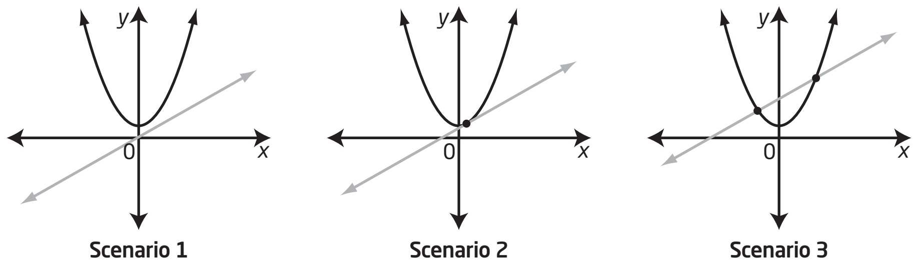
Chapter 8 Systems of Equations

8.1 Solving Systems of Equations Graphically

KEY IDEAS

- A system of equations is two or more different equations involving the same variabes.
- Determining the solution to a system of equations means determining point(s) that are common to both equations. A common point is called a point of intersection.
- Graphing a linear equation and a quadratic equation can produce one of three possible scenarios:

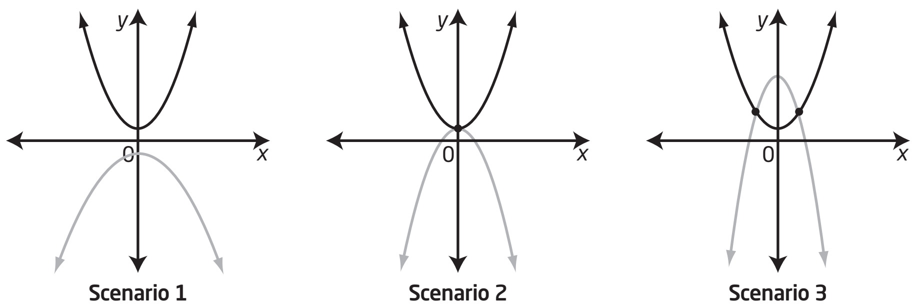


  – In Scenario 1, the line and parabola do not intersect. There is no solution.

  – In Scenario 2, the line and parabola intersect at one point. There is one solution.

  – In Scenario 3, the line and parabola intersect at two different points. There are two solutions.

- Graphing two quadratic equations can produce one of four possible scenarios. In the scenario in which the quadratic equations are the same, there are an infinite number of solutions. The other three scenarios are illustrated:



  – In Scenario 1, the parabolas do not intersect. There is no solution.

  – In Scenario 2, the parabolas intersect at one point. There is one solution.

  – In Scenario 3, the parabolas intersect at two different points. There are two solutions.

- Once you have determined the point of intersection, remember to check the ordered pair in both of the original equations.

Working Example 1: Solve a System of Linear-Quadratic Equations Graphically

a) Solve the following system of equations graphically.

$y + 3x^2 - 2x - 4 = 0$
$y + x + 2 = 0$

b) Verify the solution.

Solution

a) Isolate y in the first equation. Perform the opposite operations to both sides of the equation to isolate y.

$y + 3x^2 - 2x - 4 = 0$

Why do you have to perform inverse operations to both sides of the equation?

New equation: ______________________________

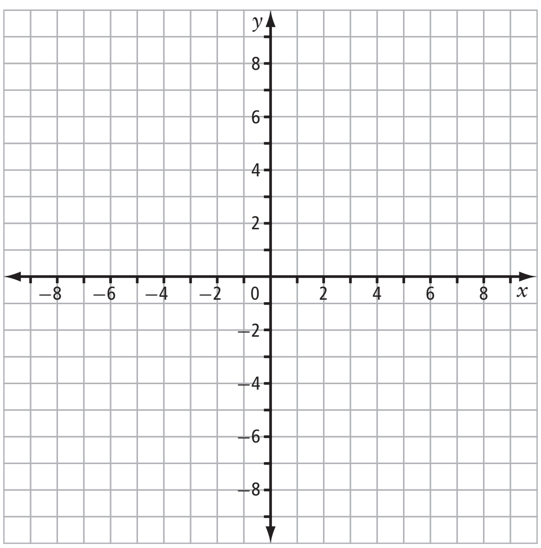
Isolate y in the second equation.

$y + x + 2 = 0$

New equation: ______________________________

Enter the equations into your graphing calculator. Use the intersect feature to determine the point(s) of intersection.

Sketch the graph of each equation and label the point(s) of intersection.



How many points of intersection are there? How many solutions are there?

b) Verify your solution by substituting the first solution into both of the original equations.

Why do you have to use the original equations to verify your solution(s)?

