Express your answers as exact values.

   a) $x^2 - 8x = 9$

   Determine the value that must be added to make the left side a perfect square.

   $\left(\frac{1}{2}(8)\right)^2 =$ ________

   What value in the equation have you multiplied by half and then squared?

   Add this value to both sides of the equation.

   $x^2 - 8x + 16 = 9 +$ ________

   $(x -$ ________$)^2 = 25$

   Take the square root of both sides.

   $x -$ ________ $= \pm$ ________________

   So, $x -$ ________ $=$ ________ or $x -$ ________ $=$ ________.

   $x =$ ________ or $x =$ ________

   b) $x^2 + 6x = 27$

   c) $x^2 + 10x + 5 = 0$

   d) $x^2 - 2x - 39 = 0$

   e) $x^2 + 3x = \frac{15}{4}$

3. Solve each of the following. Express your answers as exact values.

a) $(x - 2)^2 = 9$

b) $(x + 5)^2 = 49$

c) $\left(x + \frac{2}{3}\right)^2 = 1$

d) $\left(x - \frac{7}{5}\right)^2 = \frac{36}{25}$

4. Solve. Express your answers as exact values.

a) $2x^2 - 12x = 110$

Remove a common factor of ________.

________$(x^2 -$ ________$) = 110$

Divide both sides of the equation by 2.

$x^2 - 6x =$ ________

Complete the square and solve for x.

b) $3x^2 - 12x = 18$

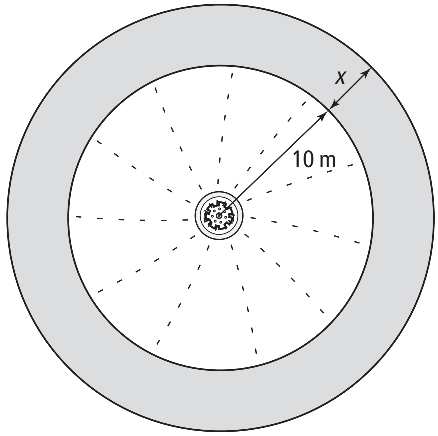
c) $0.5x^2 + 8.5 = 5x$

d) $\frac{1}{4}x^2 + x - \frac{7}{2} = 0$

Completing #3 and 4 will help you answer #6 and 7 on page 241 of *Pre-Calculus 11.*

Apply

5. A sprinkler waters a circular area of diameter 10 m. A designer wants to make a sprinkler that will spray over a larger circular area.



   a) If the radius of the new circle will be x metres more than the radius of first circle, write a quadratic expression for the area of the larger circle.

   $A = \pi r^2$

   b) If the new circle is to have an area of 150 m^2, determine the amount that the radius needs to increase, to the nearest tenth of a metre.

6. A rectangular storage room measures 6 feet by 8 feet. Aimee wants to triple the area of the room by moving each wall by the same amount.

   a) Sketch and label a diagram for this situation.

   b) Write an equation to determine the new area.

   c) Solve your equation to determine the dimensions of the new storage room. Express your answer to the nearest tenth of a foot.

Your work on this question should help you solve 8 on page 241 of *Pre-Calculus 11*.

7. Kara solved the equation $x^2 - x - 5 = 0$ as shown. Identify Kara's error. Then, write the correct solution to the quadratic as an exact value.

$$x^2 - x = 5$$
$$x^2 - x + 1 = 5 + 1$$
$$(x - 1)^2 = 6$$
$$x = \sqrt{6} + 1 \text{ or } x = -\sqrt{6} + 1$$

8. A Pythagorean triple is a set of three natural numbers that satisfy the Pythagorean Theorem. For example, the numbers 3, 4, and 5 form a Pythagorean triple because $3^2 + 4^2 = 5^2$. Another Pythagorean triple can be written as $8^2 + x^2 = (x + 2)^2$. Determine the numbers in this Pythagorean triple.

9. The height of a golf ball, in yards, is $h(d) = -0.02d^2 + 2d$, where d is the horizontal distance the ball has travelled, in yards, after being struck. Determine how far the ball travels before it first strikes the ground.

$-0.02d^2 + 2d =$ ________

How high is the ball when it hits the ground?