Left Side	Right Side
$y + 3x^2 - 2x - 4$	0

Left Side	Right Side
$y + x + 2$	0

Verify your solution by substituting the second solution into both of the original equations.

Left Side	Right Side
$y + 3x^2 - 2x - 4$	0

Left Side	Right Side
$y + x + 2$	0

Working Example 2: Solve a System of Quadratic-Quadratic Equations Graphically

a) Solve the following system of equations graphically.

$y - 3x^2 + 12x = 16$
$y + 4x^2 - 16x = -12$

b) Verify the solution.

Solution

a) Isolate y in the first equation.

$y - 3x^2 + 12x = 16$

New equation: ______________________

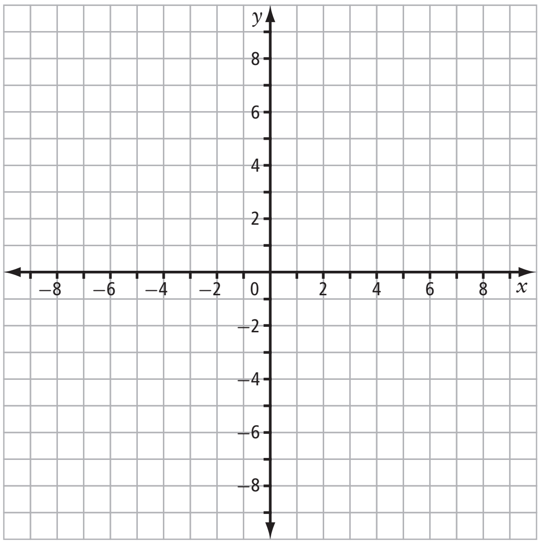
Isolate y in the second equation.

$y + 4x^2 - 16x = -12$

New equation: ______________________

Enter the equations into your graphing calculator. Use the intersect feature to determine the point(s) of intersection.

Sketch the graph of each equation and label the point(s) of intersection.



How many points of intersection are there? How many solutions are there?

b) Verify your solution by substituting the solution into both of the original equations.

Left Side	Right Side
$y - 3x^2 + 12x$	16

Left Side	Right Side
$y + 4x^2 - 16x$	−12

Working Example 3: Model a Situation Using a System of Equations

Two numbers have a sum of 17. The square of the first number plus 5 equals the second number. What are the two numbers?

Solution

Identify and define the unknowns in the scenario.

Let x represent the ____________. Let y represent the ____________.

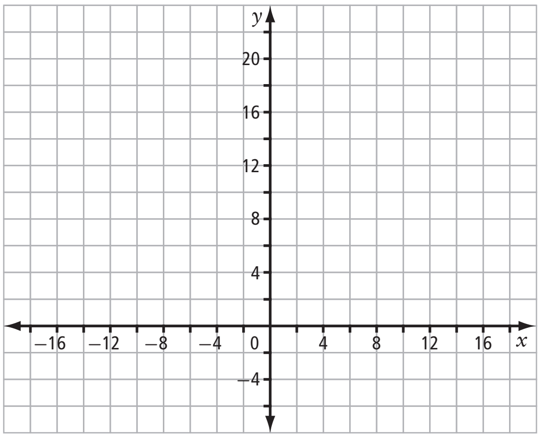
Using the variables as you have defined them, fill in the blanks to define the system of equations.

$\square + \square = 17$ The sum of the two numbers is 17.

$\square^2 + 5 = \square$ The square of the first number plus 5 equals the second number.

Isolate y in each equation.

Graph the system.



Interpret the point(s) of intersection.

The two numbers are ________ and ________.

This example should help you complete #13 on page 437 of *Pre-Calculus 11*.

Check Your Understanding

Practise

1. Check to determine which point is a solution to the given equation.

> Choose one point. Substitute the *x*-coordinate for *x* in the equation and the *y*-coordinate for *y* in the equation.

a) $y = 2x^2 + 5x - 3$; (0, −3) or (0, 3)

$y = 2x^2 + 5x - 3$

________ = 2________² + 5________ − 3

________ = 2________ + ________ − 3

________ = ________ + ________ − 3

________ = ________ − 3

________ = ________ **True** **False** (*circle one*)

$y = 2x^2 + 5x - 3$

________ = 2________² + 5________ − 3

________ = 2________ + ________ − 3

________ = ________ + ________ − 3

________ = ________ − 3

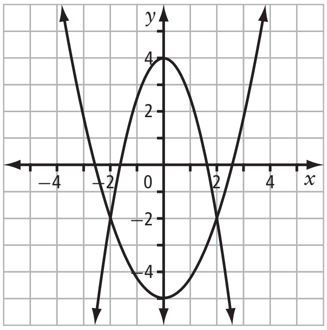
________ = ________ **True** **False** (*circle one*)

b) $y + 3x^2 - 4x = 12$; (−7, 19) or (−1, 5)

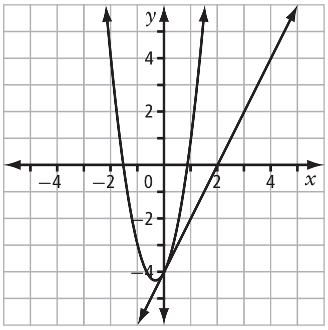
c) $2y - x^2 + 5x = 8$; (3, 1) or (−2, 11)

2. What type of system of equations is represented in each graph? Give the solution(s) to each system.

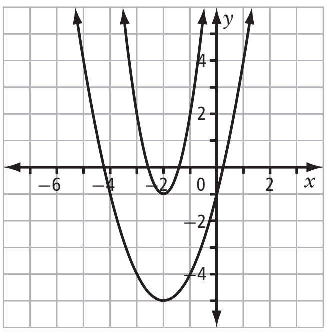
a)



b)

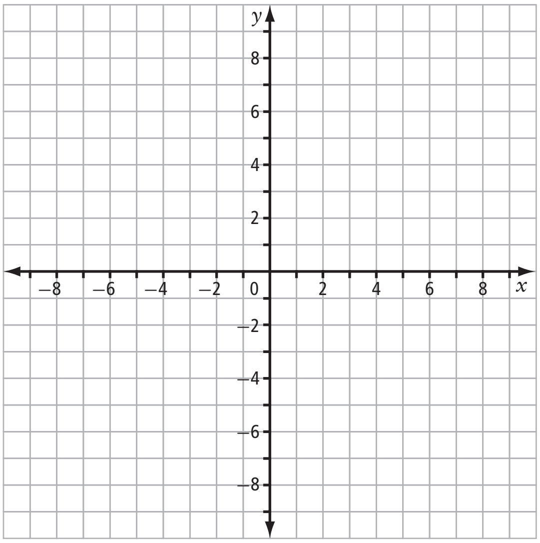


c)



3. Solve each system by graphing. Verify your solutions.

a) $2y - x^2 - 4x + 6 = 0$
$y - x - 1 = 0$



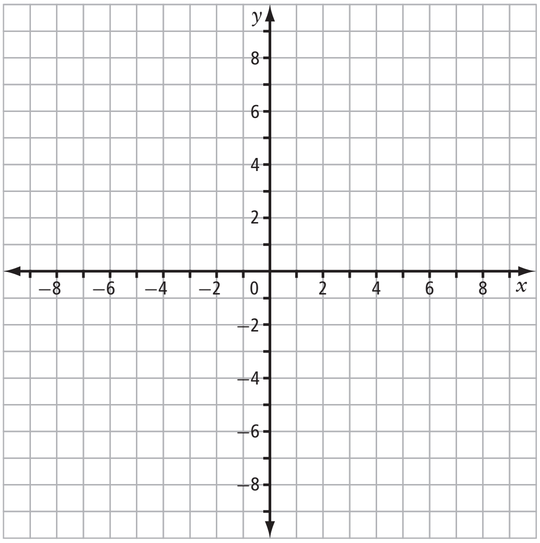
Enter the equations into your graphing calculator. Use the intersect feature to determine the point(s) of intersection. Sketch a graph of the equations and label the point(s) of intersection.

Verify by substituting any solution(s) into each of the original equations.

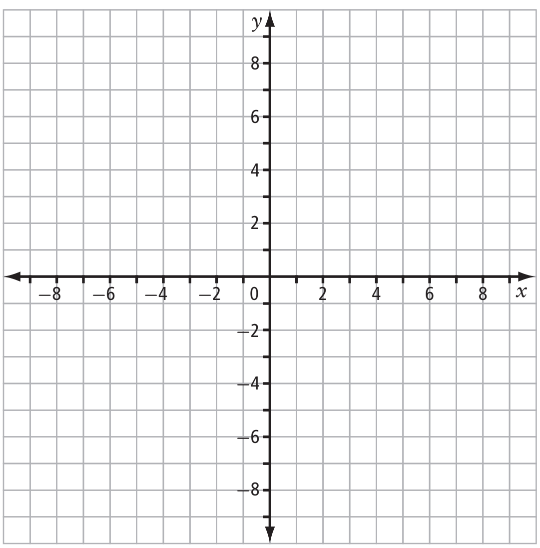
There is/are ________ points of intersection.