Connect

10. Write a quadratic equation that cannot be solved by completing the square. Justify your answer.

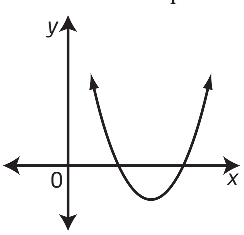
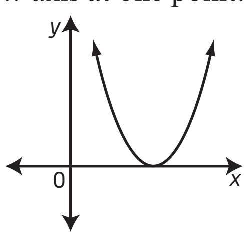
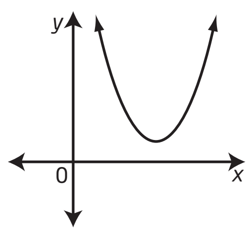
11. In Chapters 3 and 4, you have used the process of completing the square in different ways. Explain how you know when to complete the square to find solutions to quadratic equations and when to complete the square to find other properties of quadratic functions.

4.4 The Quadratic Formula

KEY IDEAS

Definition		
Term	**Description**	**Examples**
Quadratic formula	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ • You can use this formula to solve a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$.	For the quadratic equation $2x^2 - 12x - 4 = 0$, $a = 2$, $b = -12$, and $c = -4$. $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ $x = \frac{-(\mathbf{-12}) \pm \sqrt{(\mathbf{-12})^2 - 4(\mathbf{2})(\mathbf{-4})}}{2(\mathbf{2})}$ $x = \frac{12 \pm \sqrt{144 - (-32)}}{4}$ $x = \frac{12 \pm \sqrt{176}}{4}$ $x \approx 6.32$ or $x \approx -0.32$
Discriminant	• the expression under the radical sign in the quadratic formula: $b^2 - 4ac$	For the above example: $(\mathbf{-12})^2 - 4(\mathbf{2})(-4) = 144 + 32$ $= 176$

Working With the Quadratic Formula	Example
If exact values are required, use your skills for reducing radicals to write the value resulting from $\sqrt{b^2 - 4ac}$ in simplest radical form. Then, write any resulting fractions in lowest terms.	$x = \frac{12 \pm \sqrt{176}}{4}$ $x = \frac{12 \pm \sqrt{(16)(11)}}{4}$ $x = \frac{12 \pm 4\sqrt{11}}{4}$ $x = 3 \pm \sqrt{11}$

Working With the Discriminant		
If $b^2 - 4ac > 0$, there are two distinct real roots. The graph intersects the x-axis at two points.  	If $b^2 - 4ac = 0$, there is one distinct real root or two equal real roots. The graph intersects the x-axis at one point.  	If $b^2 - 4ac < 0$, there are no real roots. The graph does not intersect the x-axis. 

Working Example 1: Use the Discriminant to Determine the Nature of the Roots

Determine the nature of the roots for each quadratic equation. Check your answer.

a) $5x^2 - 7x + 4 = 0$

b) $x^2 + 8 = -6x$

c) $4x^2 = 20x - 25$

> Note that the question does not ask you to determine the roots. The question just asks you to determine how many roots there are.

Solution

a) For $5x^2 - x + 4 = 0$, $a =$ ________, $b =$ ________, and $c =$ ________.
Substitute the values of a, b, and c into the discriminant.

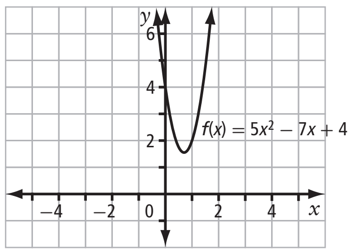
$b^2 - 4ac =$ (________)$^2 - 4$(________)(________)

$b^2 - 4ac =$ ________

Since the discriminant is ________ 0,
(=, <, or >)

the equation has ________ root(s).

Check your answer by graphing the corresponding function $f(x) = 5x^2 - 7x + 4$. You can see that there are no x-intercepts, thus there are no real roots.



b) First, rewrite the equation in the form $ax^2 + bx + c = 0$.

$$x^2 + 8 = -6x$$

$x^2 +$ ________ $+$ ________ $= 0$

So, $a =$ ________, $b =$ ________, and $c =$ ________.

Calculate the value of the discriminant.

$b^2 - 4ac =$ (________)$^2 - 4$(________)(________)

$b^2 - 4ac =$ ________

Since the value of the discriminant is ________ 0, the equation has ________ root(s).
(=, <, or >)

Check your answer by factoring.

$$x^2 + 6x + 8 = 0$$

(________)(________) $= 0$

This confirms that the equation has ________ roots. Note that you do not need to know the values of the roots in this case; this check simply confirms how many roots exist.

c) First, rewrite the equation in the form $ax^2 + bx + c = 0$.