The solutions are ________________.

b) $y - 2x^2 + 4x - 5 = 0$
$y + x - 4 = 0$

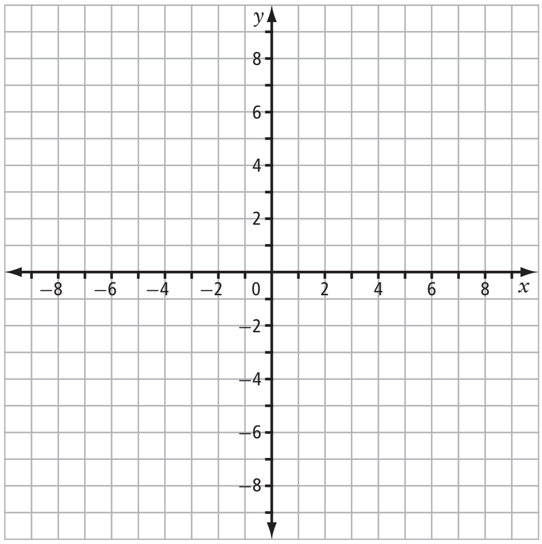


c) $y - 4x^2 - 7x + 5 = 0$
$10y - 32x + 90 = 0$



4. Solve each system by graphing. Verify your solutions.

a) $y + 4x^2 - x = 12$
$y - 3x^2 + 6x = -2$

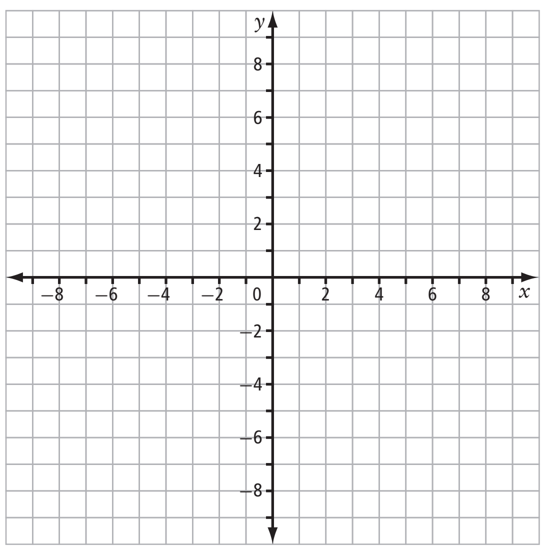


Verify by substituting any solution(s) into both of the original equations.

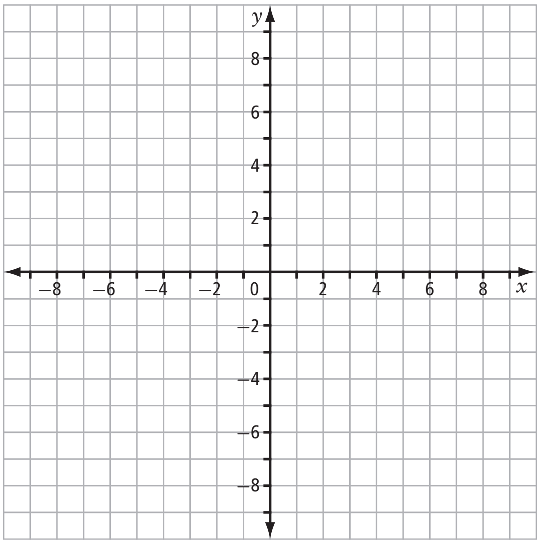
There are ________ points of intersection.

The solutions are ________________.

b) $y - 3x^2 + 24 = 30$
$y + 2x^2 + 8x = -9$

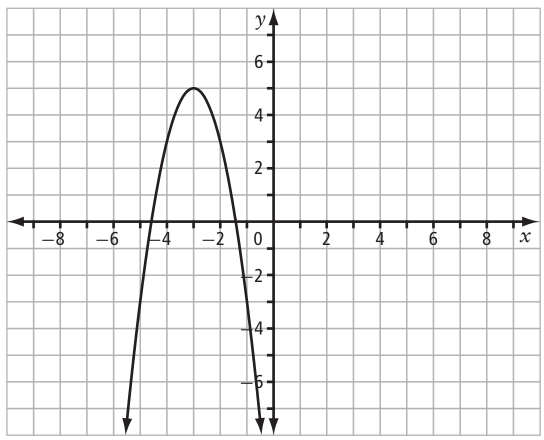


c) $y - x^2 + 6x = 13$
$y + 2x^2 - 12x = -14$



Apply

5. The graph of a quadratic function is shown.



a) Draw a line on the graph so that a system of equations with one solution is represented.

b) Choose two points on the line and determine the equation of the line.

First point: (________, ________)

Second point: (________, ________)

Determine the slope, m, of the line using the points you have selected.

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$m = \frac{\square}{\square}$$

$m =$ ________

Use the slope-intercept form to determine the equation of the line.

$y = mx + b$

$y =$ ________$x +$ ________

Recall that b is the y-intercept of the line. What is the y-intercept of the line you drew?

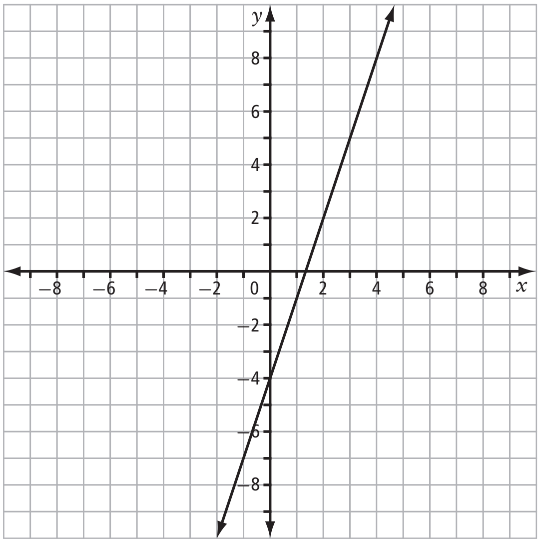
c) Determine the solution to the system of equations you created.

6. The graph of a linear function is shown.

a) Draw a parabola on the graph so that a system of equations with two solutions is represented.

b) Determine the equation of the parabola.

> Recall that the vertex form of a quadratic equation is $y = a(x - p)^2 + q$. What are the values of p and q for the parabola that you drew?



c) Determine the solution(s) to the system of equations you created.

7. Two positive numbers have a difference of 17. When the square of the smaller number is added to that number, the result is 13 less than the larger number. What are the two numbers?

Let x represent the ______________. Let y represent the ______________.

Using the variables as you have defined them, fill in the blanks.

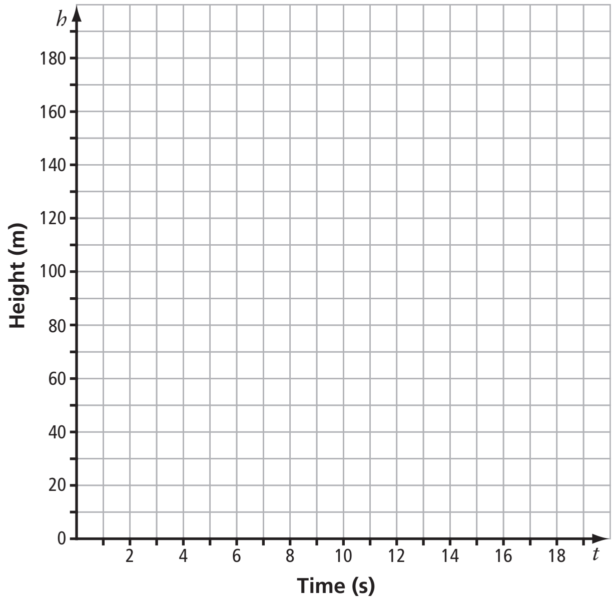
$\square - \square = 17$ The difference of the two numbers is 17.

$\square + \square^2 = \square - 13$ The sum of the smaller number and its square is 13 less than the larger number.

a) Solve the system graphically and interpret the solution(s).

b) Is there a solution that cannot be used in this question? Explain.

8. Two projectiles are launched at the same time. The height of the first projectile is modelled by the equation $h = -5t^2 + 50t - 50$. The height of the second projectile is given by $h = -5t^2 + 100t - 300$. In each equation, h represents height above ground, in metres, and t is time, in seconds, following the launch. Graph the path of both projectiles. Then, determine and interpret the point(s) of intersection.



Connect

9. Without graphing, use your knowledge of linear and quadratic equations to determine whether each system has no solution, one solution, two solutions, or an infinite number of solutions. Explain how you know.

a) $y = x^2$
$y = x - 1$

b) $y = (x - 3)^2 - 1$
$y = x^2 - 6x + 8$

c) $y = -3(x + 2)^2 + 4$
$y = 3(x + 2)^2 + 4$

d) $y = \frac{1}{2}(x - 2)^2 + 4$
$y = (x - 2)^2 + 2$

8.2 Solving Systems of Equations Algebraically

KEY IDEAS

- Algebraically, solving a system of equations in two variables means determining values of any ordered pairs (x, y) that satisfy both of the equations in the system.
- To solve a linear-quadratic system of equations, you may use either a substitution method or an elimination method. Both methods will give you the same solution.

Example

$y = x^2 - x - 6$
$y = x - 3$

Method 1: Use Substitution

Since both equations equal y, they must equal each other. Substitute one expression for y into the other equation. Then, solve for x.

$$x^2 - x - 6 = x - 3$$
$$x^2 - 2x - 3 = 0$$
$$(x - 3)(x + 1) = 0$$

Set each factor equal to zero.