$4x^2 - 20x + 25 = 0$

$a =$ ________, $b =$ ________, $c =$ ________

The value of the discriminant, $b^2 - 4ac$, is

$(\_\_\_\_\_\_\_\_)^2 - 4(\_\_\_\_\_\_\_\_)(\_\_\_\_\_\_\_\_) =$ ________.

Since the value of the discriminant is ______________ 0, the equation has ________ root(s).
(=, <, or >)

You can check your solution by writing the quadratic function in vertex form, $y = a(x - p)^2 + q$.

$y = 4x^2 - 20x + 25$

$y = (4x^2 - 20x) + 25$

$y = \_\_\_\_\_\_\_\_(x^2 - \_\_\_\_\_\_\_\_x) + 25$

$y = 4(x^2 - 5x + 6.25 - 6.25) + 25$

$y = 4(x - \_\_\_\_\_\_\_\_)^2 - 4(6.25) + 25$

$y =$ ________

The coordinates of the vertex are (p, q) in the vertex form. Therefore, the vertex for this parabola is at (________). Since the vertex of the parabola is on the ________-axis, you know that the graph has one real root or ______________ roots.

For additional examples about the nature of roots, see pages 246–247 of *Pre-Calculus 11*.

Working Example 2: Use the Quadratic Formula to Solve Quadratic Equations

Use the quadratic formula to solve each of the following. Express your answers as exact values.

a) $x^2 + 6x = 1$ **b)** $3x^2 = 8x - 3$

Solution

a) First, rewrite the equation in the form $ax^2 + bx + c = 0$.

$x^2 + 6x - 1 = 0$

$a =$ ________, $b =$ ________, $c =$ ________

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Substitute the values of a, b, and c in the quadratic formula.

$$x = \frac{-6 \pm \sqrt{\square^2 - 4(\square)(\square)}}{2(\square)}$$

$$x = \frac{-6 \pm \sqrt{40}}{2}$$

Write $\sqrt{40}$ as a mixed radical.

What factors of 40 are used to rewrite $\sqrt{40}$ as a mixed radical?

$$x = \frac{-6 \pm \_\_\_\_\sqrt{\_\_\_\_}}{2}$$

$x = -3 \pm$ ________ Divide through by 2.

The two roots of the equation are $x = -3 +$ ________ or $x = -3 -$ ________.

b) First, rewrite the equation in the form $ax^2 + bx + c = 0$.

____________ $= 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$$

$$x = \frac{(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$$

$$x = \frac{8 \pm \sqrt{28}}{6}$$

Reduce the radical in the expression to a mixed radical.

$$x = \frac{8 \pm \square\sqrt{7}}{\square}$$

$$x = \frac{\square \pm \sqrt{\square}}{3}$$ Divide through by 2.

So, $x =$ ____________ or $x =$ ____________.

Working Example 3: Use a Quadratic to Model a Situation

The drama club is painting a set of parabolic arches on a backdrop. To make sure that the arches are identical, they use the equation $y = 2x^2 - 3.66x + 0.93025$, where all measurements are in metres. They need to find the width of the bottom of each arch to know if the arches will properly fit on the backdrop.

a) Determine the roots of the quadratic equation. Express your answer to the nearest hundredth of a metre.

b) State the width of each arch where it touches the floor. Express your answer to the nearest hundredth of a metre.

Solution

a) $-2x^2 + 3.66x - 0.93025 = 0$

$a =$ ________, $b =$ ________, $c =$ ________

To find the roots, substitute the values of a, b, and c into the quadratic formula and solve.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(\square) \pm \sqrt{(\square)^2 - 4(\square)(\square)}}{2(\square)}$$

$$x = \frac{-3.66 \pm \sqrt{13.3956 - \square}}{-4}$$

$$x = \frac{\square \pm \sqrt{\square}}{\square}$$

$x =$ ________ or $x =$ ________

How do you make sure that your calculator performs the operations in the intended order?

b) To calculate the distance across the bottom of the arch, subtract the roots.

________ − ________ = ________

Why does subtracting the two roots result in the width of the arch at its base?

The width of each arch at its widest point, to the nearest hundredth of a metre, is ________ m.