$x - 3 = 0$ or $x + 1 = 0$
$x = 3$ $\quad$ $x = -1$

Substitute these values into the original linear equation to determine the corresponding values of y.

When $x = 3$:	When $x = -1$:
$y = x - 3$	$y = x - 3$
$y = \mathbf{3} - 3$	$y = \mathbf{-1} - 3$
$y = 0$	$y = -4$

The two solutions are (3, 0) and (−1, −4).

Method 2: Use Elimination

Since both equations equal y, the second equation can be subtracted from the first equation to eliminate y. Then, solve for x.

Align the terms with the same degree and subtract the second equation from the first.

$y = x^2 - x - 6$
$y = \quad\quad x - 3$

$0 = x^2 - 2x - 3$

Then, solve the equation $x^2 - 2x - 3 = 0$ by factoring, as in the substitution method at left, to obtain the two solutions, (3, 0) and (−1, −4).

- To solve a quadratic-quadratic system of equations, you may use a substitution method or an elimination method. When solving this type of system using elimination, you cannot eliminate x, so you should always eliminate y.

  Example
  $y = 4x^2 + 8x + 4$
  $y = 3x^2 - 2x - 5$

  Since both equations equal y, the second equation can be subtracted from the first equation to eliminate y.

  $$\begin{array}{r} y = 4x^2 + 8x + 4 \\ -y = -3x^2 + 2x + 5 \\ \hline 0 = x^2 + 10x + 9 \end{array}$$

  Since both equations equal y, the substitution may also be used. Setting the quadratic expressions $4x^2 + 8x + 4$ and $3x^2 - 2x - 5$ equal to each other and then solving produces the same quadratic equation as the elimination method.

  To solve for x, factor $x^2 + 10x + 9$.
  $0 = (x + 9)(x + 1)$

  Set each factor equal to zero. Then, solve.
  $x + 9 = 0$ or $x + 1 = 0$
  $x = -9$ $\quad$ $x = -1$

  Substitute these values into either of the original equations to determine the corresponding values of y.

When $x = -9$:	When $x = -1$:
$y = 4x^2 + 8x + 4$	$y = 4x^2 + 8x + 4$
$y = 4(\mathbf{-9})^2 + 8(\mathbf{-9}) + 4$	$y = 4(\mathbf{-1})^2 + 8(\mathbf{-1}) + 4$
$y = 324 - 72 + 4$	$y = 4 - 8 + 4$
$y = 256$	$y = 0$

  The two solutions are (256, –9) and (–1, 0).

- For both types of systems, remember to take the values you determine and substitute them back into the original equation to find the corresponding values of the other variable. You should always verify solutions by substituting into both of the original equations.

- To solve linear-quadratic and quadratic-quadratic systems, you must determine the solution to a quadratic equation. Remember that to solve a quadratic equation means to determine the x-intercepts of the graph of the corresponding quadratic function or the zeros of the function. This can be done by factoring and setting each factor equal to zero to solve for x. In the case of un-factorable quadratics, the quadratic formula can be used. Once you determine a value of x, substitute it into one of the original equations to determine the corresponding value of y.

Working Example 1: Solve a System of Linear-Quadratic Equations Algebraically

a) Solve the following system of equations algebraically.

$x^2 - x - y = 6$
$2x - y = 2$

b) Verify your solution.

Solution

a) Rewrite each equation so that y is isolated on one side of the equal sign.

$x^2 - x - y = 6$ ⟶ $y = \square^2 - \square - \square$ ①

$2x - y = 2$ ⟶ $y = \square - \square$ ②

Since each equation is equal to y, you can use either elimination or substitution.

Method 1: Use Elimination

You could eliminate ________ by ________________ ② from ①. You would
(x or y)
end up with a quadratic equation with only ________ variable.

Method 2: Use Substitution

You could substitute the expression for ________ from the first equation into the second equation.

Solve using either method.

State your solution(s) as ordered pair(s).

b) Verify both solutions in both of the original equations.

Working Example 2: Solve a System of Quadratic-Quadratic Equations Algebraically

a) Solve the following system of equations algebraically.

$3x^2 - 24x - y = -52$
$-2x^2 - y + 16x - 28 = 0$

b) Verify the solution(s) in the original equations.

Solution

a) Rewrite each equation so that y is isolated on one side of the equal sign.

$3x^2 - 24x - y = -52 \longrightarrow y = \square^2 - \square + \square$ ①

$-2x^2 - y + 16x - 28 = 0 \longrightarrow y = \square^2 + \square - \square$ ②

To solve this system using elimination, you could eliminate ______ (x or y) by ____________ ② from ①. You would end up with a quadratic equation with only ______ variable.

① – ②:

Can you divide each term by a common factor to make the quadratic easier to work with?

Factor your quadratic:

How many solutions are there?

b) Verify the solution(s) by substituting into both of the original equations.

Working Example 3: Model a Situation With a System of Equations

Two golfers are practising their swing at a driving range. Each hits a ball at the same time. The path of the first ball can be modelled by the equation $y + 27 = -5x^2 + 40x$. The path of the second ball can be modelled by the equation $y + 8x^2 = 40x$. In each equation, y is the height above ground, in metres, after x seconds. Determine and interpret the solution(s) to the system of equations.

Solution

Write each equation as a "$y =$" equation.

Eliminate y.

Solve the resulting equation for x.

Why is there only one possible answer for *x*?

Use your value for x to determine the corresponding value for y.

State your answer as a point.

What does the solution mean in the context of the question?

Verify your solution in the original equations.

See pages 443–446 of *Pre-Calculus 11* for more examples.

Check Your Understanding

Practise

1. Solve the following systems of linear-quadratic equations algebraically. Verify your solution(s).

a) $x^2 - y - 2x = -2$
$y - 2x = -2$

Isolate y in each equation.

Solve using elimination or substitution.

State the point(s) of your solution.

Check your solution(s) in both of the original equations.

b) $x^2 - y = -1$
$y - x = 1$

c) $3x - y = -5$
$x^2 - y + 2x = 1$

d) $-2x^2 - y + 3x = -4$
$2x - y + 3 = 0$

2. Solve the following systems of quadratic-quadratic equations algebraically, and verify your solution(s).

a) $-4x^2 - y - 2x = -5$
$3x^2 - 4y - 46x - 37 = 0$

Isolate the y-term in each equation.

How can you make the y-values in each equation match?

Solve using elimination.

State the point(s) of your solution.

Check your solution(s) in both of the original equations.

b) $4x^2 - y + 8x = -2$
$y + 2 = 4x^2 - 8x$

c) $2x^2 + 12x - y = -17$
$y = -x^2 - x + 7$

d) $y = \frac{1}{2}x^2 - 4x + 12$
$y = -2x^2 - 12x - 23$

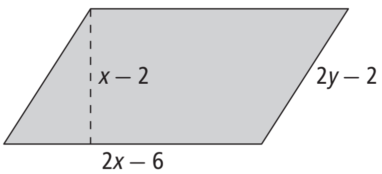
This question should help you complete #4 on page 452 of *Pre-Calculus 11*.

Apply

3. Determine the values of m and n if (3, 4) is a solution to the following system of equations.

$mx^2 - y = 32$
$mx^2 - 5y = n$

4. The perimeter of a parallelogram is 28 cm. Its area is $10y$ cm^2.



a) Write a simplified expression for the parallelogram's perimeter.

b) Write a simplified expression for the parallelogram's area.

c) Write a system of equations and solve the system for x and y.

d) Interpret your solution.

5. You have two unknown integers. Double the larger number increased by triple the smaller number is 46. Squaring the larger number and increasing it by four times itself gives the same result as multiplying the smaller number by 20 and adding 5. What are the two integers?

Connect

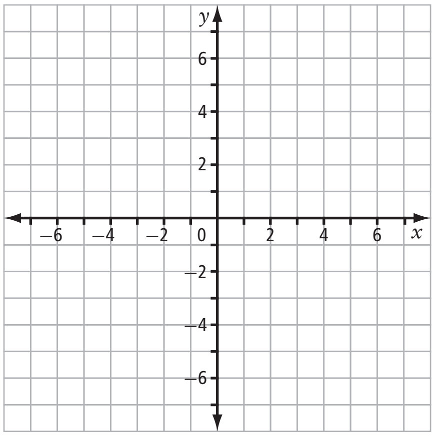
6. A parabola has its vertex at (3, −1) and passes through the point (1, 7). A second parabola has its vertex at (1, 4) and has *y*-intercept 3. What are the approximate coordinates of the point(s) at which these parabolas intersect?

Chapter 8 Review

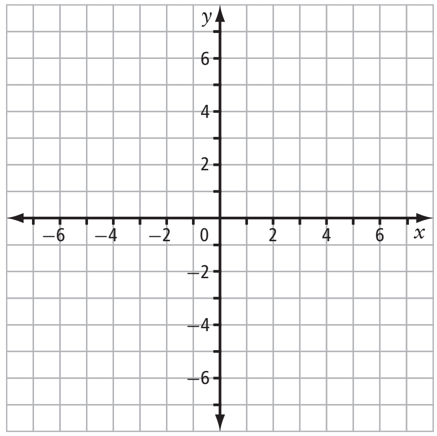
8.1 Solving Systems of Equations Graphically, pages 333–344