For additional examples using the quadratic formula, see pages 248–252 of *Pre-Calculus 11*.

Check Your Understanding

Practise

1. Use the discriminant to determine the number of roots for each. Use technology to check your answers graphically.

   a) $16x^2 + 8x + 1 = 0$

   b) $2x^2 + 7x - 3 = 0$

   c) $-2x^2 + 7x - 9 = 0$

   d) $3x^2 - 6x + 3 = 0$

2. Use the quadratic formula to solve each of the following. Express your answers as exact values.

   a) $3x^2 + 2 = 6x$

   b) $6x^2 + x = 2$

   c) $9x^2 = 12x - 2$

   d) $4x^2 - 5x = 2$

3. Solve each of the following using the quadratic formula. Express your answers as approximate values, to the nearest tenth.

a) $-x^2 + 1 = -x$

b) $3x^2 + 1 = 7x$

c) $2x^2 - 17x - 30 = 0$

d) $0.25x^2 - 0.5x = 1.3$

For more practice using the quadratic formula, see page 254 of *Pre-Calculus 11*.

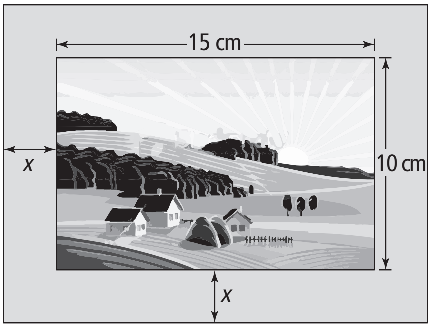
Apply

Use the quadratic formula to answer the following. For questions 4 to 8, decide whether to express your answer as an exact value or a decimal approximation.

4. Five times a number less the square of the number is equal to the negative of the number. Determine the number.

5. Two numbers have a sum of 15 and a product of 36. Determine the numbers.

6. Tee-Jay has a photo that measures 10 cm by 15 cm. He wants to fix it to a mat, so that there is a border of uniform width around the photo. The mat and photo will have an area of 500 cm^2. What should the width of the mat be?



7. A batter hits a baseball straight up. The ball's height, h, in metres, is given by $h(t) = -4.9t^2 + 21t + 1.2$, where t is time, in seconds. (Hint: Draw a sketch of this situation.)

a) How long, to the nearest hundredth of a second, will the ball be in the air before it lands on the ground?

b) How long is the ball in the air, to the nearest hundredth of a second, if a fielder catches it when it is 1 m from the ground?

c) Why do you disregard one of the roots in parts a) and b)?

8. The path of a firework that is launched from the ground can be approximated by the quadratic function $h = -0.05d^2 + 3d + 15$, where d is the horizontal distance, in metres. Determine the width of the opening of the parabola formed by the smoke trail of the firework. Express your answer to the nearest hundredth of a metre.

Connect

9. A store sells 100 calculators every year at $125 each. The revenue that these sales create is equal to the price of the calculator times the number of calculators sold. The store's manager wants to raise the price, but has learned that for every $5 price increase he can expect to sell two fewer calculators.

a) If n is the number of $5 price increases, fill in the following table. In the last cell, you will have a quadratic equation that can be used to calculate the revenue from calculator sales for any number of price increases.

	Present Price ($)	Changed Price ($)
Cost of a calculator	________	(125 + ________n)
Number sold	________	(100 − ________n)
Revenue	r = (________)(________)	$r(n)$ = (________)(________)

b) Explain the expression in each cell in the third column of the table.

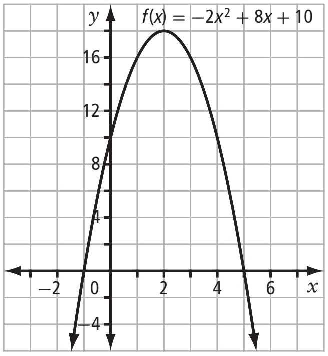
c) Explain how you would determine the range of prices at which calculators must be sold if the store needs to generate at least $14 000 annually from calculator sales. Do the calculation and state the range of prices.