1. Sketch graphs to show the possible solutions to each system of equations.

a) linear-quadratic

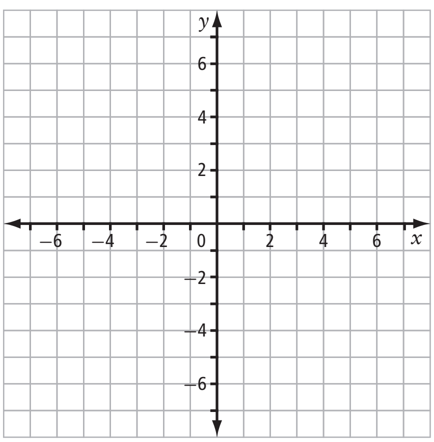


b) quadratic-quadratic

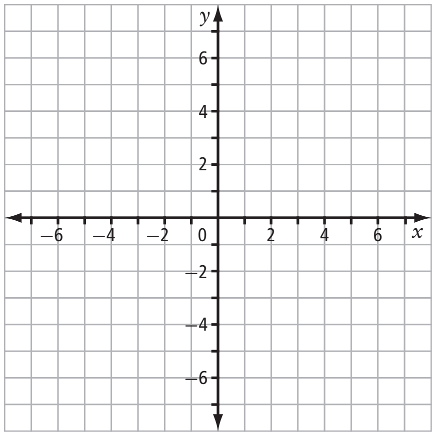


2. Solve each system of equations by graphing. Verify your solutions.

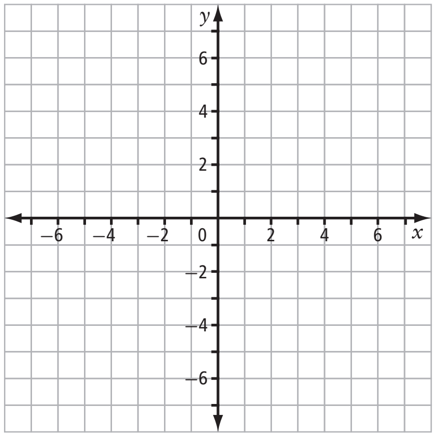
a) $2y + 6x^2 + x - 8 = 0$
$3y + x - 12 = 0$



b) $y - x^2 + 2x + 4 = 0$
$y - 3x + 8 = 0$

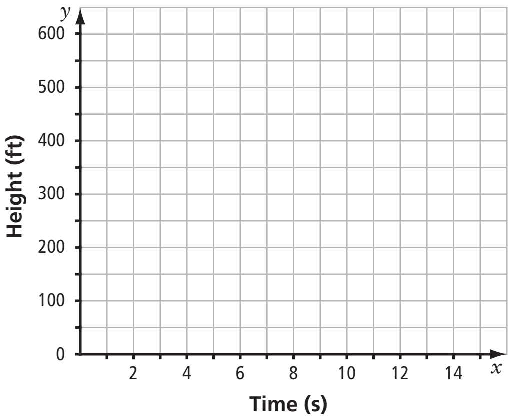


c) $y - 3x^2 - 12x = 6$
$y + 4x^2 + 16x = -15$



3. A model rocket is launched from a field. The height of the rocket, y, in feet above the ground, after x seconds is modelled by the equation $y = -16x^2 + 177x + 4$. From the 10th floor of a nearby building, a boy looks out a window when he hears the rocket fired. The boy's line of sight is given by the equation $y = 65x + 100$.

a) Graph the path of the rocket and the boy's line of sight on the same set of axes.



b) Determine and interpret the point(s) of intersection.

8.2 Solving Systems of Equations Algebraically, pages 345–355

4. Solve each system algebraically. Verify your solutions.

a) $4x^2 - y - 2x = -7$
$-6x - y + 15 = 0$

b) $2x^2 - y + 16x + 29 = 0$
$-2x^2 - 16x - y = 35$

c) $3x^2 - y - 4x = -3$
$-2x^2 + 2x - y + 8 = 0$

5. Determine two whole numbers such that the first number increased by triple the second number is 24. If the first number is squared and decreased by five times itself, the result is 13 less than the second number.

6. Two players are throwing basketballs back and forth. Standing about 9 m apart and facing each other, each player throws a ball at the same time. In one exchange, the path of one basketball is represented by the equation $y = -x^2 + 12x - 28$. The path of the other ball is modelled by the equation $y = -x^2 + 4x + 2$. In each equation, x is the horizontal distance a ball travels, in metres, and y is the vertical distance travels, also in metres. Determine the point(s) of intersection algebraically and interpret the solution.

Chapter 8 Skills Organizer

Complete the table for solving non-linear systems of equations.

Type of System	Possible Number of Solutions	How to Solve Graphically	How to Solve Algebraically
Linear-quadratic			
Quadratic-quadratic			