Chapter 4 Review

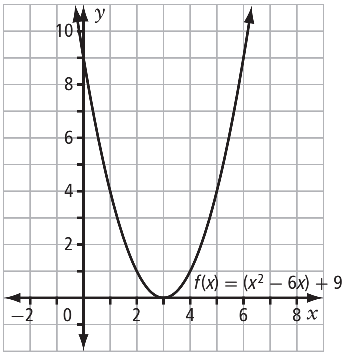
4.1 Graphical Solutions of Quadratic Equations, pages 147–155

1. Use the graph to state the roots of each equation.

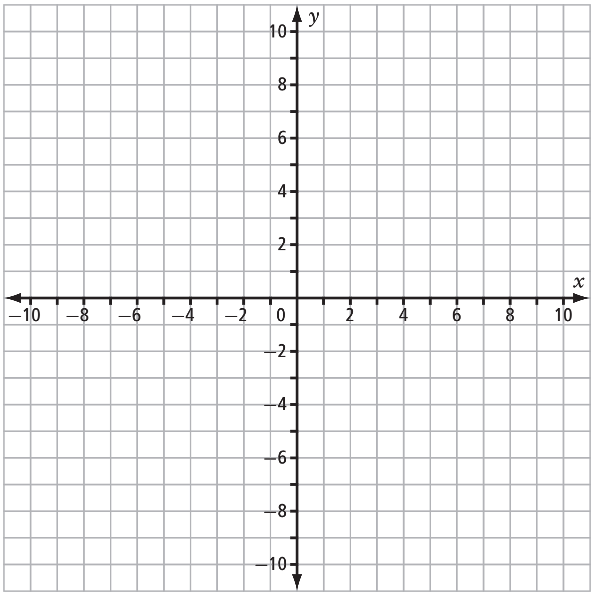
a)



b)



2. Explain which properties dictate the number of x-intercepts for each of the following. Then, sketch a sample of each type of graph on the same set of axes.



a) two distinct real roots

b) one real root

c) no real roots

3. Graph the following. From your graph, state the roots to the nearest tenth.

a) $y = x^2 - 10x + 20$

b) $y = 0.5(x - 11)^2 - 3$

c) $y = -5x^2 + 16x - 2$

4.2 Factoring Quadratic Equations, pages 156–164

4. Factor each of the following completely.

a) $(a + 5)^2 - 49(b - 9)^2$

b) $(x - 6)^2 + 10(x - 6) + 9$

c) $\frac{9m^2}{16} - \frac{100n^2}{81}$

5. Solve each of the following equations by factoring. Verify your answers.

a) $x^2 + 6x + 8 = 0$

b) $3x^2 - 5x + 2 = 0$

c) $4x^2 + 27 = 24x$

d) $36x^2 - 81 = 0$

6. One side of an envelope is 3 inches longer than the other side. The area of the envelope is 108 in.^{2}. Determine the dimensions of the envelope. (Sketch a diagram to help you with your solution.)

4.3 Solving Quadratic Equations by Completing the Square, pages 165–171

7. Solve each of the following. State your answers as exact values.

a) $x^2 = 169$

b) $(x + 7)^2 = 121$

c) $(x - 12)^2 = 80$

d) $-3(x + 1)^2 = -48$

8. Solve each of the following by completing the square. State your answers as exact values and as approximations to the nearest tenth.

a) $x^2 + 8x = 7$

b) $2x^2 - 20x + 14 = 0$

9. The profit, p, earned from the sale of a particular product by a business is given by $p(d) = -0.25d^2 + 5d + 80$, where d is the number of days the product has been for sale. Solve this equation by completing the square to determine the last day on which the product will be profitable.

4.4 The Quadratic Formula, pages 172–180

10. Use the discriminant to decide the nature of the roots for each of the following.

> What is the discriminant? What can it tell you?

a) $2x^2 + 5x = 8$

b) $x^2 = x + 12$

c) $16x^2 + 49 = -56x$

d) $7x^2 = 3x - 2$

11. Use the quadratic formula to solve each of the following. State your answers as exact values and as approximations to the nearest tenth.

a) $x^2 + 10 = 10x$

b) $5x^2 = 8 - 2x$

12. Solve each of the following using an algebraic method. Explain your choice of method.

a) $x^2 + 4x = 21$

b) $5x^2 - 13x - 6 = 0$

c) $2x^2 + 9x = -3$

Chapter 4 Skills Organizer

Complete the missing information in the graphic organizer.

To solve graphically, I … Example:	To solve by factoring, I … Example:
To determine the nature of the roots, I … Example:	To solve with the quadratic formula, I … Example:

A quadratic equation is …